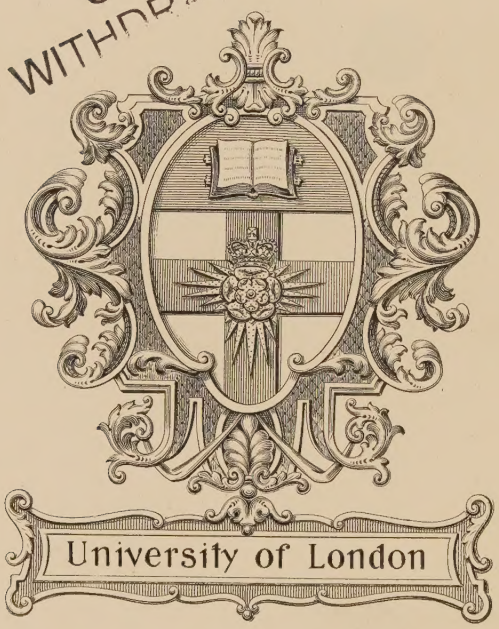
P.R. [PRIVATE dm.]
DEPOSITORY

SHL
WITHDRAWN



PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 40

LONDON:
PRINTED AND PUBLISHED FOR THE SOCIETY BY
C. F. HODGSON & SON, LTD.,
2 NEWTON STREET, KINGSWAY, W.C.2

1936

BIBL.
LONDIN.
UNIV.

CONTENTS.

Papers published in the *Proceedings*, pp. 1–555.

Index of Papers published in this Volume arranged according to the names of Authors, p. 556.

Index according to Subjects, p. 557.

Corrigenda, p. 558.

Digitized by the Internet Archive
in 2026

https://archive.org/details/london-mathematical-society-proceedings_1936_40

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

INTEGRALDARSTELLUNGEN AUS DER THEORIE DER BESSELSCHEN FUNKTIONEN

Von C. S. MEIJER.

[Received 22 May, 1934.—Read 14 June, 1934.]

Einleitung.

Es bezeichnen $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$ und $H_\nu^{(2)}(z)$ die Besselschen und Hankelschen Funktionen. Es sei weiter

$$I_\nu(z) = e^{-\frac{1}{2}\nu\pi i} J_\nu(ze^{\frac{1}{2}\pi i}), \tag{1}$$

$$K_\nu(z) = \tfrac{1}{2}\pi i\, e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(ze^{\frac{1}{2}\pi i}), \tag{2}$$

$$V_{\nu,\mu}(z) = \tfrac{1}{2}\{e^{\frac{1}{2}(\nu-\mu)\pi i} H_\nu^{(1)}(z) H_\mu^{(2)}(z) + e^{\frac{1}{2}(\mu-\nu)\pi i} H_\mu^{(1)}(z) H_\nu^{(2)}(z)\} \tag{3}$$

und

$$U_{\nu,\mu}(z) = \frac{1}{2i}\{e^{\frac{1}{2}(\nu-\mu)\pi i} H_\nu^{(1)}(z) H_\mu^{(2)}(z) - e^{\frac{1}{2}(\mu-\nu)\pi i} H_\mu^{(1)}(z) H_\nu^{(2)}(z)\}. \tag{4}$$

Es bezeichne ferner

$${}_pF_q(a_1, \ldots, a_p;\ r_1, \ldots, r_q;\ z)$$

die verallgemeinerte hypergeometrische Funktion und es sei schliesslich $S_{\mu,\nu}(z)$ die Lommelsche† und $W_{k,m}(z)$ die Whittakersche‡ Funktion.

† Man vergl. E. Lommel, *Math. Annalen*, 9 (1876), 425–444. Siehe auch G. N. Watson, *A treatise on the theory of Bessel functions* (1922), 345–352; das letztere Werk wird in der Folge mit B.F. zitiert.

‡ Siehe E. T. Whittaker and G. N. Watson, *A course of modern analysis*, ch. 16. Dieses Werk wird in der Folge mit M.A. zitiert.

In der vorliegenden Abhandlung werde ich unter Annahme der obigen Bezeichnungen die folgenden sechs Sätze beweisen. Die Sätze 3, 5 und 6 sind Spezialfälle von Satz 1. Satz 4 ist ein Spezialfall von Satz 2.

SATZ 1†. *Es sei*

$$n \text{ ganz} \geqslant 2,$$

$$\zeta \neq 0, \quad -2\pi < \arg\zeta < 2\pi,$$

$$\max(-\tfrac{1}{2}\pi, -\tfrac{1}{2}\pi+\tfrac{1}{2}\arg\zeta) < \tau < \min(\tfrac{1}{2}\pi, \tfrac{1}{2}\pi+\tfrac{1}{2}\arg\zeta), \tag{5}$$

$$r_j \neq 0, \quad -1, \quad -2, \quad \ldots \quad (j = 1, \ldots, n-2)\ddagger,$$

$$a_j - a_h \text{ nicht ganz } (j = 1, \ldots, n+1; \ h = 1, \ldots, n+1; \ j \neq h).$$

Behauptung§:

$$\sum_{h=1}^{n+1} \frac{\prod_{\substack{j=1 \\ j\neq h}}^{n+1} \Gamma(a_j - a_h)/\Gamma(a_j)}{\prod_{j=1}^{n-2} \Gamma(r_j - a_h)} \zeta^{a_h} \tag{6}$$

$$\times {}_{n-1}F_n(a_h, 1+a_h-r_1, \ldots, 1+a_h-r_{n-2}; \ 1+a_h-a_1, \overset{*}{\ldots}, 1+a_h-a_{n+1}; \ -\zeta)$$

$$= \frac{4}{\Gamma(\alpha)\Gamma(\beta)\prod_{j=1}^{n-2}\Gamma(r_j)} \int_0^{\infty e^{i\tau}} u^{\alpha+\beta-1} K_{\alpha-\beta}(2u)$$

$$\times {}_{n+1}F_n\left(a_1, \ldots, a_{n+1}; \ r_1, \ldots, r_{n-2}, \alpha, \beta; \ -\frac{u^2}{\zeta}\right) du.$$

Hierin sind α *und* β *beliebige Zahlen mit* $R(\alpha) > 0$, $R(\beta) > 0$.

SATZ 2. *Es sei*

$$n \text{ ganz} \geqslant 2,$$

$$\zeta \neq 0, \quad -\pi < \arg\zeta < \pi,$$

$$r_j \neq 0, \quad -1, \quad -2, \quad \ldots \quad (j = 1, \ldots, n-2),$$

$$a_j - a_h \text{ nicht ganz } (j = 1, \ldots, n+1; \ h = 1, \ldots, n+1; \ j \neq h),$$

$$\min_{h=1, \ldots, n+1} \{R(a_h)\} > -\tfrac{3}{4}.$$

---

† Einige verwandte Sätze kommen vor in meiner Arbeit: "Uber Whittakersche bezw. Besselsche Funktionen und deren Produkte". Diese Arbeit wird demnächst im *Nieuw Archief voor Wiskunde* erscheinen.

‡ Die Zahlen r_j kommen nicht vor, wenn $n = 2$ ist.

§ Ein leeres Produkt wird gleich 1 gesetzt.

Der Stern bedeutet, dasz die Zahl $1+a_h-a_h$ in der Reihe $1+a_h-a_1, \ldots, 1+a_h-a_{n+1}$ nicht vorkommt. Für $n = 2$ besteht die Reihe $a_h, 1+a-r_1, \ldots, 1+a_h-r_{n-2}$ nur aus der Zahl a_h, und ebenso die Reihe $r_1, \ldots, r_{n-2}, \alpha, \beta$ nur aus den Zahlen α und β.

Behauptung:

$$(7)\quad \frac{1}{\Gamma(a_{n+1})}\sum_{h=1}^{n}\frac{\prod_{\substack{j=1\\ j\neq h}}^{n}\Gamma(a_j-a_h)/\Gamma(a_j)}{\Gamma(1+a_h-a_{n+1})\prod_{j=1}^{n-2}\Gamma(r_j-a_h)}\zeta^{a_h}$$

$$\times {}_{n-1}F_n(a_h,\ 1+a_h-r_1,\ \ldots,\ 1+a_h-r_{n-2};\ 1+a_h-a_1,\ \overset{*}{\ldots},\ 1+a_h-a_{n+1};\ \zeta)$$

$$= \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)\prod_{j=1}^{n-2}\Gamma(r_j)}\int_0^\infty v^{\alpha+\beta-1}\{e^{(a_{n+1}-\beta)\pi i}H^{(1)}_{\alpha-\beta}(2v)+e^{(\beta-a_{n+1})\pi i}H^{(2)}_{\alpha-\beta}(2v)\}$$

$$\times {}_{n+1}F_n\left(a_1,\ \ldots,\ a_{n+1};\ r_1,\ \ldots,\ r_{n-2},\ \alpha,\ \beta;\ -\frac{v^2}{\zeta}\right)dv.$$

Hierin sind α und β beliebige Zahlen, die den Bedingungen

$$R(\alpha)>0,\quad R(\beta)>0,\quad R(\alpha+\beta)<\tfrac{3}{2}+2\min_{h=1,\ldots,n+1}\{R(a_h)\}$$

genügen†.

SATZ 3. *Es sei $z\neq 0$, $-\pi<\arg z<\pi$ und*

$$(8)\qquad \max(-\tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi+\arg z)<\tau<\min(\tfrac{1}{2}\pi,\ \tfrac{1}{2}\pi+\arg z).$$

Behauptungen:

$$(9)\quad V_{\nu,\mu}(z)=\frac{8}{\pi z\,\Gamma(\alpha)\,\Gamma(\beta)}\int_0^{\infty e^{i\tau}}u^{\alpha+\beta-1}K_{\alpha-\beta}(2u)$$

$$\times {}_4F_3\left(\frac{1+\nu+\mu}{2},\ \frac{1-\nu+\mu}{2},\ \frac{1+\nu-\mu}{2},\ \frac{1-\nu-\mu}{2};\ \tfrac{1}{2},\ \alpha,\ \beta;\ -\frac{u^2}{z^2}\right)du,$$

$$(10)\quad U_{\nu,\mu}(z)=\frac{4(\nu^2-\mu^2)}{\pi z^2\,\Gamma(\alpha)\,\Gamma(\beta)}\int_0^{\infty e^{i\tau}}u^{\alpha+\beta-1}K_{\alpha-\beta}(2u)$$

$$\times {}_4F_3\left(\frac{2+\nu+\mu}{2},\ \frac{2-\nu+\mu}{2},\ \frac{2+\nu-\mu}{2},\ \frac{2-\nu-\mu}{2};\ \tfrac{3}{2},\ \alpha,\ \beta;\ -\frac{u^2}{z^2}\right)du.$$

Hierin sind α und β beliebige Zahlen mit $R(\alpha)>0$, $R(\beta)>0$.

---

† Die folgenden Spezialfälle von Satz 2 sind von Herrn Watson (B.F., §13.52 und §13.6) bewiesen worden:

$$n=2,\quad a_1=\alpha=\frac{\rho+\nu}{2},\quad a_2=\beta=\frac{\rho-\nu}{2},\quad a_3=1+\mu,\quad \zeta=\frac{a^2k^2}{4},\quad v=\frac{ax}{2};$$

$$n=2,\quad a_1=\alpha=\frac{\rho+\nu}{2},\quad a_2=1+\mu,\quad a_3=\beta=\frac{\rho-\nu}{2},\quad \zeta=\frac{a^2k^2}{4},\quad v=\frac{ax}{2}.$$

Man beachte hierbei, dasz

$${}_3F_2\left(\alpha,\ \beta,\ 1+\mu;\ \alpha,\ \beta;\ -\frac{x^2}{k^2}\right)=k^{2+2\mu}(x^2+k^2)^{-1-\mu}$$

ist.

Man vergl. ferner auch N. Nielsen, *Handbuch der Theorie der Cylinderfunktionen* (1904), 211–221.

SATZ 4. *Es sei* $z \neq 0$ *und* $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi$.

Behauptungen†:

1. *Ist* $|R(\nu)|+|R(\mu)| < \frac{5}{2}$, *dann ist*

$$\begin{aligned}(11)\quad &\{-2I_\nu(z)\,I_\mu(z)\sin\tfrac{1}{2}(\nu+\mu)\pi\cos\tfrac{1}{2}(\nu-\mu)\pi \\ &\quad +I_{-\nu}(z)\,I_\mu(z)\sin\mu\pi+I_\nu(z)\,I_{-\mu}(z)\sin\nu\pi\}\cos\tfrac{1}{2}(\nu+\mu)\pi \\ &=\frac{2i\sin\nu\pi\sin\mu\pi}{z\,\Gamma(\alpha)\,\Gamma(\beta)}\int_0^\infty v^{\alpha+\beta-1}\{e^{-\frac{1}{2}(\nu+\mu)\pi i-\beta\pi i}H^{(1)}_{\alpha-\beta}(2v)-e^{\frac{1}{2}(\nu+\mu)\pi i+\beta\pi i}H^{(2)}_{\alpha-\beta}(2v)\} \\ &\quad\times{}_4F_3\left(\frac{1+\nu+\mu}{2},\frac{1-\nu+\mu}{2},\frac{1+\nu-\mu}{2},\frac{1-\nu-\mu}{2};\ \tfrac{1}{2},\alpha,\beta;\ -\frac{v^2}{z^2}\right)dv.\end{aligned}$$

Hierin sind α *und* β *beliebige Zahlen, die den Bedingungen*

$$R(\alpha)>0,\quad R(\beta)>0,\quad R(\alpha+\beta)<\tfrac{5}{2}-|R(\nu)|-|R(\mu)|$$

genügen.

2. *Ist* $|R(\nu)|+|R(\mu)| < \frac{7}{2}$, *dann ist*

$$\begin{aligned}&\{-2I_\nu(z)\,I_\mu(z)\cos\tfrac{1}{2}(\nu+\mu)\pi\sin\tfrac{1}{2}(\nu-\mu)\pi \\ &\quad -I_{-\nu}(z)\,I_\mu(z)\sin\mu\pi+I_\nu(z)\,I_{-\mu}(z)\sin\nu\pi\}\sin\tfrac{1}{2}(\nu+\mu)\pi \\ &=\frac{(\nu^2-\mu^2)\sin\nu\pi\sin\mu\pi}{z^2\,\Gamma(\alpha)\,\Gamma(\beta)}\int_0^\infty v^{\alpha+\beta-1}\{e^{-\{\frac{1}{2}(\nu+\mu)+\beta\}\pi i}H^{(1)}_{\alpha-\beta}(2v)+e^{\{\frac{1}{2}(\nu+\mu)+\beta\}\pi i}H^{(2)}_{\alpha-\beta}(2v)\} \\ &\quad\times{}_4F_3\left(\frac{2+\nu+\mu}{2},\frac{2-\nu+\mu}{2},\frac{2+\nu-\mu}{2},\frac{2-\nu-\mu}{2};\ \tfrac{3}{2},\alpha,\beta;\ -\frac{v^2}{z^2}\right)dv.\end{aligned}$$

Hierin sind α *und* β *beliebige Zahlen, die den Bedingungen*

$$R(\alpha)>0,\quad R(\beta)>0,\quad R(\alpha+\beta)<\tfrac{7}{2}-|R(\nu)|-|R(\mu)|$$

genügen.

SATZ 5. *Ist* $z \neq 0$, $-\pi < \arg z < \pi$ *und genügt* τ *der Bedingung* (8), *dann ist*

$$\begin{aligned}(12)\quad S_{\mu,\nu}(z)&=\frac{4z^{\mu-1}}{\Gamma(\alpha)\,\Gamma(\beta)}\int_0^{\infty e^{i\tau}}u^{\alpha+\beta-1}K_{\alpha-\beta}(2u) \\ &\quad\times{}_3F_2\left(1,\frac{1+\nu-\mu}{2},\frac{1-\nu-\mu}{2};\ \alpha,\beta;\ -\frac{4u^2}{z^2}\right)du.\end{aligned}$$

Hierin sind α *und* β *beliebige Zahlen mit* $R(\alpha)>0$, $R(\beta)>0$.

---

† Ein sehr einfacher Spezialfall ist Formel (71) von § 4.

SATZ 6. *Ist* $z \neq 0$, $-\pi < \arg z < \pi$ *und genügt* τ *der Bedingung* (8), *dann ist*†

$$(13)\quad W_{k,m}(iz)\,W_{k,m}(-iz) = \frac{4z^{2k}}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^{\infty e^{i\tau}} u^{\alpha+\beta-1} K_{\alpha-\beta}(2u) \times {}_4F_3\left(\tfrac{1}{2}+m-k,\ \tfrac{1}{2}-m-k,\ \tfrac{1}{2}-k,\ 1-k;\ 1-2k,\ \alpha,\ \beta;\ -\frac{4u^2}{z^2}\right) du.$$

Hierin sind α *und* β *beliebige Zahlen mit* $R(\alpha) > 0$, $R(\beta) > 0$.

Diese Sätze waren, soviel ich weisz, bis jetzt nicht bekannt. Viele Spezialfälle sind aber schon gegeben worden. In den Paragraphen 4 und 6 werde ich zeigen, dasz z. B. die Beziehungen (47) bis (51), (76) und (77), die man bisher mit Hilfe von andern Methoden bewiesen hat, nur Spezialfälle der Sätze 3 und 5 sind.

Die in der vorliegenden Abhandlung gefundenen Integraldarstellungen erinnern in vielen Hinsichten an die Integralformeln, die ich vor kurzem in drei anderen Arbeiten‡ für Whittakersche und Besselsche Funktionen abgeleitet habe. Jedoch sind die hier erhaltenen Ergebnisse komplizierter als die damaligen.

1. *Beweis von Satz* 1.

Setzt man die linke Seite von (6) der Kürze halber gleich $G(\zeta)$, dann hat man nach Herrn Barnes§

$$(14)\quad 2\pi i\, G(\zeta) \prod_{j=1}^{n+1} \Gamma(a_j) = \int_L \left(\Gamma(s) \prod_{j=1}^{n+1} \Gamma(a_j - s) \Big/ \prod_{j=1}^{n-2} \Gamma(r_j - s)\right) \zeta^s\, ds.$$

Diese Beziehung gilt, wenn die folgenden Bedingungen erfüllt sind:

$$(15)\quad n \text{ ganz} \geqslant 2,$$

$$(16)\quad \zeta \neq 0, \quad |\arg \zeta| < 2\pi,$$

$$(17)\quad a_j \neq 0,\ -1,\ -2,\ \ldots \quad (j = 1, \ldots, n+1),$$

$$(18)\quad a_j - a_h \text{ nicht ganz} \quad (j = 1, \ldots, n+1;\ h = 1, \ldots, n+1;\ j \neq h).$$

---

† Man hat

$${}_4F_3(a, b, \tfrac{1}{2}-k, 1-k;\ 1-2k, \alpha, \beta;\ w) = 1 + \frac{ab(1-k)}{1!\,\alpha\beta}\,\frac{w}{2} + \frac{a(a+1)\,b(b+1)(\tfrac{3}{2}-k)(2-k)}{2!\,\alpha(\alpha+1)\,\beta(\beta+1)}\left(\frac{w}{2}\right)^2 + \ldots.$$

Die rechte Seite von (13) ist also nicht sinnlos, falls $1-2k = 0, -1, -2, \ldots$ ist.

‡ "Über die Integraldarstellungen der Whittakerschen Funktion $W_{k,m}(z)$ und der Hankelschen und Besselschen Funktionen", *Nieuw Archief voor Wiskunde* (2e Reeks), 18 (1934), 35–57. "Einige Integraldarstellungen für Whittakersche und Besselsche Funktionen", und "Noch einige Integraldarstellungen für die Whittakersche Funktion", *Proc. Koninklijke Akademie van Wetenschappen te Amsterdam*, 37 (1934), 805–812, und 38 (1935), 528–535.

§ E. W. Barnes, *Proc. London Math. Soc.* (2), 5 (1907), 65–67.

Der Integrationsweg L läuft von $-\infty i$ nach ∞i, und zwar auf derartige Weise, dasz die Punkte a_j, a_j+1, a_j+2, ... $(j=1, \ldots, n+1)$ auf der rechten, die Punkte $0, -1, -2, \ldots$ aber auf der linken Seite von L liegen†.

Weiter hat man gleichfalls nach Herrn Barnes‡

$$(19)\quad \frac{2\pi i \prod_{j=1}^{n+1}\Gamma(a_j)}{\prod_{j=1}^{n}\Gamma(r_j)}\,{}_{n+1}F_n(a_1, \ldots, a_{n+1};\ r_1, \ldots, r_n;\ z) = \int_L \left(\Gamma(s)\prod_{j=1}^{n+1}\Gamma(a_j-s)\Big/\prod_{j=1}^{n}\Gamma(r_j-s)\right)(-z)^{-s}\,ds.$$

Hierin wird n ganz $\geqq 0$, $z\neq 0$ und $|\arg(-z)|<\pi$ vorausgesetzt und es wird angenommen, dasz die Zahlen $a_j\neq 0, -1, -2, \ldots$ $(j=1, \ldots, n+1)$ und $r_j\neq 0, -1, -2, \ldots$ $(j=1, \ldots, n)$ sind.

Ferner gilt [man vergl. B.F., 388, Formel (8)], falls $R(\alpha-s)>0$, $R(\beta-s)>0$ und $-\frac{1}{2}\pi<\tau<\frac{1}{2}\pi$ ist,

$$(20)\quad \int_0^{\infty e^{i\tau}} u^{\alpha+\beta-2s-1}K_{\alpha-\beta}(2u)\,du = \tfrac{1}{4}\Gamma(\alpha-s)\,\Gamma(\beta-s).$$

Ich betrachte jetzt das Integral

$$(21)\quad I=\frac{2\pi i\prod_{j=1}^{n+1}\Gamma(a_j)}{\Gamma(\alpha)\,\Gamma(\beta)\prod_{j=1}^{n-2}\Gamma(r_j)}\int_0^{\infty e^{i\tau}} u^{\alpha+\beta-1}K_{\alpha-\beta}(2u) \times {}_{n+1}F_n\left(a_1, \ldots, a_{n+1};\ r_1, \ldots, r_{n-2}, \alpha, \beta;\ -\frac{u^2}{\zeta}\right)du.$$

Ich nehme hierbei an, dasz die Bedingungen (15), (16), (17) und (18) erfüllt sind und, dasz überdies noch gilt:

$$R(\alpha)>0,\quad R(\beta)>0,$$

$$(22)\quad \max(-\tfrac{1}{2}\pi, -\tfrac{1}{2}\pi+\tfrac{1}{2}\arg\zeta)<\tau<\min(\tfrac{1}{2}\pi, \tfrac{1}{2}\pi+\tfrac{1}{2}\arg\zeta)$$

und

$$r_j\neq 0, -1, -2, \ldots \quad (j=1, \ldots, n-2).$$

Ist $\arg u=\tau$, dann folgt aus (22),

$$\left|\arg\left(\frac{u^2}{\zeta}\right)\right|<\pi.$$

† Dies ist möglich, wegen (17).

‡ *Proc. London Math. Soc.* (2), 6 (1908), 146. Siehe auch M.A , § 14.5. Herr Barnes gibt Beziehung (19) nur für $n=1$.

Für $u \neq 0$ hat man also wegen (19)

$$(23)\quad \frac{2\pi i \prod_{j=1}^{n+1} \Gamma(a_j)}{\Gamma(\alpha)\,\Gamma(\beta) \prod_{j=1}^{n-2} \Gamma(r_j)}\, {}_{n+1}F_n\left(a_1, \ldots, a_{n+1};\ r_1, \ldots, r_{n-2},\ \alpha,\ \beta;\ -\frac{u^2}{\zeta}\right)$$

$$= \int_L \left(\Gamma(s) \prod_{j=1}^{n+1} \Gamma(a_j - s) \Big/ \Gamma(\alpha - s)\,\Gamma(\beta - s) \prod_{j=1}^{n-2} \Gamma(r_j - s)\right) \left(\frac{u^2}{\zeta}\right)^{-s} ds.$$

Ich wähle nun den Integrationsweg L derart, dasz $R(\alpha - s) > 0$ und $R(\beta - s) > 0$ ist für alle Punkte s von L†. Aus (21), (23), (20) und (14) ergibt sich also, falls $\delta > 0$ ist‡,

$$I = \lim_{\delta \to 0} \int_{\delta e^{i\tau}}^{\infty e^{i\tau}} u^{\alpha+\beta-1} K_{\alpha-\beta}(2u)\, du$$

$$\times \int_L \left(\Gamma(s) \prod_{j=1}^{n+1} \Gamma(a_j - s) \Big/ \Gamma(\alpha - s)\,\Gamma(\beta - s) \prod_{j=1}^{n-2} \Gamma(r_j - s)\right) \left(\frac{u^2}{\zeta}\right)^{-s} ds$$

$$= \lim_{\delta \to 0} \int_L \left(\Gamma(s) \prod_{j=1}^{n+1} \Gamma(a_j - s) \Big/ \Gamma(\alpha - s)\,\Gamma(\beta - s) \prod_{j=1}^{n-2} \Gamma(r_j - s)\right) \zeta^s\, ds$$

$$\times \int_{\delta e^{i\tau}}^{\infty e^{i\tau}} u^{\alpha+\beta-2s-1} K_{\alpha-\beta}(2u)\, du$$

$$= \int_L \left(\Gamma(s) \prod_{j=1}^{n+1} \Gamma(a_j - s) \Big/ \Gamma(\alpha - s)\,\Gamma(\beta - s) \prod_{j=1}^{n-2} \Gamma(r_j - s)\right) \zeta^s\, ds$$

$$\times \int_0^{\infty e^{i\tau}} u^{\alpha+\beta-2s-1} K_{\alpha-\beta}(2u)\, du$$

$$= \tfrac{1}{4} \int_L \left(\Gamma(s) \prod_{j=1}^{n+1} \Gamma(a_j - s) \Big/ \prod_{j=1}^{n-2} \Gamma(r_j - s)\right) \zeta^s\, ds = \frac{\pi i}{2}\, G(\zeta) \prod_{j=1}^{n+1} \Gamma(a_j).$$

Hieraus und aus (21) folgt Beziehung (6), da $G(\zeta)$ gleich der linken Seite von (6) ist. Satz 1 ist also bewiesen für den Fall, dasz $a_j \neq 0, -1, -2, \ldots$ ist

---

† Dies ist möglich wegen $R(\alpha) > 0$, $R(\beta) > 0$.

‡ Die Vertauschung der Integrationsfolge ist erlaubt; man vergl. T. J. I'A. Bromwich, *An introduction to the theory of infinite series*, § 177. Siehe auch N. Nielsen, *Handbuch der Theorie der Gammafunktion* (1906), § 38.

Es ist ferner leicht einzusehen, dasz

$$\lim_{\delta \to 0} \int_L ds \int_{\delta e^{i\tau}}^{\infty e^{i\tau}} du = \int_L ds \int_0^{\infty e^{i\tau}} du$$

ist. [Man vergl. auch die Beziehungen (32) und (33) von § 2.]

$(j = 1, \ldots, n+1)$. Da aber beide Seiten von (6) stetige Funktionen von a_j sind, so gilt Formel (6) auch noch, falls $a_j = 0, -1, -2, \ldots$ ist.

2. *Beweis von Satz* 2.

Ich gehe von Formel (6) aus; ich nehme also an, dasz alle Voraussetzungen von Satz 1 erfüllt sind. Ferner nehme ich noch an, dasz

$$\min_{h=1, \ldots, n+1} \{R(a_h)\} > -\tfrac{3}{4}$$

ist. Ich setze zur Abkürzung

$$(24) \qquad {}_{n+1}F_n(a_1, \ldots, a_{n+1};\ r_1, \ldots, r_{n-2},\ \alpha, \beta;\ z) = \Psi(z).$$

Es seien jetzt α und β zwei beliebige Zahlen, die den Bedingungen

$$R(\alpha) > 0, \quad R(\beta) > 0,$$

$$(25) \qquad R(\alpha+\beta) < \tfrac{3}{2} + 2 \min_{h=1, \ldots, n+1} \{R(a_h)\}$$

genügen. Ich werde zeigen, dasz dann

$$(26) \qquad \int_0^{\infty e^{ir}} u^{\alpha+\beta-1} K_{\alpha-\beta}(2u)\, \Psi\left(-\frac{u^2}{\zeta}\right) du = \int_0^{\infty e^{\frac{1}{2}\pi i}} u^{\alpha+\beta-1} K_{\alpha-\beta}(2u)\, \Psi\left(-\frac{u^2}{\zeta}\right) du$$

ist, falls auszerdem $0 < \arg \zeta < 2\pi$ ist und, dasz

$$(27) \qquad \int_0^{\infty e^{ir}} u^{\alpha+\beta-1} K_{\alpha-\beta}(2u)\, \Psi\left(-\frac{u^2}{\zeta}\right) du = \int_0^{\infty e^{-\frac{1}{2}\pi i}} u^{\alpha+\beta-1} K_{\alpha-\beta}(2u)\, \Psi\left(-\frac{u^2}{\zeta}\right) du$$

ist, falls auszerdem $-2\pi < \arg \zeta < 0$ ist.

Der Beweis von (26) geht folgendermaszen:

Wegen (5) und $0 < \arg \zeta < 2\pi$ hat man

$$(28) \qquad -\tfrac{1}{2}\pi < -\tfrac{1}{2}\pi + \tfrac{1}{2} \arg \zeta < \tau < \tfrac{1}{2}\pi.$$

Es sei nun $0 < \delta < 1 < D$. Ich betrachte in der u-Ebene die Kurve N, bestehend aus den Geraden g_1 $(\arg u = \tau,\ \delta \leqslant |u| \leqslant D)$ und g_2 $(\arg u = \frac{1}{2}\pi,\ \delta \leqslant |u| \leqslant D)$ und den Kreisbogen C_1 $(|u| = \delta,\ \tau \leqslant \arg u \leqslant \frac{1}{2}\pi)$ und C_2 $(|u| = D,\ \tau \leqslant \arg u \leqslant \frac{1}{2}\pi)$. Liegt u im Innern oder auf dem Rande von N, dann hat man wegen (28)

$$(29) \qquad -\pi < 2\tau - \arg \zeta \leqslant \arg\left(\frac{u^2}{\zeta}\right) \leqslant \pi - \arg \zeta < \pi,$$

so dasz

$$u^{\alpha+\beta-1} K_{\alpha-\beta}(2u)\, \Psi\left(-\frac{u^2}{\zeta}\right)$$

für alle Punkte u im Innern und auf dem Rande von N analytisch ist†. Um Beziehung (26) zu beweisen, brauche ich daher nur zu zeigen, dasz

$$\lim_{\delta\to 0}\int_{C_1} u^{\alpha+\beta-1}K_{\alpha-\beta}(2u)\,\Psi\left(-\frac{u^2}{\zeta}\right)du = 0 \tag{30}$$

und

$$\lim_{D\to\infty}\int_{C_2} u^{\alpha+\beta-1}K_{\alpha-\beta}(2u)\,\Psi\left(-\frac{u^2}{\zeta}\right)du = 0 \tag{31}$$

ist.

Nun hat man für kleine Werte von $|u|$‡

$$K_0(2u) = O(\log|u|) \tag{32}$$

und, falls $\nu \neq 0$ ist,

$$K_\nu(2u) = O(|u|^{-|R(\nu)|}). \tag{33}$$

Wegen $R(\alpha) > 0$, $R(\beta) > 0$ folgt hieraus Relation (30).

Aus (28) ergibt sich, dasz $-\frac{1}{2}\pi < \arg u \leqslant \frac{1}{2}\pi$ ist für alle Punkte u von C_2. Es gilt also (siehe M.A., §6.222)

$$\lim_{D\to\infty}\int_{C_2} |u|^\lambda |e^{-2u}||du| = 0, \quad \text{falls } \lambda < 0 \text{ ist.} \tag{34}$$

Weiter hat man§, falls $-\pi < \arg(-z) < \pi$ ist,

$$\begin{aligned}{}_{n+1}F_n(a_1, \ldots, a_{n+1};\ r_1, \ldots, r_n;\ z) \\ = \sum_{h=1}^{n+1}\left\{\prod_{j=1}^{n}\frac{\Gamma(r_j)}{\Gamma(r_j-a_h)}\right\}\left\{\prod_{\substack{j=1\\ j\neq h}}^{n+1}\frac{\Gamma(a_j-a_h)}{\Gamma(a_j)}\right\}(-z)^{-a_h} \\ \times {}_{n+1}F_n\left(a_h,\ 1+a_h-r_1, \ldots, 1+a_h-r_n;\ 1+a_h-a_1, \overset{*}{\ldots}, 1+a_h-a_{n+1};\ \frac{1}{z}\right).\end{aligned} \tag{35}$$

† Der Punkt $z = 1$ ist wegen (24) die einzige im Endlichen liegende singuläre Stelle von $\Psi(z)$; aus (29) folgt, dasz $-u^2/\zeta \neq 1$ ist.

‡ Siehe B.F., 77, Formel (2); 78, Formel (6); und 80, Formeln (14) und (15).

§ Man vergl. Barnes, *Proc. London Math. Soc.* (2), 6 (1908), 146. Relation (35) gilt für $-\pi < \arg(-z) < \pi$, wofern $r_j \neq 0, -1, -2, \ldots$ $(j = 1, \ldots, n)$ und

$$a_j - a_h \quad (j = 1, \ldots, n+1;\ h = 1, \ldots, n+1;\ j \neq h)$$

nicht ganz ist. Der Stern bedeutet, dasz die Zahl $1+a_h-a_h$ in der Reihe

$$1+a_h-a_1, \ldots, 1+a_h-a_{n+1}$$

nicht vorkommt.

Wegen (29) darf ich hierin $z = -u^2/\zeta$ setzen. Da $K_\nu(2u)$ für grosze Werte von $|u|$ gleich

$$\left(\frac{\pi}{4u}\right)^{\frac{1}{2}} e^{-2u}\left\{1+O\left(\frac{1}{u}\right)\right\}$$

ist, folgt Beziehung (31) also aus (24), (25) und (34).

Hiermit ist Formel (26) bewiesen.

Der Beweis von (27) geht auf analoge Weise.

Nun hat man, mit Rücksicht auf (2),

$$K_{\alpha-\beta}(2ve^{-\frac{1}{2}\pi i}) = \tfrac{1}{2}\pi i\, e^{\frac{1}{2}(\alpha-\beta)\pi i} H^{(1)}_{\alpha-\beta}(2v); \tag{36}$$

aus†

$$H^{(1)}_\nu(ze^{\pi i}) = -e^{-\nu\pi i} H^{(2)}_\nu(z) \tag{37}$$

folgt daher

$$K_{\alpha-\beta}(2ve^{\frac{1}{2}\pi i}) = -\tfrac{1}{2}\pi i\, e^{\frac{1}{2}(\beta-\alpha)\pi i} H^{(2)}_{\alpha-\beta}(2v). \tag{38}$$

Setzt man die linke Seite von (6) der Kürze halber gleich $G(\zeta)$, dann ist also wegen (6), (24), (26) und (38) [man setze $u = ve^{\frac{1}{2}\pi i}$ in der rechten Seite von (26)]

$$G(\zeta) = -\frac{2\pi i\, e^{\beta\pi i}}{\Gamma(\alpha)\Gamma(\beta)\prod_{j=1}^{n-2}\Gamma(r_j)} \int_0^\infty v^{\alpha+\beta-1} H^{(2)}_{\alpha-\beta}(2v)\,\Psi\left(\frac{v^2}{\zeta}\right) dv, \tag{39}$$

falls $0 < \arg\zeta < 2\pi$ ist, und wegen (6), (24), (27) und (36) [man setze $u = ve^{-\frac{1}{2}\pi i}$ in der rechten Seite von (27)]

$$G(\zeta) = \frac{2\pi i e^{-\beta\pi i}}{\Gamma(\alpha)\Gamma(\beta)\prod_{j=1}^{n-2}\Gamma(r_j)} \int_0^\infty v^{\alpha+\beta-1} H^{(1)}_{\alpha-\beta}(2v)\,\Psi\left(\frac{v^2}{\zeta}\right) dv, \tag{40}$$

falls $-2\pi < \arg\zeta < 0$ ist.

Für $-\pi < \arg\zeta < \pi$ gilt also, mit Rücksicht auf (39),

$$-\frac{e^{-\pi i a_{n+1}}}{2\pi i} G(\zeta e^{\pi i}) = \frac{e^{(\beta-a_{n+1})\pi i}}{\Gamma(\alpha)\Gamma(\beta)\prod_{j=1}^{n-2}\Gamma(r_j)} \int_0^\infty v^{\alpha+\beta-1} H^{(2)}_{\alpha-\beta}(2v)\,\Psi\left(-\frac{v^2}{\zeta}\right) dv \tag{41}$$

und, mit Rücksicht auf (40),

$$\frac{e^{\pi i a_{n+1}}}{2\pi i} G(\zeta e^{-\pi i}) = \frac{e^{(a_{n+1}-\beta)\pi i}}{\Gamma(\alpha)\Gamma(\beta)\prod_{j=1}^{n-2}\Gamma(r_j)} \int_0^\infty v^{\alpha+\beta-1} H^{(1)}_{\alpha-\beta}(2v)\,\Psi\left(-\frac{v^2}{\zeta}\right) dv. \tag{42}$$

† Für (37) vergl. man B.F., 75, Formel (5) mit $m = 1$.

Aus der Definition der Funktion $G(\zeta)$ [$G(\zeta) =$ linke Seite von (6)] ergibt sich sofort†, dasz

$$\frac{1}{2\pi i}\{e^{\pi i a_{n+1}} G(\zeta e^{-\pi i}) - e^{-\pi i a_{n+1}} G(\zeta e^{\pi i})\}$$

gleich der linken Seite von (7) ist, so dasz Satz 2 jetzt aus (41), (42), und (24) folgt.

3. *Beweis der Sätze* 3 *und* 4.

Bekanntlich gilt (siehe B.F., 147)

$$(43)\quad J_\nu(z)\,J_\mu(z) = \frac{1}{\Gamma(1+\nu)\,\Gamma(1+\mu)}\left(\frac{z}{2}\right)^{\nu+\mu} \times {}_2F_3\left(\frac{1+\nu+\mu}{2}, \frac{2+\nu+\mu}{2};\ 1+\nu,\ 1+\mu,\ 1+\nu+\mu;\ -z^2\right),$$

also wegen (1)

$$(44)\quad I_\nu(z)\,I_\mu(z) = \frac{1}{\Gamma(1+\nu)\,\Gamma(1+\mu)}\left(\frac{z}{2}\right)^{\nu+\mu} \times {}_2F_3\left(\frac{1+\nu+\mu}{2}, \frac{2+\nu+\mu}{2};\ 1+\nu,\ 1+\mu,\ 1+\nu+\mu;\ z^2\right).$$

Weiter hat man, falls ν und μ nicht ganz sind (siehe B.F., 74),

$$e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(z) = \frac{e^{\frac{1}{2}\nu\pi i} J_{-\nu}(z) - e^{-\frac{1}{2}\nu\pi i} J_\nu(z)}{i \sin \nu\pi},$$

$$e^{-\frac{1}{2}\mu\pi i} H_\mu^{(2)}(z) = \frac{e^{\frac{1}{2}\mu\pi i} J_\mu(z) - e^{-\frac{1}{2}\mu\pi i} J_{-\mu}(z)}{i \sin \mu\pi}.$$

Hieraus ergibt sich, dasz für die durch (3) und (4) definierten Funktionen $V_{\nu,\mu}(z)$ und $U_{\nu,\mu}(z)$ die Beziehungen

$$(45)\quad V_{\nu,\mu}(z) = \frac{\{J_\nu(z)\,J_\mu(z) + J_{-\nu}(z)\,J_{-\mu}(z)\}\cos\tfrac{1}{2}(\nu-\mu)\pi - \{J_{-\nu}(z)\,J_\mu(z) + J_\nu(z)\,J_{-\mu}(z)\}\cos\tfrac{1}{2}(\nu+\mu)\pi}{\sin\nu\pi\,\sin\mu\pi},$$

$$U_{\nu,\mu}(z) = -\frac{\{J_\nu(z)\,J_\mu(z) - J_{-\nu}(z)\,J_{-\mu}(z)\}\sin\tfrac{1}{2}(\nu-\mu)\pi + \{J_{-\nu}(z)\,J_\mu(z) - J_\nu(z)\,J_{-\mu}(z)\}\sin\tfrac{1}{2}(\nu+\mu)\pi}{\sin\nu\pi\,\sin\mu\pi}$$

gelten.

† Man beachte hierbei, dasz

$$\frac{\Gamma(a_{n+1}-a_h)}{2\pi i}\{e^{(a_{n+1}-a_h)\pi i} - e^{(a_h-a_{n+1})\pi i}\} = \frac{\Gamma(a_{n+1}-a_h)\sin(a_{n+1}-a_h)\pi}{\pi} = \frac{1}{\Gamma(1+a_h-a_{n+1})}$$

ist.

Ich nehme jetzt an, dasz ν, μ und $\nu \pm \mu$ nicht ganz sind. Ich darf also in Satz 1 $n = 3$ mit

$$(46)\quad \begin{cases} a_1 = \dfrac{1+\nu+\mu}{2}, \quad a_2 = \dfrac{1-\nu+\mu}{2}, \quad a_3 = \dfrac{1+\nu-\mu}{2}, \quad a_4 = \dfrac{1-\nu-\mu}{2}, \\ r_1 = \tfrac{1}{2}, \quad \zeta = z^2 \end{cases}$$

setzen. Das erste Glied der linken Seite von (6) wird dann gleich

$$\frac{\Gamma(-\nu)\,\Gamma(-\mu)\,\Gamma(-\nu-\mu)\,z^{1+\nu+\mu}}{\Gamma\left(-\dfrac{\nu+\mu}{2}\right)\Gamma\left(\dfrac{1-\nu+\mu}{2}\right)\Gamma\left(\dfrac{1+\nu-\mu}{2}\right)\Gamma\left(\dfrac{1-\nu-\mu}{2}\right)}$$

$$\times {}_2F_3\left(\frac{1+\nu+\mu}{2}, \frac{2+\nu+\mu}{2};\ 1+\nu,\ 1+\mu,\ 1+\nu+\mu;\ -z^2\right)$$

$$= \frac{\sqrt{\pi}\cos\tfrac{1}{2}(\nu-\mu)\pi}{\Gamma(1+\nu)\,\Gamma(1+\mu)\sin\nu\pi\sin\mu\pi}\left(\frac{z}{2}\right)^{1+\nu+\mu}$$

$$\times {}_2F_3\left(\frac{1+\nu+\mu}{2}, \frac{2+\nu+\mu}{2};\ 1+\nu,\ 1+\mu,\ 1+\nu+\mu;\ -z^2\right)$$

$$= \frac{z\sqrt{\pi}\cos\tfrac{1}{2}(\nu-\mu)\pi}{2\sin\nu\pi\sin\mu\pi}\,J_\nu(z)\,J_\mu(z),$$

wegen (43). Auf analoge Weise findet man, dasz das zweite, dritte und vierte Glied der linken Seite von (6) gleich

$$-\frac{z\sqrt{\pi}\cos\tfrac{1}{2}(\nu+\mu)\pi}{2\sin\nu\pi\sin\mu\pi}\,J_{-\nu}(z)\,J_\mu(z) \quad \text{bez.} \quad -\frac{z\sqrt{\pi}\cos\tfrac{1}{2}(\nu+\mu)\pi}{2\sin\nu\pi\sin\mu\pi}\,J_\nu(z)\,J_{-\mu}(z)$$

oder

$$\frac{z\sqrt{\pi}\cos\tfrac{1}{2}(\nu-\mu)\pi}{2\sin\nu\pi\sin\mu\pi}\,J_{-\nu}(z)\,J_{-\mu}(z)$$

werden. Die linke Seite von (6) wird also wegen (45) gleich

$$\tfrac{1}{2}z\sqrt{\pi}\,V_{\nu,\mu}(z).$$

Man sieht jetzt leicht ein, dasz Beziehung (9) aus (6) folgt, wenn man darin $n = 3$ nimmt und für a_1, a_2, a_3, a_4, r_1 und ζ die durch (46) gegebenen Werte einsetzt. Die erste Behauptung von Satz 3 ist also bewiesen, falls ν, μ und $\nu \pm \mu$ nicht ganz sind. Da aber beide Seiten von (9) stetige Funktionen von ν und μ sind, so gilt Formel (9) auch noch, falls ν, μ und $\nu \pm \mu$ ganz sind.

Die zweite Behauptung von Satz 3 wird auf analoge Weise bewiesen. Man setze jetzt

$$a_1 = \frac{2+\nu+\mu}{2}, \quad a_2 = \frac{2-\nu+\mu}{2}, \quad a_3 = \frac{2+\nu-\mu}{2}, \quad a_4 = \frac{2-\nu-\mu}{2}, \quad r_1 = \tfrac{3}{2}.$$

Der Beweis von Satz 4 ist dem Beweis von Satz 3 analog. Man braucht nur Satz 2 statt Satz 1 und (44) statt (43) anzuwenden.

4. *Spezialfälle der Sätze* 3 *und* 4.

Die folgenden Integraldarstellungen sind schon auf verschiedene Weisen bewiesen worden†:

$$(47) \qquad K_\nu(z)\,K_\mu(z) = 2\int_0^\infty K_{\nu+\mu}(2z\cosh w)\cosh(\nu-\mu)\,w\,dw,$$

$$(48) \quad J_\nu(z)\,J_\mu(z)+Y_\nu(z)\,Y_\mu(z) = \frac{4}{\pi^2}\int_0^\infty K_{\nu+\mu}(2z\sinh w)(e^{(\nu-\mu)w}\cos\mu\pi+e^{(\mu-\nu)w}\cos\nu\pi)\,dw,$$

$$(49) \quad J_\mu(z)\,Y_\nu(z)-J_\nu(z)\,Y_\mu(z) = \frac{4}{\pi^2}\int_0^\infty K_{\nu+\mu}(2z\sinh w)(e^{(\nu-\mu)w}\sin\mu\pi-e^{(\mu-\nu)w}\sin\nu\pi)\,dw,$$

$$(50) \quad J_\nu(z)\,J_\mu(z)+Y_\nu(z)\,Y_\mu(z) = \frac{4}{\pi^2}\int_0^\infty K_{\nu-\mu}(2z\sinh w)\{e^{(\nu+\mu)w}+e^{-(\nu+\mu)w}\cos(\nu-\mu)\,\pi\}\,dw,$$

$$(51) \quad J_\mu(z)\,Y_\nu(z)-J_\nu(z)\,Y_\mu(z) = \frac{4\sin(\mu-\nu)\,\pi}{\pi^2}\int_0^\infty K_{\nu-\mu}(2z\sinh w)\,e^{-(\nu+\mu)w}\,dw.$$

In diesen Relationen ist $|\arg z| < \frac{1}{2}\pi$; in (48) und (49) wird überdies $|R(\nu+\mu)| < 1$, in (50) und (51) aber $|R(\nu-\mu)| < 1$ vorausgesetzt.

In diesem Paragraphen werde ich neue, sehr einfache Beweise für die Beziehungen (47) *bis* (51) *ableiten. Ich werde nämlich zeigen, dasz diese Formeln nur Spezialfälle von Satz* 3 *sind.*

---

† Man vergl.: B.F., 440–445; J. W. Nicholson, *Phil. Mag.* (6), 19 (1910), 234; *Quart. J. of Math.*, 42 (1911), 221 und 43 (1912), 83; G. H. Hardy, *Proc. London Math. Soc.* (2), 23 (1924), lxi–lxiii; A. L. Dixon and W. L. Ferrar, *Quart. J. of Math.* (Oxford), 1 (1930), 137–145 und 236–238; T. W. Chaundy, *Quart. J. of Math.* (Oxford), 2 (1931), 144–154.

Ich beweise zunächst, dasz die vier folgenden Integraldarstellungen Spezialfälle der Relationen (9) und (10) sind:

$$(52)\quad V_{\nu,\mu}(z)=\frac{8}{\pi^2}\cos\tfrac{1}{2}(\nu+\mu)\pi\int_0^{\infty+i(\tau-\arg z)}K_{\nu+\mu}(2z\sinh w)\cosh(\nu-\mu)w\,dw,$$

$$(53)\quad V_{\nu,\mu}(z)=\frac{8}{\pi^2}\cos\tfrac{1}{2}(\nu-\mu)\pi\int_0^{\infty+i(\tau-\arg z)}K_{\nu-\mu}(2z\sinh w)\cosh(\nu+\mu)w\,dw,$$

$$(54)\quad U_{\nu,\mu}(z)=\frac{8}{\pi^2}\sin\tfrac{1}{2}(\nu+\mu)\pi\int_0^{\infty+i(\tau-\arg z)}K_{\nu+\mu}(2z\sinh w)\sinh(\nu-\mu)w\,dw,$$

$$(55)\quad U_{\nu,\mu}(z)=\frac{8}{\pi^2}\sin\tfrac{1}{2}(\nu-\mu)\pi\int_0^{\infty+i(\tau-\arg z)}K_{\nu-\mu}(2z\sinh w)\sinh(\nu+\mu)w\,dw.$$

In diesen Beziehungen ist $\arg z$ beliebig, τ ist eine beliebige Zahl, die der Bedingung $-\frac{1}{2}\pi<\tau<\frac{1}{2}\pi$ genügt, und in (52) bez. (53) oder (54) oder (55) wird $|R(\nu+\mu)|<1$ bez. $|R(\nu-\mu)|<1$ oder $|R(\nu+\mu)|<2$ oder $|R(\nu-\mu)|<2$ vorausgesetzt.

Der Beweis der Formeln (52), (53), (54) und (55) ist ganz einfach. Man hat nämlich†

$$(56)\qquad {}_2F_1(\tfrac{1}{2}+\lambda,\ \tfrac{1}{2}-\lambda;\ \tfrac{1}{2};\ -\sinh^2 w)=\frac{\cosh 2\lambda w}{\cosh w}$$

und analog

$$(57)\qquad 2\lambda\sinh w\,{}_2F_1(1+\lambda,\ 1-\lambda;\ \tfrac{3}{2};\ -\sinh^2 w)=\frac{\sinh 2\lambda w}{\cosh w}.$$

Ich setze nun in der rechten Seite von (9)

$$\alpha=\tfrac{1}{2}(1+\nu+\mu),\quad \beta=\tfrac{1}{2}(1-\nu-\mu)\quad \{|R(\nu+\mu)|<1\}$$

und $u=z\sinh w$, also

$$(58)\qquad \frac{u^{\alpha+\beta-1}du}{\pi z\,\Gamma(\alpha)\,\Gamma(\beta)}=\frac{1}{\pi^2}\cos\tfrac{1}{2}(\nu+\mu)\pi\cosh w\,dw$$

und, wegen (56),

$$(59)\quad {}_4F_3\Big(\frac{1+\nu+\mu}{2},\ \frac{1-\nu+\mu}{2},\ \frac{1+\nu-\mu}{2},\ \frac{1-\nu-\mu}{2};\ \tfrac{1}{2},\ \alpha,\ \beta;\ -\frac{u^2}{z^2}\Big)$$
$$={}_2F_1\Big(\frac{1-\nu+\mu}{2},\ \frac{1+\nu-\mu}{2};\ \tfrac{1}{2};\ -\sinh^2 w\Big)=\frac{\cosh(\nu-\mu)w}{\cosh w}.$$

† Siehe Formel (42) meiner Arbeit: "Über die Integraldarstellungen der Whittakerschen Funktion", *Nieuw Archief voor Wiskunde* (2e Reeks), 18 (1934), 35–57.

Läuft u von 0 nach $\infty e^{i\tau}$, so läuft w von 0 nach $\infty + i(\tau - \arg z)$ [in (9) genügt τ der Bedingung (8), so dasz $-\frac{1}{2}\pi < \tau - \arg z < \frac{1}{2}\pi$ ist]. Aus (9) mit $\alpha = \frac{1}{2}(1+\nu+\mu)$, $\beta = \frac{1}{2}(1-\nu-\mu)$ und $u = z \sinh w$, (58) und (59) folgt daher Beziehung (52). Ist $-\pi < \arg z < \pi$ und genügt τ der Bedingung (8), dann ist also der Beweis von (52) geliefert. Da aber beide Seiten von (52), falls $-\frac{1}{2}\pi < \tau < \frac{1}{2}\pi$ ist, analytische Funktionen von z sind, so gilt Relation (52) nicht nur für $-\pi < \arg z < \pi$, sondern für alle Werte von $\arg z$.

Der Beweis von (53), (54) und (55) geht auf analoge Weise. Man findet (53), wenn man $\alpha = \frac{1}{2}(1+\nu-\mu)$ und $\beta = \frac{1}{2}(1-\nu+\mu)$ setzt in der rechten Seite von (9); man erhält (54) und (55), wenn man in der rechten Seite von (10) für α und β die Werte $\alpha = \frac{1}{2}(2+\nu+\mu)$, $\beta = \frac{1}{2}(2-\nu-\mu)$ bez. $\alpha = \frac{1}{2}(2+\nu-\mu)$, $\beta = \frac{1}{2}(2-\nu+\mu)$ einsetzt, und überdies Beziehung (57) benutzt.

Nun hat man, mit Rücksicht auf (3) und (4),

$$H_\nu^{(1)}(z)\, H_\mu^{(2)}(z) = e^{\frac{1}{2}(\mu-\nu)\pi i}\{V_{\nu,\mu}(z) + iU_{\nu,\mu}(z)\}, \tag{60}$$

$$H_\mu^{(1)}(z)\, H_\nu^{(2)}(z) = e^{\frac{1}{2}(\nu-\mu)\pi i}\{V_{\nu,\mu}(z) - iU_{\nu,\mu}(z)\}. \tag{61}$$

Weiter gilt

$$J_\nu(z)\, J_\mu(z) + Y_\nu(z)\, Y_\mu(z) = \tfrac{1}{2}\{H_\nu^{(1)}(z)\, H_\mu^{(2)}(z) + H_\mu^{(1)}(z)\, H_\nu^{(2)}(z)\},$$

$$J_\mu(z)\, Y_\nu(z) - J_\nu(z)\, Y_\mu(z) = \frac{1}{2i}\{H_\nu^{(1)}(z)\, H_\mu^{(2)}(z) - H_\mu^{(1)}(z)\, H_\nu^{(2)}(z)\},$$

also, wegen (60) und (61),

$$J_\nu(z)\, J_\mu(z) + Y_\nu(z)\, Y_\mu(z) = V_{\nu,\mu}(z) \cos\tfrac{1}{2}(\nu-\mu)\pi + U_{\nu,\mu}(z) \sin\tfrac{1}{2}(\nu-\mu)\pi, \tag{62}$$

$$J_\mu(z)\, Y_\nu(z) - J_\nu(z)\, Y_\mu(z) = -V_{\nu,\mu}(z) \sin\tfrac{1}{2}(\nu-\mu)\pi + U_{\nu,\mu}(z) \cos\tfrac{1}{2}(\nu-\mu)\pi. \tag{63}$$

Wegen

$$\cos\tfrac{1}{2}(\nu-\mu)\pi \cos\tfrac{1}{2}(\nu+\mu)\pi \cosh(\nu-\mu)w + \sin\tfrac{1}{2}(\nu-\mu)\pi \sin\tfrac{1}{2}(\nu+\mu)\pi \sinh(\nu-\mu)w$$
$$= \tfrac{1}{2}e^{(\nu-\mu)w}\cos\mu\pi + \tfrac{1}{2}e^{(\mu-\nu)w}\cos\nu\pi$$

folgt jetzt aus (62), (52) und (54)

$$J_\nu(z)\, J_\mu(z) + Y_\nu(z)\, Y_\mu(z) = \frac{4}{\pi^2}\int_0^{\infty + i(\tau - \arg z)} K_{\nu+\mu}(2z \sinh w)(e^{(\nu-\mu)w}\cos\mu\pi + e^{(\mu-\nu)w}\cos\nu\pi)\, dw.$$

In dieser Beziehung ist $|R(\nu+\mu)|<1$, arg z beliebig und τ ist eine beliebige Zahl, die der Bedingung $-\frac{1}{2}\pi<\tau<\frac{1}{2}\pi$ genügt. Hiermit ist auch Formel (48) bewiesen. Der Beweis von (49), (50) und (51) geht auf analoge Weise. Man braucht nur (63), (52) und (54) bez. (62), (53) und (55) oder (63), (53) und (55) anzuwenden.

Um nun noch Relation (47) zu beweisen, verfahre ich wie folgt:

Man hat†

$$2H_\mu^{(1)}(z)\cos\mu\pi=-e^{-\mu\pi i}H_\mu^{(2)}(z)+H_\mu^{(1)}(ze^{-\pi i}) \tag{64}$$

und

$$H_\nu^{(1)}(z)=-e^{-\nu\pi i}H_\nu^{(2)}(ze^{-\pi i}), \tag{65}$$

also

$$2H_\nu^{(1)}(z)H_\mu^{(1)}(z)\cos\mu\pi=-e^{-\mu\pi i}H_\nu^{(1)}(z)H_\mu^{(2)}(z)-e^{-\nu\pi i}H_\mu^{(1)}(ze^{-\pi i})H_\nu^{(2)}(ze^{-\pi i}). \tag{66}$$

Nun folgt aus (60), (52) und (54), falls $|R(\nu+\mu)|<1$ ist ($w=v-\frac{1}{2}\pi i$),

$$H_\nu^{(1)}(z)H_\mu^{(2)}(z) \tag{67}$$

$$=\frac{8e^{\frac{1}{2}(\mu-\nu)\pi i}}{\pi^2}\int_0^{\infty+i(\tau-\arg z)}K_{\nu+\mu}(2z\sinh w)\cosh(\nu w-\mu w+\tfrac{1}{2}\nu\pi i+\tfrac{1}{2}\mu\pi i)\,dw$$

$$=\frac{8e^{\frac{1}{2}(\mu-\nu)\pi i}}{\pi^2}\left(\int_{\frac{1}{2}\pi i}^{0}+\int_0^{\infty+i(\tau-\arg z+\frac{1}{2}\pi)}\right)K_{\nu+\mu}(-2iz\cosh v)\cosh(\nu v-\mu v+\mu\pi i)\,dv.$$

Auf analoge Weise findet man (man setze jetzt $w=v+\frac{1}{2}\pi i$)

$$H_\mu^{(1)}(ze^{-\pi i})H_\nu^{(2)}(ze^{-\pi i}) \tag{68}$$

$$=\frac{8e^{\frac{1}{2}(\nu-\mu)\pi i}}{\pi^2}\int_0^{\infty+i(\tau-\arg z+\pi)}K_{\nu+\mu}(-2z\sinh w)\cosh(\mu w-\nu w+\tfrac{1}{2}\mu\pi i+\tfrac{1}{2}\nu\pi i)\,dw$$

$$=\frac{8e^{\frac{1}{2}(\nu-\mu)\pi i}}{\pi^2}\left(\int_{-\frac{1}{2}\pi i}^{0}+\int_0^{\infty+i(\tau-\arg z+\frac{1}{2}\pi)}\right)K_{\nu+\mu}(-2iz\cosh v)\cosh(\mu v-\nu v+\mu\pi i)\,dv.$$

Es ist leicht einzusehen, dasz

$$\int_{\frac{1}{2}\pi i}^{0}K_{\nu+\mu}(-2iz\cosh v)\cosh(\nu v-\mu v+\mu\pi i)\,dv$$

$$+\int_{-\frac{1}{2}\pi i}^{0}K_{\nu+\mu}(-2iz\cosh v)\cosh(\mu v-\nu v+\mu\pi i)\,dv=0$$

† Beziehung (65) folgt aus (37). Für (64) vergl. man B.F., 75, Formel (5) mit $m=-1$.

ist. Aus (66), (67) und (68) ergibt sich daher

$$(69)\quad H_\nu^{(1)}(z)\,H_\mu^{(1)}(z)\cos\mu\pi$$

$$= -\frac{4e^{-\frac{1}{2}(\nu+\mu)\pi i}}{\pi^2}\int_0^{\infty+i(\tau-\arg z+\frac{1}{2}\pi)} K_{\nu+\mu}(-2iz\cosh v)$$

$$\times\{\cosh(\nu v-\mu v+\mu\pi i)+\cosh(\mu v-\nu v+\mu\pi i)\}\,dv$$

$$= -\frac{8e^{-\frac{1}{2}(\nu+\mu)\pi i}\cos\mu\pi}{\pi^2}\int_0^{\infty+i(\tau-\arg z+\frac{1}{2}\pi)} K_{\nu+\mu}(-2iz\cosh v)\cosh(\nu-\mu)v\,dv.$$

Ist $\mu+\frac{1}{2}$ nicht ganz und $|R(\nu+\mu)|<1$, dann ist also wegen (2) und (69)

$$(70)\quad K_\nu(z)\,K_\mu(z) = 2\int_0^{\infty+i(\tau-\arg z)} K_{\nu+\mu}(2z\cosh v)\cosh(\nu-\mu)v\,dv.$$

Da aber beide Seiten von (70) analytische Funktionen von ν und μ sind, so gilt Beziehung (70) für alle Werte von ν und μ.

Ich werde jetzt noch eine Integraldarstellung für $I_\nu(z)\,K_\nu(z)$ ableiten. Ich betrachte dazu den Spezialfall† mit $\mu=\nu$ von (11). Man hat nämlich (siehe B.F., 78)

$$K_\nu(z)\sin\nu\pi = \tfrac{1}{2}\pi\{I_{-\nu}(z)-I_\nu(z)\}.$$

Setzt man nun $\mu=\nu$ in (11) $\{|R(\nu)|<\frac{5}{4}\}$, dann wird die linke Seite also gleich

$$\frac{4\sin^2\nu\pi\cos\nu\pi}{\pi}\,I_\nu(z)\,K_\nu(z)$$

und man findet somit

$$(71)\quad I_\nu(z)\,K_\nu(z)\cos\nu\pi$$

$$= \frac{\pi i}{2z\,\Gamma(\alpha)\,\Gamma(\beta)}\int_0^\infty v^{\alpha+\beta-1}\{e^{-(\nu+\beta)\pi i}H_{\alpha-\beta}^{(1)}(2v)-e^{(\nu+\beta)\pi i}H_{\alpha-\beta}^{(2)}(2v)\}$$

$$\times {}_3F_2\left(\tfrac{1}{2}+\nu,\ \tfrac{1}{2}-\nu,\ \tfrac{1}{2};\ \alpha,\ \beta;\ -\frac{v^2}{z^2}\right)dv.$$

Diese Beziehung gilt für $|\arg z|<\frac{1}{2}\pi$ und $|R(\nu)|<\frac{5}{4}$; α und β sind beliebige Zahlen, die den Bedingungen

$$R(\alpha)>0,\quad R(\beta)>0,\quad R(\alpha+\beta)<\tfrac{5}{2}-2|R(\nu)|$$

genügen.

Die Spezialfälle mit $\alpha=\frac{1}{2}+\nu$, $\beta=\frac{1}{2}-\nu$ bez. $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}-\nu$ von (71) sind auch von Herrn Watson (B.F., 435) gegeben worden.

---

† Andere Spezialfälle von Satz 4 sind von Nielsen (*Handbuch der Theorie der Cylinderfunktionen*, 213–218) gegeben worden.

5. *Beweis von Satz* 5.

Bekanntlich hat man bei beliebigem λ

$$\frac{1}{\Gamma(1+\nu)}\left(\frac{z}{2}\right)^{\nu} {}_1F_2\left(\lambda;\ \lambda,\ 1+\nu;\ -\frac{z^2}{4}\right) = J_\nu(z). \tag{72}$$

Ich nehme nun an, dasz die Zahlen $\frac{1}{2}(1+\nu+\mu)$, $\frac{1}{2}(1-\nu+\mu)$ und ν nicht ganz sind. Ich darf also $n=2$ mit $a_1=1$, $a_2=\frac{1}{2}(1+\nu-\mu)$, $a_3=\frac{1}{2}(1-\nu-\mu)$ und $\zeta=\frac{1}{4}z^2$ setzen in Satz 1. Die rechte Seite von (6) wird dann gleich $z^{1-\mu}$ mal der rechten Seite von (12); die linke Seite von (6) wird aber wegen (72) gleich

$$\frac{z^2}{(1-\nu+\mu)(1+\nu+\mu)}\ {}_1F_2\left(1;\ \frac{3-\nu+\mu}{2},\ \frac{3+\nu+\mu}{2};\ -\frac{z^2}{4}\right)$$

$$-\left(\frac{z}{2}\right)^{1-\mu}\frac{\Gamma\{\frac{1}{2}(1-\nu+\mu)\}\,\Gamma\{\frac{1}{2}(1+\nu+\mu)\}}{\sin\nu\pi}$$

$$\times\{J_\nu(z)\cos\tfrac{1}{2}(\nu+\mu)\pi - J_{-\nu}(z)\cos\tfrac{1}{2}(\nu-\mu)\pi\}$$

$$= z^{1-\mu}S_{\mu,\nu}(z) \quad \text{(siehe B.F., 347)}.$$

Beziehung (12) folgt daher aus Satz 1. Sind $\frac{1}{2}(1+\nu+\mu)$, $\frac{1}{2}(1-\nu+\mu)$ und ν nicht ganz, dann ist also der Beweis von Satz 5 geliefert. Durch Grenzübergang findet man aber, dasz Formel (12) für alle Werte von ν und μ gültig ist.

6. *Spezialfälle von Satz* 5†.

1. *Die Struvesche Funktion* $\mathbf{H}_\nu(z)$. Diese Funktion wird definiert durch

$$\mathbf{H}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1}\sum_{m=0}^{\infty}\frac{(-1)^m z^{2m}}{2^{2m}\,\Gamma(\frac{3}{2}+m)\,\Gamma(\frac{3}{2}+\nu+m)}.$$

und man hat [man vergl. B.F., 346, Formel (10), und 347, Formel (3)]

$$\mathbf{H}_\nu(z) - Y_\nu(z) = \frac{2^{1-\nu}}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\nu)}\,S_{\nu,\nu}(z),$$

also wegen (12)

$$\mathbf{H}_\nu(z) - Y_\nu(z) = \frac{2^{3-\nu}z^{\nu-1}}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\nu)\,\Gamma(\alpha)\,\Gamma(\beta)} \tag{73}$$

$$\times\int_0^{\infty e^{ir}} u^{\alpha+\beta-1}K_{\alpha-\beta}(2u)\,{}_3F_2\left(1,\ \tfrac{1}{2},\ \tfrac{1}{2}-\nu;\ \alpha,\ \beta;\ -\frac{4u^2}{z^2}\right)du.$$

† Andere Anwendungen von Satz 5 kommen vor in meiner Arbeit: "Integraldarstellungen für Lommelsche und Struvesche Funktionen". Diese Arbeit wird demnächst im *Proc. Koninklijke Akademie van Wetenschappen te Amsterdam* erscheinen.

Die Spezialfälle mit $\alpha = 1$, $\beta = \frac{1}{2}$ bez. $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2} - \nu$ von (73) sind auch von Herrn Watson [B.F., 331, Formel (4), und 426, Formel (10)] gegeben worden.

2. *Integraldarstellungen des Neumannschen Polynoms $O_n(z)$ und des Schläflischen Polynoms $S_n(z)$*†.

Diese Funktionen können folgendermaszen definiert werden‡:

1. Ist n gerade, so ist

$$O_n(z) = \frac{1}{z} S_{1,n}(z), \quad S_n(z) = 2n\, S_{-1,n}(z). \tag{74}$$

2. Ist n ungerade, so ist

$$O_n(z) = \frac{n}{z} S_{0,n}(z), \quad S_n(z) = 2 S_{0,n}(z). \tag{75}$$

Sie besitzen die folgenden Integraldarstellungen (siehe B.F., 278 und 288)

$$O_n(z) = \tfrac{1}{2}\int_0^{\infty + i(\tau - \arg z)} e^{-z\sinh w}\{e^{nw} + (-1)^n e^{-nw}\}\cosh w\, dw, \tag{76}$$

$$S_n(z) = \int_0^{\infty + i(\tau - \arg z)} e^{-z\sinh w}\{e^{nw} - (-1)^n e^{-nw}\}\, dw. \tag{77}$$

Hierin ist $-\frac{1}{2}\pi < \tau < \frac{1}{2}\pi$.

Es ist nicht schwer zu zeigen, dasz diese Beziehungen Spezialfälle von Satz 5 sind.

Denn man hat

$$K_{\frac{1}{2}}(2u) = \tfrac{1}{2}\sqrt{(\pi/u)}\, e^{-2u}.$$

Wegen

$${}_2F_1(a, b;\ c;\ x) = (1-x)^{c-a-b}\, {}_2F_1(c-b, c-a;\ c;\ x)$$

folgt also aus (12) mit $\alpha = 1$ und $\beta = \frac{1}{2}$,

$$S_{1,\nu}'(z) = 2\int_0^{\infty e^{i\tau}} e^{-2u}\, {}_2F_1\left(\frac{\nu}{2}, -\frac{\nu}{2};\ \tfrac{1}{2};\ -\frac{4u^2}{z^2}\right) du \tag{78}$$

$$= 2\int_0^{\infty e^{i\tau}} e^{-2u}\left(1 + \frac{4u^2}{z^2}\right)^{\frac{1}{2}} {}_2F_1\left(\frac{1+\nu}{2}, \frac{1-\nu}{2};\ \tfrac{1}{2};\ -\frac{4u^2}{z^2}\right) du.$$

† C. Neumann, *Theorie der Besselschen Funktionen*, 8–16; L. Schläfli, *Math. Annalen*, 3 (1871), 134–149.

‡ Siehe B.F., 350 und 271–290. $O_n(z)$ und $S_n(z)$ sind Polynome in $1/z$, und zwar $O_n(z)$ vom Grade $|n|+1$, $S_n(z)$ aber vom Grade $|n|$.

Ebenso findet man ($\alpha = \frac{3}{2}$, $\beta = 1$)

$$(79) \qquad S_{-1,\nu}(z) = \frac{4}{z^2}\int_0^{\infty e^{ir}} ue^{-2u}\,{}_2F_1\left(1+\frac{\nu}{2},\ 1-\frac{\nu}{2};\ \tfrac{3}{2};\ -\frac{4u^2}{z^2}\right)du,$$

und ($\alpha = 1$, $\beta = \frac{1}{2}$)

$$(80) \qquad S_{0,\nu}(z) = \frac{2}{z}\int_0^{\infty e^{ir}} e^{-2u}\,{}_2F_1\left(\frac{1+\nu}{2},\ \frac{1-\nu}{2};\ \tfrac{1}{2};\ -\frac{4u^2}{z^2}\right)du.$$

Setzt man nun $u = \frac{1}{2}z\sinh w$ in (78), (79) und (80), dann erhält man [man vergl. den Beweis der Beziehungen (52) und (54)]

$$(81) \qquad S_{1,\nu}(z) = z\int_0^{\infty+i(\tau-\arg z)} e^{-z\sinh w}\cosh \nu w\cosh w\,dw,$$

$$(82) \qquad \nu S_{-1,\nu}(z) = \int_0^{\infty+i(\tau-\arg z)} e^{-z\sinh w}\sinh \nu w\,dw,$$

$$(83) \qquad S_{0,\nu}(z) = \int_0^{\infty+i(\tau-\arg z)} e^{-z\sinh w}\cosh \nu w\,dw.$$

Diese Relationen gelten, falls $-\frac{1}{2}\pi < \tau < \frac{1}{2}\pi$ ist, für beliebige Werte von $\arg z$. Mittels partieller Integration folgt noch aus (83)

$$(84) \qquad \nu S_{0,\nu}(z) = z\int_0^{\infty+i(\tau-\arg z)} e^{-z\sinh w}\sinh \nu w\cosh w\,dw.$$

Man sieht nun leicht ein, wegen (74) und (75), dasz die Beziehungen (76) und (77) Spezialfälle von (81), (82), (83) und (84) sind.

7. *Beweis von Satz* 6.

Bekanntlich gilt†

$$(85) \quad {}_1F_1(\alpha;\ \rho;\ x)\times{}_1F_1(\alpha;\ \rho;\ -x) = {}_2F_3(\alpha,\ \rho-\alpha;\ \rho,\ \tfrac{1}{2}\rho,\ \tfrac{1}{2}\rho+\tfrac{1}{2};\ \tfrac{1}{4}x^2)$$

und

$$(86) \quad {}_1F_1(\alpha;\ \rho;\ x)\times{}_1F_1(\alpha-\rho+1;\ 2-\rho;\ x)$$
$$= {}_2F_3(\alpha-\tfrac{1}{2}\rho+\tfrac{1}{2},\ \tfrac{1}{2}\rho-\alpha+\tfrac{1}{2};\ \tfrac{1}{2}\rho+\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{3}{2}-\tfrac{1}{2}\rho;\ \tfrac{1}{4}x^2)$$
$$+\frac{(2\alpha-\rho)(1-\rho)}{\rho(2-\rho)}\,x\,{}_2F_3(\alpha-\tfrac{1}{2}\rho+1,\ \tfrac{1}{2}\rho-\alpha+1;\ \tfrac{1}{2}\rho+1,\ \tfrac{3}{2},\ 2-\tfrac{1}{2}\rho;\ \tfrac{1}{4}x^2).$$

† Man vergl. W. N. Bailey, *Proc. London Math. Soc.* (2), 28 (1928), 246.

Weiter hat man (siehe M.A., § 16.41 und § 16.1), falls $2m$ nicht ganz und $|\arg z| < \frac{3}{2}\pi$ ist,

$$W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)}\, z^{\frac{1}{2}+m} e^{-\frac{1}{2}z}\, {}_1F_1(\tfrac{1}{2}+m-k;\ 1+2m;\ z)$$
$$+ \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\, z^{\frac{1}{2}-m} e^{-\frac{1}{2}z}\, {}_1F_1(\tfrac{1}{2}-m-k;\ 1-2m;\ z).$$

Hieraus folgt mit Rücksicht auf (85) und (86)

$$(87)\quad \frac{z^{-2k}}{\Gamma(1-2k)}\, W_{k,m}(iz)\, W_{k,m}(-iz)$$
$$= \frac{\Gamma(-2m)\,\Gamma(-m)\,\Gamma(\frac{1}{2}-m)}{\{\Gamma(\frac{1}{2}-m-k)\}^2\,\Gamma(\frac{1}{2}-k)\,\Gamma(1-k)}\,\phi_1(z)$$
$$+ \frac{\Gamma(2m)\,\Gamma(m)\,\Gamma(\frac{1}{2}+m)}{\{\Gamma(\frac{1}{2}+m-k)\}^2\,\Gamma(\frac{1}{2}-k)\,\Gamma(1-k)}\,\phi_2(z)$$
$$+ \frac{\Gamma(m)\,\Gamma(-m)\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-k)\,\Gamma(\frac{1}{2}+m-k)\,\Gamma(\frac{1}{2}-m-k)\,\Gamma(1-k)}\,\phi_3(z)$$
$$+ \frac{\Gamma(-\frac{1}{2}+m)\,\Gamma(-\frac{1}{2}-m)\,\Gamma(-\frac{1}{2})}{\Gamma(-k)\,\Gamma(\frac{1}{2}+m-k)\,\Gamma(\frac{1}{2}-m-k)\,\Gamma(\frac{1}{2}-k)}\,\phi_4(z),$$

worin

$$(88)\quad \phi_1(z) = \left(\frac{z}{2}\right)^{1+2m-2k}$$
$$\times {}_2F_3\left(\tfrac{1}{2}+m-k,\ \tfrac{1}{2}+m+k;\ 1+2m,\ \tfrac{1}{2}+m,\ 1+m;\ -\frac{z^2}{4}\right)$$

$$(89)\quad \phi_2(z) = \left(\frac{z}{2}\right)^{1-2m-2k}$$
$$\times {}_2F_3\left(\tfrac{1}{2}-m-k,\ \tfrac{1}{2}-m+k;\ 1-2m,\ \tfrac{1}{2}-m,\ 1-m;\ -\frac{z^2}{4}\right),$$

$$(90)\quad \phi_3(z) = \left(\frac{z}{2}\right)^{1-2k} {}_2F_3\left(\tfrac{1}{2}-k,\ \tfrac{1}{2}+k;\ 1+m,\ \tfrac{1}{2},\ 1-m;\ -\frac{z^2}{4}\right),$$

$$(91)\quad \phi_4(z) = \left(\frac{z}{2}\right)^{2-2k} {}_2F_3\left(1-k,\ 1+k;\ \tfrac{3}{2}+m,\ \tfrac{3}{2},\ \tfrac{3}{2}-m;\ -\frac{z^2}{4}\right).$$

Ich wende nun Satz 1 an, und zwar mit $n=3$, $a_1 = \frac{1}{2}+m-k$, $a_2 = \frac{1}{2}-m-k$, $a_3 = \frac{1}{2}-k$, $a_4 = 1-k$, $r_1 = 1-2k$ und $\zeta = \frac{1}{4}z^2$ ($2m$ nicht

ganz, $k \neq \frac{1}{2}, 1, \frac{3}{2}, \ldots$). Die rechte Seite von (6) wird dann gleich $z^{-2k}/\{\Gamma(1-2k)\}$ mal der rechten Seite von (13). Die linke Seite von (6) wird aber wegen (87), (88), (89), (90) und (91) gleich

$$\frac{z^{-2k}}{\Gamma(1-2k)} W_{k,m}(iz)\, W_{k,m}(-iz),$$

so dasz Beziehung (13) aus (6) folgt. Hiermit ist Satz 6 bewiesen für den Fall, dasz $2m$ nicht ganz und $k \neq \frac{1}{2}, 1, \frac{3}{2}, \ldots$ ist. Da aber beide Seiten von (13) stetige Funktionen von k und m sind, so gilt Formel (13) für alle Werte von k und m.

Bemerkung. Anwendungen von Satz 6 kommen vor in meiner Arbeit: "Einige Integraldarstellungen für Produkte von Whittakerschen Funktionen" (Diese Arbeit wird demnächst im *Quart. J. of Math.* erscheinen.

Tuinbouwstraat 40,
Groningen, Holland.

CONDITIONS SATISFIED BY THE VORTICITY AND THE STREAM-FUNCTION IN A VISCOUS LIQUID MOVING IN TWO DIMENSIONS BETWEEN FIXED PARALLEL PLANES

By J. L. SYNGE.

[Received 6 June, 1934.—Read 14 June, 1934.]

Contents.

1. Introduction.
2. The kinematical condition.
3. Conditions satisfied by the coefficients in the Fourier series for the vorticity on account of the kinematical condition.
4. The dynamical condition.
5. Conditions satisfied by the stream-function of the small disturbance of steady flow.
6. An explanation.

1. *Introduction.*

In a two-dimensional motion of an *inviscid* liquid, confined by fixed walls, the distribution of vorticity at any instant may be arbitrarily selected without violation of the boundary conditions. For, if $\omega(x, y)$ is the assigned vorticity, the stream-function $\psi(x, y)$ of the motion which has this vorticity is merely required to satisfy the partial differential equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 2\omega(x, y), \tag{1.1}$$

with the boundary condition

$$\psi = \text{const. on each wall.} \tag{1.2}$$

The determination of ψ is therefore a problem of Dirichlet, and in general there is a solution. When the liquid is confined by the fixed walls $y = \pm b$, the velocity-components at an instant t, when the vorticity is

$\omega(x, y, t)$, are given by

$$(1.3)\quad u(x, y, t) - iv(x, y, t) = \chi(t) + \frac{i}{4b}\iint \omega(x_1, y_1, t)\left\{\tanh\frac{\pi(z-\bar{z}_1)}{4b} - \coth\frac{\pi(z-z_1)}{4b}\right\} dx_1\, dy_1,$$

where $\chi(t)$ is an arbitrary real function,

$$(1.4)\qquad z = x+iy, \quad z_1 = x_1+iy_1, \quad \bar{z}_1 = x_1-iy_1,$$

and the integral is taken over the whole region between the planes*.

If the liquid is *viscous*, and the vorticity at time t is assigned, the vanishing of the *normal* velocity on $y = \pm b$ again leads us to the expression (1.3) for the components of velocity. But there is now an additional condition to satisfy, namely

$$(1.5)\qquad u = 0 \quad \text{for} \quad y = \pm b,$$

a condition which is by no means automatically satisfied by (1.3). In fact, *the distribution of vorticity in a viscous liquid flowing between fixed planes cannot be arbitrarily assigned at an instant, but must satisfy certain conditions.* The purpose of the present paper is to investigate those conditions. We shall deal with the case where the boundaries are the fixed planes $y = \pm b$, but it is obvious that in the case of more general boundary conditions also there must exist relations satisfied by the vorticity.

We shall confine ourselves to motions in which the velocity-components (and hence the vorticity) are periodic in x with period $2a$.

2. *The kinematical condition.*

THEOREM I. *If a viscous liquid moves two-dimensionally in the (x, y) plane between the fixed walls $y = \pm b$, and if the motion is periodic in x with period $2a$, then at any instant the vorticity ω satisfies*

$$(2.1)\qquad \int_{x=-a}^{a}\int_{y=-b}^{b} \omega(x, y, t)\, W(x, y)\, dx\, dy = 0,$$

where W is any harmonic function, periodic in x with period $2a$, and such that

$$(2.2)\qquad \int_{x=-a}^{a} \frac{\partial W}{\partial y}\, dx = 0 \quad \textit{for} \quad y = \pm b.$$

We shall refer to (2.1) as the "kinematical condition".

---

* J. L. Synge, *Phil. Trans. Royal Soc.* (A), 234 (1934), 52.

To establish this theorem, let us write down the equations

$$(2.3)\qquad \begin{cases} u = -\dfrac{\partial \psi}{\partial y}, \qquad v = \dfrac{\partial \psi}{\partial x}, \\ \omega = \frac{1}{2}\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right) = \frac{1}{2}\Delta\psi, \qquad \Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}. \end{cases}$$

Then

$$(2.4)\qquad 2\iint \omega W\, dx\, dy = \iint W \Delta\psi\, dx\, dy$$

$$= \iint \psi \Delta W\, dx\, dy + \int W \frac{\partial \psi}{\partial n}\, ds - \int \psi \frac{\partial W}{\partial n}\, ds,$$

where the double integral is taken over the interior of the rectangle $x = \pm a$, $y = \pm b$, and the single integrals are taken round its boundary, ∂n being directed outwards. The double integral on the right vanishes since W is harmonic, and the contributions to the single integrals from $x = \pm a$ cancel out on account of the periodicity of the motion and of W. Hence

$$(2.5)\qquad 2\iint \omega W\, dx\, dy = \int_{x=-a}^{a} \left[W \frac{\partial \psi}{\partial y} \right]_{y=-b}^{y=b} dx - \int_{x=-a}^{a} \left[\psi \frac{\partial W}{\partial y} \right]_{y=-b}^{y=b} dx.$$

The first integral vanishes since

$$(2.6)\qquad \frac{\partial \psi}{\partial y} = 0 \quad \text{for} \quad y = \pm b,$$

and the second integral vanishes, by virtue of (2.2), since ψ is constant along each wall. Thus Theorem I is established.

If the walls are not fixed, but sliding parallel to the x-axis, the condition (2.1) again holds, but for a more restricted harmonic W, namely one which satisfies

$$(2.7)\qquad \int_{-a}^{a} W\, dx = 0 \qquad (y = \pm b),$$

in addition to (2.2).

Returning to the case of fixed walls, we observe that $W = 1$ satisfies the conditions, and hence

$$(2.8)\qquad \iint \omega\, dx\, dy = 0,$$

a result which is obvious since the circulation round the fundamental rectangle is easily seen to be zero.

We shall now establish the converse of Theorem I.

THEOREM II. *If a liquid moves in the (x, y) plane with a vorticity ω which is periodic in x with period $2a$ and satisfies the conditions*

$$(2.9)\qquad \begin{cases} \iint \omega(x, y) \cosh(s\pi y/a) \cos(s\pi x/a)\, dx\, dy = 0, \\ \iint \omega(x, y) \cosh(s\pi y/a) \sin(s\pi x/a)\, dx\, dy = 0, \\ \iint \omega(x, y) \sinh(s\pi y/a) \cos(s\pi x/a)\, dx\, dy = 0, \\ \iint \omega(x, y) \sinh(s\pi y/a) \sin(s\pi x/a)\, dx\, dy = 0 \end{cases} \qquad (s = 0, 1, 2, \ldots),$$

the double integrals being taken over the interior of $x = \pm a$, $y = \pm b$, and if $v = 0$ on $y = \pm b$, then u takes a single constant value on $y = \pm b$.

Let us take the first of (2.9), which may be written

$$(2.10)\qquad \iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \cosh(s\pi y/a) \cos(s\pi x/a)\, dx\, dy = 0.$$

Taking $s = 0$, we obtain

$$(2.11)\qquad \int_{-b}^{b} \Big[v\Big]_{x=-a}^{x=a} dy - \int_{-a}^{a} \Big[u\Big]_{y=-b}^{y=b} dx = 0.$$

But, since $v = 0$ on $y = \pm b$, the velocity components are given by (1.3), and the periodicity of ω obviously implies the periodicity of u and v. Hence the first integral in (2.11) vanishes, and we have

$$(2.12)\qquad \int_{-a}^{a} \Big[u\Big]_{y=-b}^{y=b} dx = 0.$$

Taking $s \neq 0$ in (2.10), we obtain, on integration by parts, again using the periodicity of v,

$$(2.13)\qquad (s\pi/a) \iint v \cosh(s\pi y/a) \sin(s\pi x/a)\, dx\, dy + (s\pi/a) \iint u \sinh(s\pi y/a) \cos(s\pi x/a)\, dx\, dy - \int_{-a}^{a} \Big[u\Big]_{y=-b}^{y=b} \cosh(s\pi b/a) \cos(s\pi x/a)\, dx = 0.$$

When we use the fact that $v = 0$ for $y = \pm b$, the first of these integrals

becomes, on integration by parts with respect to y,

$$-\iint \frac{\partial v}{\partial y} \sinh(s\pi y/a) \sin(s\pi x/a)\, dx\, dy, \tag{2.14}$$

while the second integral becomes, on integration by parts with respect to x,

$$-\iint \frac{\partial u}{\partial x} \sinh(s\pi y/a) \sin(s\pi x/a)\, dx\, dy. \tag{2.15}$$

But the sum of (2.14) and (2.15) is zero by reason of the equation of continuity. Hence (2.13) becomes

$$\int_{-a}^{a} \Big[u\Big]_{y=-b}^{y=b} \cos(s\pi x/a)\, dx = 0. \tag{2.16}$$

When we employ the second of (2.9) in a similar way, we get

$$\int_{-a}^{a} \Big[u\Big]_{y=-b}^{y=b} \sin(s\pi x/a)\, dx = 0. \tag{2.17}$$

Now $\Big[u\Big]_{y=-b}^{y=b}$ is a periodic function of x, and may be expanded in a Fourier series. But (2.12), (2.16), (2.17) show that all the coefficients in this Fourier series are zero. Hence

$$(u)_{y=b} = (u)_{y=-b}. \tag{2.18}$$

When we employ the last two of (2.9), we obtain

$$\int_{-a}^{a} \{(u)_{y=b} + (u)_{y=-b}\} \begin{matrix}\cos\\ \sin\end{matrix} (s\pi x/a)\, dx = 0 \quad (s = 1, 2, \ldots), \tag{2.19}$$

and hence

$$(u)_{y=b} + (u)_{y=-b} = \text{const.} \tag{2.20}$$

Combining this with (2.18), the theorem is established.

If we restate Theorem II, with the omission of the value $s = 0$, we get the same conditions as above, except for (2.12), which does not occur. Hence in this case we deduce that u is a constant along each of $y = \pm b$: this corresponds to the case of sliding walls.

Since the functions multiplying ω in (2.9) satisfy the condition (2.2), *the conditions* (2.9) *are necessary and sufficient conditions that the assumed periodic distribution of vorticity* ω *may be consistent with the motion of a viscous liquid between the fixed walls* $y = \pm b$.

Since any harmonic function, periodic in x with period $2a$, and satisfying (2.2), may be expanded in a series of terms*

$$\begin{matrix}\cosh \\ \sinh\end{matrix} (s\pi y/a) \begin{matrix}\cos \\ \sin\end{matrix} (s\pi x/a),$$

we may make the following statement of our result.

THEOREM III. *The Poisson equation*

$$\Delta\psi = 2\omega(x, y)\ , \tag{2.21}$$

where ω is periodic in x with period $2a$, possesses a solution in the strip $y = \pm b$ satisfying the boundary conditions

$$\frac{\partial\psi}{\partial x} = 0, \quad \frac{\partial\psi}{\partial y} = 0 \quad \textit{for} \quad y = \pm b, \tag{2.22}$$

if, and only if, ω satisfies

$$\int_{x=-a}^{a}\int_{y=-b}^{b} \omega W\, dx\, dy = 0, \tag{2.23}$$

where W is any harmonic function, periodic in x with period $2a$, and satisfying

$$\int_{-a}^{a} \frac{\partial W}{\partial y}\, dx = 0 \quad \textit{for} \quad y = \pm b. \tag{2.24}$$

3. *Conditions satisfied by the coefficients in the Fourier series for the vorticity on account of the kinematical condition.*

For any motion periodic in x with period $2a$ between the fixed walls $y = \pm b$, we may expand the vorticity in the Fourier series

$$\omega(x, y, t) = \sum_s \Omega_s(y, t) \exp(s\pi xi/a) \quad \left(\sum_s = \sum_{s=-\infty}^{\infty}\right), \tag{3.1}$$

and we may expand Ω_s in the Fourier series

$$\Omega_s(y, t) = \sum_r \omega_{r,s}(t) \exp(r\pi yi/b) \quad (s = 0, \pm 1, \pm 2, \ldots), \tag{3.2}$$

so that

$$\omega(x, y, t) = \sum_s \sum_r \omega_{r,s}(t) \exp(r\pi yi/b) \exp(s\pi xi/a). \tag{3.3}$$

* If (2.2) is not satisfied, the term independent of x ($s=0$) will not be independent of y.

Now the conditions (2.9) may be written compactly as

$$\iint \omega W_{s,\epsilon}\, dx\, dy = 0, \tag{3.4}$$

where

$$W_{s,\epsilon} = \exp(\epsilon s\pi y/a) \exp(-s\pi xi/a) \tag{3.5}$$
$$(s = 0, \pm 1, \pm 2, \ldots; \quad \epsilon = \pm 1).$$

Substituting for ω in (3.4) from (3.1), we have

$$\int_{-b}^{b} \Omega_s(y, t) \exp(\epsilon s\pi y/a)\, dy = 0, \tag{3.6}$$

or, by (3.2),

$$\int_{-b}^{b} \sum_r \omega_{r,s}(t) \exp(r\pi yi/b) \exp(\epsilon s\pi y/a)\, dy = 0, \tag{3.7}$$

which gives

$$\omega_{0,0}(t) = 0, \tag{3.8}$$

$$\sum_r \omega_{r,s}(t) \frac{(-1)^r}{\lambda_s + ir} = 0, \quad \sum_r \omega_{r,s}(t) \frac{(-1)^r}{\lambda_s - ir} = 0 \quad (s = \pm 1, \pm 2, \ldots), \tag{3.9}$$

where

$$\lambda_s = bs/a. \tag{3.10}$$

The conditions (3.8), (3.9) *are necessary and sufficient for the consistency of* ω, *as given by* (3.3), *with the fixity of the walls* $y = \pm b$.

The conditions (3.9) may also be written

$$\sum_r \omega_{r,s}(t) \frac{(-1)^r}{r^2 + \lambda_s^2} = 0, \quad \sum_r \omega_{r,s}(t) \frac{r(-1)^r}{r^2 + \lambda_s^2} = 0 \tag{3.11}$$
$$(s = \pm 1, \pm 2, \ldots),$$

or

$$\begin{cases} \dfrac{\omega_{0,s}(t)}{\lambda_s^2} + \displaystyle\sum_{r=1}^{\infty} \{\omega_{r,s}(t) + \omega_{-r,s}(t)\} \frac{(-1)^r}{r^2 + \lambda_s^2} = 0, \\ \displaystyle\sum_{r=1}^{\infty} \{\omega_{r,s}(t) - \omega_{-r,s}(t)\} \frac{r(-1)^r}{r^2 + \lambda_s^2} = 0 \\ (s = \pm 1, \pm 2, \ldots). \end{cases} \tag{3.12}$$

If the walls are not fixed, but sliding parallel to the x-axis, the condition (3.8) disappears, and we have (3.9) or (3.11) or (3.12).

Since $\omega_{r,s}(t)$, as coefficient in the Fourier series (3.2), is of the order of $1/|r|$ at least, the above series are absolutely convergent.

4. *The dynamical condition.*

Everything so far deduced has resulted from purely kinematical considerations, including the condition of no slipping at the walls: the dynamical equations have not been used. It is well known that these give

$$\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=\nu\Delta\omega, \tag{4.1}$$

where ν is the kinematical viscosity. Again assuming a motion periodic in x with period $2a$, we have, as in (2.1),

$$\iint\omega W\,dx\,dy=0, \tag{4.2}$$

and, since this holds for all values of t,

$$\iint\frac{\partial\omega}{\partial t}W\,dx\,dy=0, \tag{4.3}$$

the integral being taken over the interior of the rectangle $x=\pm a$, $y=\pm b$. Substituting from (4.1), we obtain

$$\iint\left(u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}\right)W\,dx\,dy-\nu\iint W\Delta\omega\,dx\,dy=0. \tag{4.4}$$

Now

$$\iint u\frac{\partial\omega}{\partial x}W\,dx\,dy = \int_{-b}^{b}\Big[u\omega W\Big]_{x=-a}^{x=a}dy-\iint\frac{\partial u}{\partial x}\omega W\,dx\,dy-\iint u\omega\frac{\partial W}{\partial x}\,dx\,dy, \tag{4.5}$$

in which the first term on the right vanishes on account of periodicity in x. Also

$$\iint v\frac{\partial\omega}{\partial y}W\,dx\,dy = \int_{-a}^{a}\Big[v\omega W\Big]_{y=-b}^{y=b}dx-\iint\frac{\partial v}{\partial y}\omega W\,dx\,dy-\iint v\omega\frac{\partial W}{\partial y}\,dx\,dy, \tag{4.6}$$

in which the first term on the right vanishes since $v=0$ on the walls. Adding (4.5) and (4.6), and using the equation of continuity, we see that (4.4) may be written

$$\iint\omega\left(u\frac{\partial W}{\partial x}+v\frac{\partial W}{\partial y}\right)dx\,dy+\nu\iint W\Delta\omega\,dx\,dy=0. \tag{4.7}$$

Integrating by parts, using the boundary conditions for fixed walls, the periodicity, and the equation of continuity, and remembering that W is harmonic, we see that

$$
\begin{aligned}
(4.8)\quad & \iint \omega\left(u\frac{\partial W}{\partial x}-v\frac{\partial W}{\partial y}\right)dx\,dy \\
&= \tfrac{1}{2}\iint\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(u\frac{\partial W}{\partial x}-v\frac{\partial W}{\partial y}\right)dx\,dy \\
&= -\tfrac{1}{2}\iint v\frac{\partial u}{\partial x}\frac{\partial W}{\partial x}\,dx\,dy+\tfrac{1}{4}\iint v^2\frac{\partial^2 W}{\partial x\,\partial y}\,dx\,dy \\
&\qquad\qquad +\tfrac{1}{4}\iint u^2\frac{\partial^2 W}{\partial x\,\partial y}\,dx\,dy-\tfrac{1}{2}\iint u\frac{\partial v}{\partial y}\frac{\partial W}{\partial y}\,dx\,dy \\
&= \tfrac{1}{4}\iint\frac{\partial}{\partial y}(v^2)\frac{\partial W}{\partial x}\,dx\,dy+\tfrac{1}{4}\iint\frac{\partial}{\partial x}(u^2)\frac{\partial W}{\partial y}\,dx\,dy \\
&\qquad\qquad +\tfrac{1}{4}\iint v^2\frac{\partial^2 W}{\partial x\,\partial y}\,dx\,dy+\tfrac{1}{4}\iint u^2\frac{\partial^2 W}{\partial x\,\partial y}\,dx\,dy \\
&= \tfrac{1}{4}\iint\frac{\partial}{\partial y}\left(v^2\frac{\partial W}{\partial x}\right)dx\,dy+\tfrac{1}{4}\iint\frac{\partial}{\partial x}\left(u^2\frac{\partial W}{\partial y}\right)dx\,dy \\
&= 0.
\end{aligned}
$$

Consequently,

$$
(4.9)\qquad \iint\omega\left(u\frac{\partial W}{\partial x}+v\frac{\partial W}{\partial y}\right)dx\,dy=2\iint\omega v\frac{\partial W}{\partial y}\,dx\,dy,
$$

and we may state the following result:

THEOREM IV. *When a viscous liquid moves in two-dimensional motion, periodic in x with period $2a$, between fixed walls $y=\pm b$, then the vorticity ω and the velocity-component v satisfy the condition*

$$
(4.10)\qquad 2\iint\omega v\frac{\partial W}{\partial y}\,dx\,dy+\nu\iint W\Delta\omega\,dx\,dy=0,
$$

the integrals being taken over the interior of the rectangle $x=\pm a$, $y=\pm b$, and W being any harmonic function, periodic in x with period $2a$, and such that

$$
(4.11)\qquad \int_{-a}^{a}\frac{\partial W}{\partial y}\,dx=0 \quad \textit{for} \quad y=\pm b.
$$

We shall refer to (4.10) *as the "dynamical condition".*

We have not drawn attention to assumptions regarding continuity in deducing the above results, because all that need be assumed is the continuity of the velocity-components and their first derivatives: these conditions of continuity are necessarily satisfied in a viscous liquid in order to ensure finite continuous stress. The second-order derivatives of the velocity components need only be continuous in bits. But if we now assume (as we shall in future) the continuity of the second-order derivatives of the velocity-components, we have, on applying Green's theorem,

$$(4.12) \qquad \iint W\Delta\omega\, dx\, dy = \int_{-a}^{a}\left[W\frac{\partial\omega}{\partial y}\right]_{y=-b}^{y=b} dx - \int_{-a}^{a}\left[\omega\frac{\partial W}{\partial y}\right]_{y=-b}^{y=b} dx.$$

This may be used to give an alternative form to the second term in (4.10).

Since v is, by (1.3), expressible in terms of ω, the equation (4.10) represents a singly infinite set of conditions satisfied by ω, obtained by substituting $W_{s,\epsilon}$ for W, where $W_{s,\epsilon}$ is as given in (3.5).

The dynamical condition (4.10) is, of course, distinct from the kinematical condition (2.1), since ν is involved in (4.10) but not in (2.1). Thus

$$(4.13) \qquad \psi = (b^2-y^2)^2$$

gives a motion with vanishing velocity on $y = \pm b$. It gives

$$(4.14) \qquad \omega = 6y^2-2b^2,$$

and it is easily seen that (2.1) is satisfied. But with $W = 1$, the left-hand side of (4.10) is $48\nu ab$, so that (4.10) is not satisfied.

5. *Conditions satisfied by the stream-function of the small disturbance of steady flow.*

Consider the steady flow between the fixed walls $y = \pm b$, given by

$$(5.1) \qquad u_0 = k(b^2-y^2);$$

the vorticity is

$$(5.2) \qquad \omega_0 = ky.$$

Let there be superimposed on this a motion with small velocity-components u, v, periodic in x with period $2a$, and let the vorticity of this disturbance be ω. Then the velocity-components and the vorticity of the actual motion are

$$(5.3) \qquad u_0+u,\ v,\ \omega+\omega_0.$$

We shall neglect terms of the second and higher orders in u, v, ω. Let ψ be the stream-function of the disturbance, so that

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}. \tag{5.4}$$

As boundary conditions we have

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0 \quad \text{for} \quad y = \pm b. \tag{5.5}$$

Let us now consider the implications of the kinematical condition (2.1) and the dynamical condition (4.10). The condition (2.1) becomes, in the present notation,

$$\iint kyW\,dx\,dy + \tfrac{1}{2}\iint \Delta\psi W\,dx\,dy = 0, \tag{5.6}$$

in which W is any one of the functions (3.5). If W is independent of x, it is also independent of y, and in that case

$$\iint kyW\,dx\,dy = kW2a\left[\tfrac{1}{2}y^2\right]_{y=-b}^{y=b} = 0; \tag{5.7}$$

on the other hand, if W involves x, we have

$$\int_{-a}^{a} W\,dx = 0, \tag{5.8}$$

and so in either case the first integral in (5.6) vanishes. When we apply Green's theorem to the second integral, and remember that W is harmonic, it is easily seen that this integral vanishes for a disturbance periodic in x with period $2a$, satisfying the boundary conditions (5.5). Thus the kinematical condition (2.1) tells us nothing more than (5.5).

Let us now consider the dynamical condition (4.10). In the present notation and approximation, it becomes

$$2\iint \omega_0 v \frac{\partial W_{s,\epsilon}}{\partial y}\,dx\,dy + \nu \iint W_{s,\epsilon}\,\Delta\omega\,dx\,dy = 0, \tag{5.9}$$

or, using the alternative form (4.12),

$$2k\iint yv \frac{\partial W_{s,\epsilon}}{\partial y}\,dx\,dy + \nu \int_{-a}^{a}\left[W_{s,\epsilon}\frac{\partial \omega}{\partial y} - \omega \frac{\partial W_{s,\epsilon}}{\partial y}\right]_{y=-b}^{y=b} dx = 0. \tag{5.10}$$

Let us expand ψ in a Fourier series

$$\psi(x, y, t) = \sum_s \Psi_s(y, t) \exp(s\pi x i/a). \tag{5.11}$$

The conditions (5.5) give

$$(5.12)\qquad \begin{cases} \Psi_s = 0 \quad \text{for} \quad y = \pm b \quad (s = \pm 1, \pm 2, \ldots), \\ \dfrac{\partial \Psi_s}{\partial y} = 0 \quad \text{for} \quad y = \pm b \quad (s = 0, \pm 1, \ldots). \end{cases}$$

We have

$$(5.13)\qquad \begin{cases} v = \dfrac{\partial \psi}{\partial x} = \sum_s \dfrac{s\pi i}{a} \Psi_s \exp(s\pi x i/a), \\ \omega = \frac{1}{2}\Delta\psi = \frac{1}{2} \sum_s \left\{ \dfrac{\partial^2 \Psi_s}{\partial y^2} - \left(\dfrac{s\pi}{a}\right)^2 \Psi_s \right\} \exp(s\pi x i/a). \end{cases}$$

Substituting these expressions in (5.10), and substituting for W the expression (3.5), we have the following result.

THEOREM V. *When the steady flow $u_0 = k(b^2 - y^2)$ between the fixed walls $y = \pm b$ is slightly disturbed, the disturbance being periodic in x with period $2a$, the coefficients $\Psi_s(y, t)$ in the expansion* (5.11) *for the stream-function of the disturbance satisfy the conditions* (5.12) *and also the conditions*

$$(5.14)\qquad \epsilon i \frac{4k}{\nu} \left(\frac{s\pi}{a}\right)^2 \int_{-b}^{b} y \Psi_s \exp(\epsilon s\pi y/a)\, dy + \left[\left(\frac{\partial^3 \Psi_s}{\partial y^3} - \frac{\epsilon s\pi}{a} \frac{\partial^2 \Psi_s}{\partial y^2} \right) \exp(\epsilon s\pi y/a) \right]_{y=-b}^{y=b} = 0$$

$$(\epsilon = \pm 1;\ s = 0, \pm 1, \pm 2, \ldots).$$

If we introduce the dimensionless quantity

$$(5.15)\qquad \eta = y/b$$

and the Reynolds number

$$(5.16)\qquad R = \frac{1}{\nu} (\text{mean velocity} \times \text{width of channel}) = \tfrac{4}{3} k b^3/\nu,$$

the condition (5.14) may be written

$$(5.17)\qquad 3\epsilon i R \pi^2 \lambda_s^2 \int_{-1}^{+1} \eta \Psi_s \exp(\epsilon \pi \lambda_s \eta)\, d\eta + \left[\left(\frac{\partial^3 \Psi_s}{\partial \eta^3} - \epsilon \pi \lambda_s \frac{\partial^2 \Psi_s}{\partial \eta^2} \right) \exp(\epsilon \pi \lambda_s \eta) \right]_{\eta=-1}^{\eta=+1} = 0$$

$$(\epsilon = \pm 1;\ s = 0, \pm 1, \pm 2, \ldots;\ \lambda_s = bs/a).$$

For $s = 0$, the condition becomes

$$\left[\frac{\partial^3 \Psi_0}{\partial y^3}\right]_{y=-b}^{y=b} = 0. \tag{5.18}$$

6. *An explanation.*

Consider the motion of a dynamical system with a finite number of degrees of freedom, subject to certain constraints. To determine the motion, we have to solve a system of ordinary differential equations, with initial conditions at some instant $t = t_0$. In assigning the initial conditions in all generality, we may choose any configuration for the system and any distribution of the velocities consistent with the constraints.

By analogy, it would appear that, in discussing the motion (periodic in x) of a viscous liquid between fixed planes $y = \pm b$, we might assign at time $t = t_0$ any distribution of velocity satisfying the stated conditions, namely the condition of periodicity in x and the condition of vanishing velocity on the walls. Such a distribution of velocity will not in general satisfy the dynamical condition (4.10). What is the meaning of this apparent contradiction? It appears to be that the instant $t = t_0$, at which the arbitrary distribution of velocity is assigned, is a peculiar or singular instant in the history of the motion. There is an instantaneous adjustment in the values of $\partial u/\partial y$ and $\partial^2 u/\partial y^2$ on the walls, and for all values of t greater than t_0 the dynamical condition is satisfied.

A similar state of affairs obtains in problems of heat conduction in a slab, to which the problem of laminar flow of a viscous liquid is mathematically equivalent. We assign at time $t = t_0$ an arbitrary distribution of temperature, but at all subsequent instants quite special conditions are satisfied by the partial derivatives of the temperature on the walls, as may be seen by inspecting the solutions*.

Problems of stability for viscous liquids have proved so baffling, and mathematical short-cuts have, in the main, led to such unsatisfactory results, that it would seem wise to take the simple and fundamental problem of the flow between fixed parallel planes and proceed carefully, avoiding all unjustifiable assumptions, such as the inclusion of an exponential time-factor, in the solution. On the basis of the assumption of periodicity in x, the Fourier expansion (5.11) and its term-by-term differentiations, as required in the preceding investigation, appear to be above reproach. The use of this Fourier expansion will reduce the problem of

* *Cf.* MacRobert, *Spherical harmonics*, 41.

the determination of the disturbance at any time to the solution of a set of partial differential equations; in the typical equation, Ψ_s is the dependent variable, and y, t are the independent variables. For further progress, it seems necessary to write an expansion for Ψ_s. It is just at this point that the argument becomes delicate, and questions of the validity of term-by-term differentiation assume major importance. If we wish to employ the method of Reynolds, as developed by Orr*, which is based on the rate of change of the kinetic energy of the disturbance, it is essential that we should calculate this rate of change, not at a singular instant in the history of the liquid, but at an ordinary instant, for which the dynamical condition (4.10) or (5.14) is satisfied. Thus, while the present paper leads to no definite hydrodynamical result affecting stability, it seems to me that the discussion of the dynamical condition is a necessary preliminary to all exact investigations of the stability problem.

University of Toronto,
Toronto, Canada.

* *Proc. Royal Irish Academy*, 27A (1907), 122.

SOME INFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS

By W. N. BAILEY.

[Received 23 June, 1934.—Read 15 November, 1934.]

1. Special cases of the integral

$$\int_0^\infty t^{\lambda-1} J_\mu(at)\, J_\nu(bt)\, K_\rho(ct)\, dt \tag{1.1}$$

have been evaluated by several writers. Macdonald* evaluated the integral in terms of Legendre functions when

$$\lambda = \rho+2, \quad \mu = \nu.$$

Watson† recently proved that, when

$$\lambda = \tfrac{3}{2}, \quad \rho = \tfrac{1}{2},$$

it is possible to express the integral as an integral of an elementary function. Still more recently Popov‡ evaluated the integral in terms of elementary functions when

$$\lambda = 2, \quad \mu = 0, \quad \rho = \nu.$$

In this paper I first consider the integral (1.1) for values of the parameters which are unrestricted apart from convergence conditions, and I show that it can be expressed very simply in terms of the fourth type of Appell's hypergeometric functions of two variables. I then consider the most interesting cases when the function F_4 reduces to functions of a simpler type.

---

* Macdonald (5). See also Watson (7), 412, (6).

† Watson (9).

‡ Popov (6). See also Watson (8), where some still more special cases are considered in detail.

Later I consider the allied integral

$$\int_0^\infty t^{\lambda-1} J_\mu(at)\, J_\nu(bt)\, J_\rho(ct)\, dt, \tag{1.2}$$

where a, b, c are real and positive.

This integral has been evaluated* in the form of an infinite series, the terms of this series involving products of hypergeometric functions. It was shown by Macdonald† that the integral could be evaluated in terms of Legendre functions when

$$\lambda+\mu=2, \quad \nu=\rho.$$

I show that (1.2), like (1.1), can be expressed in terms of Appell's function F_4, and I then give particular cases in which it can be evaluated in a comparatively simple form.

2. It is convenient to assume initially that c is real, and large compared with $|a|$ and $|b|$. Then‡

$$\int_0^\infty t^{\lambda-1} J_\mu(at)\, J_\nu(bt)\, K_\rho(ct)\, dt$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^{r+s}(\frac{1}{2}a)^{\mu+2r}(\frac{1}{2}b)^{\nu+2s}}{r!\, s!\, \Gamma(\mu+1+r)\, \Gamma(\nu+1+s)} \int_0^\infty t^{\lambda+\mu+\nu+2r+2s-1} K_\rho(ct)\, dt$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\begin{array}{r}(-1)^{r+s}\, 2^{\lambda-2}\, a^{\mu+2r}\, b^{\nu+2s} \\ \times \Gamma\{\frac{1}{2}(\lambda+\mu+\nu-\rho+2r+2s)\}\, \Gamma\{\frac{1}{2}(\lambda+\mu+\nu+\rho+2r+2s)\}\end{array}}{r!\, s!\, \Gamma(\mu+1+r)\, \Gamma(\nu+1+s)\, c^{\lambda+\mu+\nu+2r+2s}},$$

and so

$$\int_0^\infty t^{\lambda-1} J_\mu(at)\, J_\nu(bt)\, K_\rho(ct)\, dt \tag{2.1}$$

$$= \frac{2^{\lambda-2}\, a^\mu\, b^\nu\, \Gamma\{\frac{1}{2}(\lambda+\mu+\nu-\rho)\}\, \Gamma\{\frac{1}{2}(\lambda+\mu+\nu+\rho)\}}{c^{\lambda+\mu+\nu}\, \Gamma(\mu+1)\, \Gamma(\nu+1)}$$

$$\times F_4[\tfrac{1}{2}(\lambda+\mu+\nu-\rho),\ \tfrac{1}{2}(\lambda+\mu+\nu+\rho);\ \mu+1,\ \nu+1;\ -a^2/c^2,\ -b^2/c^2].$$

* Watson (7), § 13.46, (7).

† See Watson (7), § 13.46, (4) and (5).

‡ Watson (7), 388, (8).

This formula is true (by the theory of analytic continuation) if

$$R(\lambda+\mu+\nu) > |R(\rho)|$$

and each of the four numbers

$$R(c \pm ia \pm ib)$$

is positive.

3. There are known cases in which the function F_4 can be expressed in terms of ordinary hypergeometric functions*. The most general case of reduction†, in which there is only one relation to be satisfied by the parameters, is

$$(3.1)\quad F_4[\alpha, \beta;\ \gamma,\ 1+\alpha+\beta-\gamma;\ x(1-y),\ y(1-x)] = F(\alpha, \beta;\ \gamma;\ x)\, F(\alpha, \beta;\ 1+\alpha+\beta-\gamma;\ y),$$

where x and y lie in certain regions surrounding $x = 0$, $y = 0$.

This formula can be used in (2.1) when $\lambda = 1$, and we find that

$$\int_0^\infty J_\mu(ct \sin\phi)\, J_\nu(ct \sin\Phi)\, K_\rho(ct \cos\phi \cos\Phi)\, dt$$

$$= \frac{\tan^\mu\phi \tan^\nu\Phi\, \Gamma\{\frac{1}{2}(1+\mu+\nu-\rho)\}\, \Gamma\{\frac{1}{2}(1+\mu+\nu+\rho)\}}{2c \cos^{\nu+1}\phi \cos^{\mu+1}\Phi\, \Gamma(\mu+1)\, \Gamma(\nu+1)}$$

$$\times F[\tfrac{1}{2}(1+\mu+\nu-\rho),\ \tfrac{1}{2}(1+\mu+\nu+\rho);\ \mu+1;\ -\tan^2\phi]$$

$$\times F[\tfrac{1}{2}(1+\mu+\nu-\rho),\ \tfrac{1}{2}(1+\mu+\nu+\rho);\ \nu+1;\ -\tan^2\Phi],$$

which can be written more conveniently in the form

$$(3.2)\quad \int_0^\infty J_\mu(ct \sin\phi)\, J_\nu(ct \sin\Phi)\, K_\rho(ct \cos\phi \cos\Phi)\, dt$$

$$= \frac{\sin^\mu\phi \sin^\nu\Phi}{2c \cos^\rho\phi \cos^\rho\Phi} \frac{\Gamma\{\frac{1}{2}(1+\mu+\nu-\rho)\}\, \Gamma\{\frac{1}{2}(1+\mu+\nu+\rho)\}}{\Gamma(\mu+1)\, \Gamma(\nu+1)}$$

$$\times F[\tfrac{1}{2}(1+\mu+\nu-\rho),\ \tfrac{1}{2}(1+\mu-\nu-\rho);\ \mu+1;\ \sin^2\phi]$$

$$\times F[\tfrac{1}{2}(1+\mu+\nu-\rho),\ \tfrac{1}{2}(1+\nu-\mu-\rho);\ \nu+1;\ \sin^2\Phi].$$

* See, for example, Appell and Kampé de Fériet (1), 23.

† Bailey (2) and (3).

Thus the integral (1.1) is expressed in terms of ordinary hypergeometric functions when $\lambda = 1$.

When $\phi \to \frac{1}{2}\pi$, (3.2) reduces to the Weber-Schafheitlin integral. When $\phi \to 0$, it becomes equivalent to the known formula*

$$(3.21)\quad \int_0^\infty \frac{K_\mu(at)\,J_\nu(bt)}{t^\lambda}\,dt = \frac{b^\nu\,\Gamma\{\frac{1}{2}(\nu-\lambda+\mu+1)\}\,\Gamma\{\frac{1}{2}(\nu-\lambda-\mu+1)\}}{2^{\lambda+1}\,a^{\nu-\lambda+1}\,\Gamma(\nu+1)}$$
$$\times F[\tfrac{1}{2}(\nu-\lambda+\mu+1),\ \tfrac{1}{2}(\nu-\lambda-\mu+1);\ \ \nu+1;\ \ -b^2/a^2].$$

The particular case of (3.2) when $\rho = \frac{1}{2}$ is worth noting, since then the hypergeometric functions on the right reduce to Legendre functions. In this case the formula becomes

$$(3.3)\quad \int_0^\infty t^{-\frac{1}{2}}\,J_\mu(t\sin\phi)\,J_\nu(t\sin\Phi)\,e^{-t\cos\phi\cos\Phi}\,dt$$
$$= \Gamma(\mu+\nu+\tfrac{1}{2})\,P^{-\mu}_{\nu-\frac{1}{2}}(\cos\phi)\,P^{-\nu}_{\mu-\frac{1}{2}}(\cos\Phi).$$

This result should be compared with one given by Gegenbauer†, who showed effectively that the integral in (3.3) is expressible as a *series* of products of Gegenbauer polynomials.

The formula (3.3) is given in a form suitable for real values of ϕ and Φ. For general values the corresponding formula is

$$(3.4)\quad \int_0^\infty t^{-\frac{1}{2}}\,I_\mu(t\sinh\phi)\,I_\nu(t\sinh\Phi)\,e^{-t\cosh\phi\cosh\Phi}\,dt$$
$$= \Gamma(\mu+\nu+\tfrac{1}{2})\,P^{-\mu}_{\nu-\frac{1}{2}}(\cosh\phi)\,P^{-\nu}_{\mu-\frac{1}{2}}(\cosh\Phi),$$

where $|I(\phi)| < \frac{1}{2}\pi$, $|I(\Phi)| < \frac{1}{2}\pi$, and $R(\mu+\nu) > -\frac{1}{2}$.

Both (3.3) and (3.4) reduce to known results‡ when $\phi \to 0$.

4. Another formula which gives interesting results is

$$(4.1)\quad F_4\Big[\alpha,\beta;\ \gamma,\beta;\ -\frac{x}{(1-x)(1-y)},\ -\frac{y}{(1-x)(1-y)}\Big]$$
$$= (1-x)^\alpha(1-y)^\alpha\,F_1[\alpha;\ \gamma-\beta,\ 1+\alpha-\gamma;\ \gamma;\ x,\ xy].$$

---

* Watson (7), 410, (1).
† Gegenbauer (4).
‡ Watson (7), 387, (1) and (3).

This formula does not appear to have been noticed previously. To prove it* we note that the coefficient of $x^m y^n$ in the expansion of

$$(1-x)^{-\alpha}(1-y)^{-\beta} F_4\left[\alpha, \beta;\ \gamma, \gamma';\ -\frac{x}{(1-x)(1-y)},\ -\frac{y}{(1-x)(1-y)}\right]$$

is, for small enough values of $|x|$ and $|y|$,

$$\sum_{r=0}^{m}\sum_{s=0}^{n} \frac{(\alpha)_{r+s}(\beta)_{r+s}}{r!\,s!\,(\gamma)_r(\gamma')_s} \frac{(-1)^{r+s}(\alpha+r+s)_{m-r}(\beta+r+s)_{n-s}}{(m-r)!\,(n-s)!}$$

$$= \frac{(\alpha)_m(\beta)_n}{m!\,n!} \sum_{r=0}^{m}\sum_{s=0}^{n} \frac{(\alpha+m)_s(\beta+n)_r(-m)_r(-n)_s}{r!\,s!\,(\gamma)_r(\gamma')_s}$$

$$= \frac{(\alpha)_m(\beta)_n}{m!\,n!} F(\alpha+m, -n;\ \gamma')\, F(\beta+n, -m;\ \gamma)$$

$$= \frac{(\alpha)_m(\beta)_n}{m!\,n!} \frac{(\gamma'-\alpha-m)_n(\gamma-\beta-n)_m}{(\gamma')_n(\gamma)_m}.$$

Thus, when $\gamma' = \beta$, we have

$$(1-x)^{-\alpha}(1-y)^{-\beta} F_4\left[\alpha, \beta;\ \gamma, \beta;\ -\frac{x}{(1-x)(1-y)},\ -\frac{y}{(1-x)(1-y)}\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\beta-\alpha-m)_n(\gamma-\beta-n)_m}{m!\,n!\,(\gamma)_m} x^m y^n$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m(\gamma-\beta)_m x^m}{m!\,(\gamma)_m} F\left[\begin{matrix}\beta-\alpha-m,\ 1+\beta-\gamma;\ y\\ 1+\beta-\gamma-m\end{matrix}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m(\gamma-\beta)_m}{m!\,(\gamma)_m} x^m (1-y)^{\alpha-\beta} F\left[\begin{matrix}-m,\ 1+\alpha-\gamma;\ y\\ 1+\beta-\gamma-m\end{matrix}\right]$$

$$= (1-y)^{\alpha-\beta} \sum_{m=0}^{\infty}\sum_{r=0}^{m} \frac{(\alpha)_m(\gamma-\beta)_m}{m!\,(\gamma)_m} \frac{(-m)_r(1+\alpha-\gamma)_r}{r!\,(1+\beta-\gamma-m)_r} x^m y^r.$$

When we write $m = r+s$, this becomes

$$(1-y)^{\alpha-\beta} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\alpha)_{r+s}(\gamma-\beta)_s(1+\alpha-\gamma)_r}{r!\,s!\,(\gamma)_{r+s}} x^{r+s} y^r$$

$$= (1-y)^{\alpha-\beta} F_1[\alpha;\ \gamma-\beta,\ 1+\alpha-\gamma;\ \gamma;\ x,\ xy],$$

---

* *Cf.* Bailey (3).

and this proves the result for small enough values of x and y. It follows by analytic continuation that the result is true in certain simply-connected regions surrounding $x = 0$, $y = 0$.

As particular cases of (4.1) we note the following:

$$(4.2)\quad F_4\left[\alpha, \beta;\ \alpha, \beta;\ -\frac{x}{(1-x)(1-y)},\ -\frac{y}{(1-x)(1-y)}\right] = (1-xy)^{-1}(1-x)^{\beta}(1-y)^{\alpha},$$

$$(4.3)\quad F_4\left[\alpha, \beta;\ \beta, \beta;\ -\frac{x}{(1-x)(1-y)},\ -\frac{y}{(1-x)(1-y)}\right] = (1-x)^{\alpha}(1-y)^{\alpha} F[\alpha, 1+\alpha-\beta;\ \beta;\ xy],$$

$$(4.4)\quad F_4\left[\alpha, \beta;\ 1+\alpha-\beta, \beta;\ -\frac{x}{(1-x)(1-y)},\ -\frac{y}{(1-x)(1-y)}\right] = (1-y)^{\alpha} F\left[\alpha, \beta;\ 1+\alpha-\beta;\ -\frac{x(1-y)}{1-x}\right].$$

In (4.3) and (4.4), the hypergeometric functions that occur on the right can be expressed in terms of Legendre functions.

5. We now choose the parameters in (2.1) so that the function F_4 can be transformed into an F_1 by (4.1). This is the case when

$$\lambda = \mu-\nu+\rho+2,$$

and so we find that

$$(5.1)\quad \int_0^{\infty} t^{\mu-\nu+\rho+1} J_{\mu}(ct\sin\phi) J_{\nu}(ct\sin\Phi) K_{\rho}(ct\cos\phi\cos\Phi)\,dt = \frac{2^{\mu-\nu+\rho}\sin^{\mu}\phi\sin^{\nu}\Phi\cos^{\rho}\phi\cos^{\rho}\Phi\,\Gamma(\mu+\rho+1)}{c^{\mu-\nu+\rho+2}\Gamma(\nu+1)} \times F_1[\mu+\rho+1;\ \nu-\mu, \mu-\nu+\rho+1;\ \nu+1;\ \sin^2\Phi, \sin^2\phi].$$

Now the function F_1 can be expressed as an integral of an elementary function, and hence the same is true of the integral on the left of (5.1).

The formula (4.2) can be used with (2.1) when

$$\lambda = 2, \quad \rho = \nu-\mu,$$

and the formula obtained is

$$(5.2)\quad \int_0^\infty t\,J_\mu(ct\sin\phi)\,J_\nu(ct\sin\Phi)\,K_{\nu-\mu}(ct\cos\phi\cos\Phi)\,dt = \frac{\sin^\mu\phi\sin^\nu\Phi\cos^{\nu-\mu}\phi\cos^{\mu-\nu}\Phi}{c^2(1-\sin^2\phi\sin^2\Phi)}.$$

This integral reduces to one evaluated by Popov when $\mu = 0$. When $\Phi \to 0$ the formula reduces to a known result*.

We can use (4.3) with (2.1) when

$$\lambda = 2 \pm \rho, \quad \mu = \nu.$$

The first of these cases leads to Macdonald's formula, and the second gives an equivalent result.

The formula (4.4) can be used with (2.1) when

$$\lambda = \mu + 2, \quad \rho = \nu,$$

and we obtain

$$(5.3)\quad \int_0^\infty t^{\mu+1}\,J_\mu(ct\sin\phi)\,J_\nu(ct\sin\Phi)\,K_\nu(ct\cos\phi\cos\Phi)\,dt = \frac{2^\mu\sin^\mu\phi\cos^{2\mu+2}\frac{1}{2}\Psi}{c^{\mu+2}\cos^{2\mu+2}\Phi}\,\Gamma(\mu+\nu+1)\,P_\mu^{-\nu}(\cos\Psi),$$

where $\tan\frac{1}{2}\Psi = \tan\Phi\cos\phi$.

This result is in agreement with (5.2) when μ is zero.

It will be noticed that (5.3) bears a curious resemblance to Macdonald's formula. The connection can be shown by Popov's method. We have†

$$\int_0^\infty t^{\mu+1}\,J_\mu(at)\,J_\nu(bt)\,K_\nu(ct)\,dt = \int_0^\infty\int_0^\infty t^{\mu-\nu+1}\,J_\mu(at)\,J_\nu(bt)\,\frac{x^{\nu+1}\,J_\nu(cx)}{x^2+t^2}\,dt\,dx$$

$$= \int_0^\infty x^{\mu+1}\,K_\mu(ax)\,I_\nu(bx)\,J_\nu(cx)\,dx,$$

which is an integral of Macdonald's type and can be evaluated in terms of Legendre functions.

---

* Watson (7), 410, (2).

† Watson (7), 425, (4), and 430, (6).

6. Our knowledge of the integral (1.1) is now as follows.

For all values of the parameters for which the integral converges, it can be expressed in terms of Appell's function F_4.

It can be evaluated in terms of ordinary hypergeometric functions when

$$\text{(i)} \quad \lambda = 1.$$

It can be evaluated in terms of Legendre functions when

$$\text{(ii)} \quad \lambda = 1, \quad \rho = \tfrac{1}{2},$$

$$\text{(iii)} \quad \lambda = \rho + 2, \quad \mu = \nu,$$

$$\text{(iv)} \quad \lambda = \mu + 2, \quad \rho = \nu.$$

It can be evaluated in terms of elementary functions when

$$\text{(v)} \quad \lambda = 2, \quad \rho = \nu - \mu.$$

It can be expressed as an integral of an elementary function when

$$\text{(vi)} \quad \lambda = \mu - \nu + \rho + 2,$$

$$\text{(vii)} \quad \lambda = \tfrac{3}{2}, \quad \rho = \tfrac{1}{2}.$$

The case (iii) is Macdonald's result, and (vii) is Watson's. The other results appear to be new.

7. We now consider the integral (1.2). Our starting point is the formula (2.1) which is true when

$$R(\lambda + \mu + \nu) > |R(\rho)|, \quad R(c) > 0,$$

the numbers a and b now being taken to be real and positive.

If we replace c by $k + ic$, where k and c are real, and use the asymptotic expansion of $K_\rho\{(k+ic)t\}$, it is easily seen that the integral is uniformly convergent for $k \geqslant 0$, provided that it converges when $k = 0$ and $c \neq 0$. It follows that the integral is continuous when $k = 0$, and so (2.1) remains valid when c is replaced by $\pm ic$, where the new c is real and positive. Using the formula

$$\pi i J_\nu(z) = e^{-\frac{1}{2}\nu\pi i} K_\nu(-iz) - e^{\frac{1}{2}\nu\pi i} K_\nu(iz),$$

we find that

$$(7.1)\quad \int_0^\infty t^{\lambda-1} J_\mu(at)\, J_\nu(bt)\, J_\rho(ct)\, dt$$

$$= \frac{2^{\lambda-1} a^\mu b^\nu \Gamma\{\frac{1}{2}(\lambda+\mu+\nu+\rho)\}}{c^{\lambda+\mu+\nu}\, \Gamma(\mu+1)\, \Gamma(\nu+1)\, \Gamma\{1-\frac{1}{2}(\lambda+\mu+\nu-\rho)\}}$$

$$\times F_4[\tfrac{1}{2}(\lambda+\mu+\nu-\rho),\ \tfrac{1}{2}(\lambda+\mu+\nu+\rho);\ \mu+1,\ \nu+1;\ a^2/c^2,\ b^2/c^2],$$

provided that

$$R(\lambda+\mu+\nu+\rho)>0, \quad R(\lambda)<\tfrac{5}{2}, \quad c>a+b,$$

the last condition being necessary when the function F_4 is defined as a double series.

8. The formula (7.1) simplifies in several special cases.

If $\lambda=1$, we can use (3.1), and we find that

$$(8.1)\quad \int_0^\infty J_\mu(ct\sin\phi\cos\Phi)\, J_\nu(ct\cos\phi\sin\Phi)\, J_\rho(ct)\, dt$$

$$= \frac{\Gamma\{\frac{1}{2}(1+\mu+\nu+\rho)\}\sin^\mu\phi\cos^\mu\Phi\cos^\nu\phi\sin^\nu\Phi}{c^{\mu+\nu+1}\,\Gamma(\mu+1)\,\Gamma(\nu+1)\,\Gamma\{\frac{1}{2}(1-\mu-\nu+\rho)\}}$$

$$\times F[\tfrac{1}{2}(1+\mu+\nu-\rho),\ \tfrac{1}{2}(1+\mu+\nu+\rho);\ \mu+1;\ \sin^2\phi]$$

$$\times F[\tfrac{1}{2}(1+\mu+\nu-\rho),\ \tfrac{1}{2}(1+\mu+\nu+\rho);\ \nu+1;\ \sin^2\Phi],$$

where ϕ and Φ are positive angles whose sum is acute. If $\Phi\to 0$, this reduces to the Weber-Schafheitlin integral.

When $\lambda=\rho-\mu-\nu$, the series F_4 in (7.1) reduces to unity, and we obtain the formula

$$(8.2)\qquad \int_0^\infty t^{\rho-\mu-\nu-1} J_\mu(at)\, J_\nu(bt)\, J_\rho(ct)\, dt = \frac{2^{\rho-\mu-\nu-1} a^\mu b^\nu \Gamma(\rho)}{c^\rho\, \Gamma(\mu+1)\, \Gamma(\nu+1)},$$

provided that $c>a+b$ and the integral is convergent.

Another obvious case when (7.1) simplifies is obtained by taking $\lambda=\rho-\mu-\nu+2$, when the right-hand side vanishes owing to the presence of $\Gamma(0)$ in the denominator. We thus find that

$$(8.3)\qquad \int_0^\infty t^{\rho-\mu-\nu+1} J_\mu(at)\, J_\nu(bt)\, J_\rho(ct)\, dt = 0,$$

provided that $c>a+b$.

Using (4.1), we see that the integral can be expressed as an integral of an elementary function when $\lambda = \mu - \nu \pm \rho + 2$; and, using (4.2), we see that the integral is expressible in terms of elementary functions when

$$\lambda = 2, \quad \rho = \nu - \mu,$$

the formula being

$$(8.4) \quad \int_0^\infty t J_\mu(ct \sin\phi \cos\Phi) J_\nu(ct \cos\phi \sin\Phi) J_{\nu-\mu}(ct)\, dt = -\frac{2 \sin\mu\pi \sin^\mu\phi \sin^\nu\Phi}{\pi c^2 \cos^\nu\phi \cos^\mu\Phi \cos(\phi+\Phi)\cos(\phi-\Phi)}.$$

9. The formula (8.1) gives the value of the integral

$$\int_0^\infty J_\mu(at) J_\nu(bt) J_\rho(ct)\, dt$$

when a, b, c are not the sides of a triangle.

This suggests the possibility of evaluating the integral when a, b, c are the sides of a triangle. Evidently some other method of approach would have to be adopted in this case.

10. Now consider the integral

$$I = \int_0^\infty J_\rho(ct) \frac{J_\mu\{a\sqrt{(t^2+z_1^2)}\} J_\nu\{b\sqrt{(t^2+z_2^2)}\}}{(t^2+z_1^2)^{\frac{1}{2}\mu}(t^2+z_2^2)^{\frac{1}{2}\nu}} t^{\lambda-1}\, dt.$$

From Lommel's expansion*,

$$(z+h)^{-\frac{1}{2}\nu} J_\nu\{\sqrt{(z+h)}\} = \sum_{m=0}^\infty \frac{(-\frac{1}{2}h)^m}{m!} z^{-\frac{1}{2}(\nu+m)} J_{\nu+m}(\sqrt{z}),$$

we see that

$$I = \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(-\frac{1}{2}z_1^2)^m(-\frac{1}{2}z_2^2)^n a^m b^n}{m!\, n!} \times \int_0^\infty t^{\lambda-1-\mu-\nu-m-n} J_\rho(ct) J_{\mu+m}(at) J_{\nu+n}(bt)\, dt.$$

When $\lambda = \rho + 2$ the integral on the right is zero by (8.3). When $\lambda = \rho$ we

* Watson (7), 5.22, (1).

can use (8.2), and we find that

$$I = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-\frac{1}{2}z_1^2)^m(-\frac{1}{2}z_2^2)^n a^m b^n}{m!\,n!} \frac{2^{\rho-1-\mu-\nu-m-n} a^{\mu+m} b^{\nu+n} \Gamma(\rho)}{c^\rho \Gamma(\mu+m+1)\Gamma(\nu+n+1)}$$

$$= \frac{2^{\rho-1}\Gamma(\rho)}{c^\rho} \frac{J_\mu(az_1) J_\nu(bz_2)}{z_1^{\mu} z_2^{\nu}}.$$

An exactly similar argument can be used to find corresponding results when there are any number of Bessel functions in the integrand, and we obtain the formulae

$$(10.1) \quad \int_0^\infty J_\rho(bt) \frac{J_\mu\{a_1\sqrt{(t^2+z_1^2)}\}\, J_\nu\{a_2\sqrt{(t^2+z_2^2)}\} \dots J_\kappa\{a_n\sqrt{(t^2+z_n^2)}\}}{(t^2+z_1^2)^{\frac{1}{2}\mu}(t^2+z_2^2)^{\frac{1}{2}\nu} \dots (t^2+z_n^2)^{\frac{1}{2}\kappa}} \times t^{\rho+1}\,dt = 0,$$

$$(10.2) \quad \int_0^\infty J_\rho(bt) \frac{J_\mu\{a_1\sqrt{(t^2+z_1^2)}\}\, J_\nu\{a_2\sqrt{(t^2+z_2^2)}\} \dots J_\kappa\{a_n\sqrt{(t^2+z_n^2)}\}}{(t^2+z_1^2)^{\frac{1}{2}\mu}(t^2+z_2^2)^{\frac{1}{2}\nu} \dots (t^2+z_n^2)^{\frac{1}{2}\kappa}} \times t^{\rho-1}\,dt$$

$$= \frac{2^{\rho-1}\Gamma(\rho)}{b^\rho} \frac{J_\mu(a_1z_1) J_\nu(a_2z_2) \dots J_\kappa(a_nz_n)}{z_1^{\mu} z_2^{\nu} \dots z_n^{\kappa}},$$

where in each integral $b > a_1+a_2+\dots+a_n$, and $a_1, a_2, \dots a_n$ are all positive.

The corresponding generalizations of (8.3) and (8.2) are

$$(10.3) \quad \int_0^\infty J_\rho(bt)\, J_\mu(a_1t)\, J_\nu(a_2t) \dots J_\kappa(a_nt)\, t^{\rho-\mu-\nu-\dots-\kappa+1}\,dt = 0,$$

$$(10.4) \quad \int_0^\infty J_\rho(bt)\, J_\mu(a_1t)\, J_\nu(a_2t) \dots J_\kappa(a_nt)\, t^{\rho-\mu-\nu-\dots-\kappa-1}\,dt$$

$$= \frac{2^{\rho-\mu-\nu-\dots-\kappa-1}\Gamma(\rho)}{b^\rho} \frac{a_1^{\mu} a_2^{\nu} \dots a_n^{\kappa}}{\Gamma(\mu+1)\Gamma(\nu+1)\dots\Gamma(\kappa+1)},$$

where again $b > a_1+a_2+\dots+a_n$. These formulae can be obtained from (10.1) and (10.2) by making $z_1, z_2, \dots, z_n$ all tend to zero.

The particular cases of the formulae of this paragraph when $\mu=\nu=\dots=\kappa$, or when $a_1=a_2=\dots=a_n$, have been given* by Nicholson, Gegenbauer, and Kluyver.

---

* See Watson (7), 413, (8); 418, (13); 419, (15) and (16).

[*Added* 25 *July*, 1935. Since this paper was written, some of the above results have been given by S. O. Rice, "On contour integrals for the product of two Bessel functions", *Quart. J. of Math.* (Oxford), 6 (1935), 52–64. He obtains the formula (7.1) by using integrals of Pochhammer's type, and he notes the case of reduction when $\lambda = 1$. Rice's paper appears to have been written at approximately the same time as my own.]

References.

1. P. Appell and J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques* (Paris, 1926).
2. W. N. Bailey, "A reducible case of the fourth type of Appell's hypergeometric functions of two variables", *Quart. J. of Math.* (Oxford), 4 (1933), 305–308.
3. ————, "On the reducibility of Appell's function F_4", *Quart. J. of Math.* (Oxford), 5 (1934), 291–292.
4. L. Gegenbauer, "On integrals containing functions of Bessel", *Proc. Section of Sci., K. Akad. van Wet. te Amsterdam*, 4 (1902), 584–588.
5. H. M. Macdonald, "Note on the evaluation of a certain integral containing Bessel's functions", *Proc. London Math. Soc.* (2), 7 (1909), 142–149.
6. A. Popov, "Über einige bestimmte Integrale mit Zylinderfunktionen", *C. R. Acad. Sci. U.R.S.S.*, 2 (1934), 11–12.
   (The actual title of this paper is in Russian. The title quoted here is the equivalent German title as given in the *Zentralblatt für Mathematik*.)
7. G. N. Watson, *Theory of Bessel functions* (Cambridge, 1922).
8. ————, "Some integrals involving Bessel functions", *Journal London Math. Soc.*, 3 (1928), 22–27.
9. ————, "An infinite integral involving Bessel functions", *Journal London Math. Soc.*, 9 (1934), 16–22.

The University,
Manchester.

SOME PROPERTIES OF S-FUNCTIONS

By D. E. LITTLEWOOD.

[Received 12 June, 1934.—Read 14 June, 1934.]

1. This paper deals primarily with the formation of three distinct types of generating functions for S-functions*. Various results of a different nature are obtained incidentally. Thus a connection is obtained with the rational fractions used by Padé and Hadamard†. A formula is given for the S-functions associated with the product of two series, and some algebraic identities are deduced. The generalized determinants of Naegelsbach and Aitken are examined, a generating function is obtained, an expression for the determinant as a sum of S-functions, and another proof of Aitken's theorem are also given. Extensions of the formulae for the S-function as determinants, with elements h_r and a_r respectively, are obtained, and an extension of Frobenius' formula for the degree of a character of a symmetric group. The paper concludes with a partial generalization to rational fractions of the formula for the S-function as a quotient of determinants.

2. *The association of S-functions with polynomials or power series.*

It is found convenient to change the convention in associating S-functions with polynomials, so as to make possible an extension to infinite series. Instead of considering the S-functions of the zeros of a polynomial, we associate the S-functions of

$$a_1, \quad a_2, \quad \ldots, \quad a_n$$

with the series which is the reciprocal of the polynomial with zeros $1/a_r$, *i.e.*, with

$$f(x) = 1/\Pi(1-a_r x). \tag{1}$$

* These functions are defined by $n!\,\{\lambda\} = \Sigma h_\rho \chi^{(\lambda)} S_\rho$, where $\chi^{(\lambda)}$ is a character of the symmetric group of order $n!$, ρ is the class $(1^a 2^b \ldots)$ of order h_ρ, $S_\rho = S_1{}^a S_2{}^b \ldots$, S_r being the usual symmetic function, and (λ) is a partition of n. See D. E. Littlewood and A. R. Richardson, *Phil. Trans. Royal Soc.* (A), 233 (1934), 99–141.

† H. Padé, "Sur la représentations approchées d'une fonction par des fonctions rationelles", *Annales Sci. de l'Ecole Normale* (3), 9 (1892), 1–93; J. Hadamard, "Essai sur l'étude des fonctions . . .", *Journal de Math.* (4), 8 (1892), 101–286.

This series may be expressed as

$$f(x) = 1 + \sum_{1}^{\infty} h_r x^r = 1 + \sum_{1}^{\infty} \{r\} x^r, \tag{2}$$

and is connected with the polynomial

$$F(x) = \Pi(x - \alpha_r) = x^n + \sum_{r=1}^{n} (-1)^r \{1^r\} x^{n-r}$$

by the equation

$$f(x) = 1/x^n \, F(1/x).$$

We can now make the number n of the quantities α_r become infinite, provided that $\Sigma \alpha_r$ is convergent. The properties of the S-functions are in no way affected by this change, save that the formula for the S-function as a quotient of determinants becomes the limit of the quotient of determinants as the number of rows and columns becomes infinite.

A further generalisation is possible. Corresponding to any series of the form

$$f(x) = 1 + \Sigma h_n x^n$$

we have a theory of S-functions independent of the convergence or divergence of the series. S_r is defined as the coefficient of x^{r-1} in the formal quotient $f'(x)/f(x)$, *i.e.*,

$$f'(x)/f(x) = \Sigma S_r x^{r-1}, \tag{3}$$

the dash denoting differentiation.

The S-functions are defined in terms of these quantities S_r by the usual equation.

In this theory, the expression of the S-function as a quotient of determinants, and the relations between S-functions and functions of the roots of the form $\Sigma x_1^{a_1} x_2^{a_2} \ldots$ have no significance, but all other properties of S-functions must still be valid. They are true when $f(x)$ is the reciprocal of a polynomial, and it is always possible to find a polynomial whose reciprocal coincides with $f(x)$ for the first w terms. Since the S-functions of weight less than or equal to w are unaffected by the coefficients of powers of x of index greater than w, the properties of S-functions proved for a reciprocal of a polynomial must remain true for the series $f(x)$.

Some particular cases are of special importance. If $f(x)$ is a rational fraction, say

$$f(x) = \Pi(1 - \alpha_r x)/\Pi(1 - \beta_r x),$$

then

$$S_n = \Sigma \beta_r^n - \Sigma \alpha_r^n. \tag{4}$$

A partial generalization of the formula for the S-function as a quotient of determinants is obtained, later in this paper, for the case of the rational fraction.

If the numbers of poles and zeros of $f(x)$ become infinite, the equation (4) still holds, provided that $\Sigma\alpha_r$ and $\Sigma\beta_r$ are convergent.

Again, if $f(x) = e^{x^r}$, then

$$S_r = 1, \quad S_n = 0 \quad (n \neq r). \tag{5}$$

If now we take $f(x)$ as a meromorphic function of real order p, the exponential factors will affect the values of S_n for $n \leqslant p$, and these values S_n may be arbitrarily assigned, and the exponential factors adjusted, but equation (4) holds for $n > p$.

For example,

$$\Gamma(1+x) = e^{-\gamma x} \Pi (1+x/n)^{-1} e^{x/n},$$

and for this function

$$S_2 = \sum_1^\infty n^{-2}, \quad S_3 = \sum_1^\infty n^{-3}, \quad \ldots,$$

but

$$S_1 = -\gamma,$$

where γ is Euler's constant.

3. *Generating functions for the S-functions.*

(*a*) *One variable.*

If

$$f(x) = 1+\Sigma h_n x^n = 1+\Sigma\{n\} x^n,$$

it is well known that

$$1/f(x) = 1+\Sigma(-1)^n a_n x^n = 1+\Sigma(-1)^n \{1^n\} x^n. \tag{6}$$

The equation (6) may be regarded as a generating function for the S-functions $\{1^n\}$, and immediately suggests the problem of finding generating functions for other S-functions.

One difficulty is that the number of S-functions of weight w is equal to the number of partitions of w, and a simple method of associating all partitions of all numbers with powers of a variable, or variables, is not obvious, unless an infinite number of variables is taken. It is possible to find generating functions, however, if we confine ourselves to partitions of a definite form.

Consider the S-functions

$$\{n, p_1, p_2, \ldots, p_i\},$$

where $p_1, p_2, \ldots, p_i$ are fixed numbers $(p_1 \geqslant p_2 \geqslant \ldots \geqslant p_i)$, but n varies.

We have, for $n \geqslant p_1$,

$$\{n, p_1, p_2, \ldots, p_i\} = \begin{vmatrix} h_n, & h_{n+1}, & \ldots & \\ h_{p_1-1}, & h_{p_1}, & \ldots & \\ h_{p_2-2}, & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots \\ h_{p_i-i}, & \ldots & \ldots & h_{p_i} \end{vmatrix}. \tag{7}$$

Now consider the product $f(x) \cdot g(x)$, where

$$g(x) = \begin{vmatrix} x^i, & x^{i-1}, & \ldots & 1 \\ h_{p_1-1}, & h_{p_1}, & \ldots & \\ h_{p_2-2}, & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots \\ h_{p_i-i}, & \ldots & \ldots & h_{p_i} \end{vmatrix}. \tag{8}$$

The coefficient of x^{n+i} for $n \geqslant p_1$ is clearly $\{n, p_1, \ldots, p_i\}$, from (7). Hence $f(x) \cdot g(x)$ may be regarded as a generating function for S-functions of the form $\{n, p_1, p_2, \ldots, p_i\}$.

THEOREM I. $f(x) \cdot g(x) = P(x) + \sum_{n=p_1}^{\infty} \{n, p_1, p_2, \ldots, p_i\} x^{n+i}$, *where* $P(x)$ *is a polynomial of degree less than* p_1+i.

The exact form of $P(x)$ is of interest. If $n < p_1+i$, the coefficient of x^n is still given by the determinant on the right of (7), but this is not equal to the S-function on the left.

If we can find j such that

$$n-i+j = p_j,$$

the coefficient of x^n is clearly zero. Otherwise the coefficient may be exhibited as an S-function by rearranging the rows of the determinant.

THEOREM II. *In the polynomial* $P(x)$ *of Theorem I, if* $n = p_j+i-j$, *the coefficient of* x^n *is zero; if* $p_j+i-j > n > p_{j+1}+i-j-1$, *the coefficient of* x^n *is*

$$(-1)^j \{p_1-1, p_2-1, \ldots, p_j-1, n-i+j, p_{j+1}, \ldots, p_i\};$$

and if $p_i > n$, *the coefficient of* x^n *is*

$$(-1)^i \{p_1-1, p_2-1, \ldots, p_i-1, n\}.$$

A special case is of particular interest. If $p_1 = p_2 = \ldots = p_i = r+1$, then

$$P(x) = \sum_{n=0}^{r} \{r^i, n\} x^n,$$

$$f(x)\, g(x) = P(x) + O(x^{i+r+1}),$$

$$f(x) = P(x)/g(x) + O(x^{i+r+1}),$$

where $P(x)$ is a polynomial of degree r and $g(x)$ is a polynomial of degree i. This is the rational fraction approximation to the series $f(x)$, used by Padé and Hadamard.

A generalization of the Padé and Hadamard fraction presents itself.

$P(x)$ and $g(x)$ may be chosen so that $g(x)$ is a polynomial of degree i, and $P(x)$ a polynomial of degree less than or equal to $i+r$, in which the coefficients of i arbitrarily assigned powers of x are zero, and

$$f(x) = P(x)/g(x) + O(x^{i+r+1}).$$

If the missing indices are $a_1, a_2, \ldots, a_i$ $(a_1 > a_2 > \ldots > a_i)$, then take

$$p_j = a_j - j + i,$$

and define $g(x)$ and $P(x)$ by (8) and Theorem II.

The generating function for the conjugate S-function

$$\{q_1, q_2, \ldots, q_i, 1^n\}$$

may be obtained from Theorem I, by replacing $f(x)$ by $1/f(x)$ and also replacing each term h_r by a_r. The coefficients of odd powers of x then carry an extra minus sign.

(b) Generating functions in more than one variable.

Another generating function may be found which gives all the S-functions which correspond to partitions into not more than p parts.

Consider the function of p variables

$$\phi(x_1, x_2, \ldots, x_p) = \prod_{r=1}^{p} f(x_r)\, \Delta(x_1, x_2, \ldots, x_p),$$

where

$$\Delta(x_1, x_2, \ldots, x_p) = \Pi\,(x_r - x_s) \quad (r < s)$$

$$= |x_s^{p-t}|.$$

ϕ is an alternating function of the variables. Consider the coefficient of $x_1^{\lambda_1+p-1} x_2^{\lambda_2+p-2} \ldots x_p^{\lambda_p}$ $(\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p)$. There is a term in ϕ corresponding to each term in the expansion of the determinant for $\Delta(x_1, \ldots, x_p)$.

Let $x_1^{e_1} x_2^{e_2} \dots x_p^{e_p}$ be any term of this expansion. To obtain $\Pi x_i^{\lambda_i+p-i}$ this term must be multiplied by $\Pi x_i^{\lambda_i+p-i-e_i}$, and the coefficient is

$$\pm \Pi h_{\lambda_i+p-i-e_i}.$$

Hence the coefficient of $\Pi x_i^{\lambda_i+p-i}$ in ϕ is

$$|h_{\lambda_s-s+t}| \quad \text{or} \quad \{\lambda_1, \lambda_2, \dots, \lambda_p\}.$$

Hence we have

THEOREM III.

$$\Pi f(x_r)\, \Delta(x_1, x_2, \dots, x_p) = \Sigma \pm \{\lambda_1, \lambda_2, \dots, \lambda_p\} x_1^{\lambda_1+p-1} x_2^{\lambda_2+p-2} \dots x_p^{\lambda_p}.$$

The summation is taken for all partitions into not more than p parts, and for all permutations of the suffixes of the x's, the minus sign being taken for an odd permutation.

Replacing $f(x)$ by $1/f(x)$, we obtain the conjugate result, namely

THEOREM IV.

$$\Delta(x_1, x_2, \dots, x_p)/\Pi f(x_r) = \Sigma \pm \{p^{\lambda_p}(p-1)^{\lambda_{p-1}-\lambda_p} \dots 1^{\lambda_1-\lambda_2}\} x_1^{\lambda_1+p-1} x_2^{\lambda_1+p-2} \dots x_p^{\lambda_p},$$

where the summation on the right includes all partitions in which none of the parts exceed p in magnitude. The term takes a minus sign for a negative permutation of the suffixes when the weight of the coefficient is even, and for a positive permutation when the weight is odd.

An important variant of these two theorems is obtained as follows. Denote S-functions of the quantities $x_1, x_2, \dots, x_n$ by

$$\{x;\ \lambda_1, \lambda_2, \dots, \lambda_p\} \quad \text{or} \quad \{x;\ \lambda\}.$$

Then we have

$$\Sigma \pm x_1^{\lambda_1+p-1} x_2^{\lambda_2+p-2} \dots x_p^{\lambda_p} = \{x;\ \lambda_1, \lambda_2, \dots, \lambda_p\}\, \Delta(x_1, \dots, x_p).$$

Thus, from Theorems III and IV,

$$\Pi f(x_r)\, \Delta(x_1, \dots, x_p) = [1 + \Sigma \{\lambda\} \{x;\ \lambda\}]\, \Delta(x_1, \dots, x_p),$$

$$\Delta(x_1, \dots, x_p)/\Pi f(x_r) = [1 \pm \Sigma \{\lambda\} \{x;\ \bar{\lambda}\}]\, \Delta(x_1, \dots, x_p).$$

Hence we have

THEOREM V.

$$\Pi f(x_r) = 1 + \Sigma \{\lambda\} \{x;\ \lambda\},$$

$$1/\Pi f(x_r) = 1 \pm \Sigma \{\lambda\} \{x;\ \bar{\lambda}\}.$$

$(\bar{\lambda})$ denotes here, and elsewhere in this paper, the partition conjugate to (λ). Of the alternative signs, the minus is taken when the weight is odd.

This result may be compared with one mentioned by MacMahon*, easily obtained by direct multiplication:

$$\Pi f(x_r) = 1 + \Sigma h_{(\lambda)} P_{(\lambda)},$$

where $$h_{(\lambda)} = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_p},$$

and $$P_{(\lambda)} = \Sigma x_1^{\lambda_1} x_2^{\lambda_2} \dots x_p^{\lambda_p}.$$

If $$f(x) = \Pi (1 - a_r x)^{-1},$$

then $$\Pi f(x_r) = \prod_{r,s} (1 - a_r x_s)^{-1},$$

a product symmetrical with respect to the two sets of quantities $a_1, a_2, \dots$, and $x_1, x_2, \dots, x_p$. This essential symmetry is brought out by the form of Theorem V, but not by the form given by MacMahon.

Again, comparing the two results, it follows that, if

$$\{\lambda\} = \Sigma K_{\lambda\mu} h_{(\mu)},$$

then

$$\Sigma a_1^{\mu_1} a_2^{\mu_2} \dots = \Sigma K_{\lambda\mu} \{\lambda\}, \tag{9}$$

where the coefficients $K_{\lambda\mu}$ are the same in both equations.

This equality was noticed by Kostka†, who constructed tables of the coefficients.

4. *S-functions associated with a product.*

Let $$F(x) = f(x)\, g(x),$$

where $$f(x) = 1 + \Sigma h_n' x^n,$$

$$g(x) = 1 + \Sigma h_n'' x^n,$$

$$F(x) = 1 + \Sigma h_n x^n.$$

Some formulae expressing the symmetric functions associated with $F(x)$, in terms of those associated with $f(x)$ and $g(x)$ respectively, are well

---

* MacMahon, *Combinatory analysis* (1915).

† See Muir, *Theory of determinants*, 4 (1923), 145.

known. Thus

$$h_n = \Sigma h_r' h_{n-r}'',$$

$$a_n = \Sigma a_r' a_{n-r}'',$$

and

$$S_n = S_n' + S_n''.$$

We now obtain a general formula expressing any S-function associated with $F(x)$ in terms of the S-functions of $f(x)$ and $g(x)$.

Denote the S-functions of $F(x)$, $f(x)$, and $g(x)$ respectively by $\{\lambda\}$, $\{\lambda\}'$, and $\{\lambda\}''$.

Clearly

$$\Pi F(x_r) = \Pi f(x_r) \Pi g(x_r).$$

Hence, from Theorem V,

$$\begin{aligned} 1+\Sigma\{\lambda\}\{x;\ \lambda\} &= [1+\Sigma\{\lambda\}'\{x;\ \lambda\}]\,[1+\Sigma\{\lambda\}''\{x;\ \lambda\}] \\ &= 1+\Sigma\{\lambda\}'\{\mu\}''\{x;\ \lambda\}\{x;\ \mu\} \\ &= 1+\Sigma g_{\lambda\mu\nu}\{\lambda\}'\{\mu\}''\{x;\ \nu\}, \end{aligned}$$

where $g_{\lambda\mu\nu}$ is the coefficient of $\{\nu\}$ in $\{\lambda\}\{\mu\}$.

Hence we have

THEOREM VI.

$$\{\nu\} = \Sigma g_{\lambda\mu\nu}\{\lambda\}'\{\mu\}''.$$

The coefficients $g_{\lambda\mu\nu}$ may be found by the multiplication theorem for S-functions*.

Examples.

The S-functions associated with $(1-x)^{-m}$ are known†. We have

$$\{\lambda\} = \chi_0^{(\lambda)}(P_\lambda)/n!,$$

where (P_λ) is the product of the first λ_i terms from each i-th row of the set of numbers

$$\begin{array}{cccccc} m, & m+1, & m+2, & m+3, & \ldots \\ m-1, & m, & m+1, & m+2, & \ldots \\ m-2, & m-1, & m, & m+1, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

and (λ) is a partition of n.

---

* D. E. Littlewood and A. R. Richardson, *Phil. Trans. Royal Soc.* (A), 233 (1934), 119.

† D. E. Littlewood and A. R. Richardson, "Immanants of some special matrices", *Quart. J. of Math.*, 5 (1934), 269–282.

By substituting in Theorem VI for the S-functions of

$$(1-x)^{-m-n} = (1-x)^{-m}(1-x)^{-n},$$

various identities may be obtained.

Denote the product $r(r+1)(r+2)\dots(r+i-1)$ by $[r]_i$. When we put $\{\nu\} = \{p\}$, Theorem VI gives

$$[m+n]_p/p! = \sum_{r=0}^{p} [m]_r [n]_{p-r}/r!\, p-r!,$$

which is Vandermonde's theorem.

Putting $\{\nu\} = \{p^2\}$, we notice that $\{p^2\}$ appears only in products of the form $\{r, s\}\{p-s, p-r\}$, and then with coefficients unity.

Hence $$\{p^2\} = \Sigma\{r, s\}'\{p-s, p-r\}''.$$

We have $$\chi_0^{(r,s)} = (r+s)!\,(r-s+1)/r+1!\,s!,$$

whence, from Theorem VI,

$$\frac{[m+n]_p\,[m+n-1]_p}{p+1!\,p!} = \Sigma\, \frac{[m]_r\,[m-1]_s\,[n]_{p-s}\,[n-1]_{p-r}\,(r-s+1)^2}{r+1!\,s!\;p-s+1!\;p-r!}.$$

This may be simplified, and, on replacing m, n, p by x, y, n, we may write it in the form

$$\begin{aligned}(10)\quad & [x+y]_n\,[x+y-1]_n \\ &= [y]_n\,[y-1]_n + \frac{n(n+1)}{2.1}\,[x]_1\,[x-1]_1\,[y]_{n-1}\,[y-1]_{n-1} \\ &\qquad + \frac{n(n-1)(n+1)n}{2.3.1.2}\,[x]_2\,[x-1]_2\,[y]_{n-2}[y-1]_{n-2} + \dots \\ &\quad + 2^2\Big[\tfrac{1}{2}n\,[x]_1\,[y]_n\,[y-1]_{n-1} \\ &\qquad + \frac{n(n-1)(n+1)}{2.3.1}\,[x]_2\,[x-1]_1\,[y]_{n-1}\,[y-1]_{n-2} + \dots\Big] \\ &\quad + 3^2\Big[\frac{n(n-1)}{2.3}\,[x]_2\,[y]_n\,[y-1]_{n-2} \\ &\qquad + \frac{n(n-1)(n-2)(n+1)}{2.3.4.1}\,[x]_3\,[x-1]_1\,[y]_{n-1}\,[y-1]_{n-3} + \dots\Big] \\ &\quad + \dots \quad \dots \quad \dots \quad \dots \\ &\quad + (n+1)\,[x]_n\,[y]_n.\end{aligned}$$

Picking out the coefficient of $x^r y^{2n-r}$ in (10) and denoting $x(x-1)(x-2)\dots(x-i+1)$ by $(x)_i$, we obtain

$$(11)\quad \frac{r(2n)_{r-1}}{(n)_{r-1}} = r^2 + \frac{r(n+1)}{(n-r+2).1}(r-2)^2 + \frac{r(r-1)(n+1)n}{(n-r+2)(n-r+3).1.2}(r-4)^2 + \frac{r(r-1)(r-2)(n+1)n(n-1)}{(n-r+2)(n-r+3)(n-r+4).1.2.3}(r-6)^2 + \dots \text{ to } \tfrac{1}{2}r \text{ or } \tfrac{1}{2}(r+1) \text{ terms.}$$

A generalization of (10) to products of the form

$$[x+y]_n[x+y-1]_n \dots [x+y-i+1]_n$$

is easily obtained by considering the S-function $\{p^i\}$. We obtain on the right an i-dimensional series.

Another interesting identity is obtained by taking

$$\{\nu\} = \{1+p,\ 1^q\}.$$

$\{\nu\}$ appears with coefficient unity in products of the form

$$\{1+r,\ 1^s\}\{p-r,\ 1^{q-s}\} \quad \text{and} \quad \{1+r,\ 1^s\}\{1+p-r,\ 1^{q-s-1}\}.$$

Hence, from Theorem VI,

$$\{1+p,\ 1^q\} = \{1+p,\ 1^q\}' + \{1+p,\ 1^q\}'' + \Sigma\{1+r,\ 1^s\}'\{p-r,\ 1^{q-s}\}'' + \Sigma\{1+r,\ 1^s\}'\{1+p-r,\ 1^{q-s-1}\}''.$$

Remembering that $\chi_0^{(1+p,\,1^q)} = p+q!/p!\,q!$, we deduce the identity

$$(12)\quad \frac{[x+y-q]_{p+q+1}}{(p+q+1)p!\,q!} = \frac{[x-q]_{p+q+1}}{(p+q+1)p!\,q!} + y\sum_{rs}{}' \frac{[x-s]_{r+s+1}[y-q+s+1]_{p+q-r-s-1}}{(r+s+1)r!\,s!\,p-r!\,q-s!} + \frac{[y-q]_{p+q+1}}{(p+q+1)p!\,q!}.$$

Σ' excludes the case $r=p,\ s=q$.

One deduction from Theorem VI is of particular interest. If $g(x) = 1/f(x)$, then $F(x) = f(x)g(x) = 1$. Clearly all the S-functions of $F(x)$ are zero. If we denote an S-function of $f(x)$ by $\{\lambda\}$, Theorem VI gives, for any fixed S-function $\{\nu\}$,

$$(13)\quad \Sigma(-1)^w g_{\lambda\mu\nu}\{\lambda\}\{\bar{\mu}\} = 0,$$

where w is the weight of $\{\mu\}$.

If $\{\nu\}=\{p\}$, this gives the well-known result

$$h_p-h_{p-1}a_1+h_{p-2}a_2-h_{p-3}a_3+\ldots\pm a_p=0.$$

Another result analogous to (13) may be obtained as follows.

Let $$\phi=\Pi f(\rho x_r)=1+\Sigma\{\lambda\}\{x;\ \lambda\}\rho^n,$$

where (λ) is a partition of n.

Then $$\partial\phi/\partial\rho=\Sigma n\{\lambda\}\{x;\ \lambda\}\rho^{n-1}.$$

But $$1/\phi=\Sigma(-1)^m\{\bar{\mu}\}\{x;\ \mu\}\rho^m,$$

where (μ) is a partition of m. Consequently

$$\frac{1}{\phi}\frac{\partial\phi}{\partial\rho}=\Sigma(-1)^m n\{\lambda\}\{\bar{\mu}\}\{x;\ \lambda\}\{x;\ \mu\}\rho^{m+n-1}$$

$$=\Sigma(-1)^m n g_{\lambda\mu\nu}\{\lambda\}\{\bar{\mu}\}\{x;\ \nu\}\rho^{m+n-1}.$$

Again, $$\log\phi=\Sigma\log f(\rho x_r),$$

and, by differentiating,

$$\frac{1}{\phi}\frac{\partial\phi}{\partial\rho}=\Sigma\frac{x_r}{f(\rho x_r)}\frac{\partial f(\rho x_r)}{\partial(\rho x_r)}$$

$$=\sum_{rn}S_n x_r{}^n\rho^{n-1}$$

$$=\sum_{n}S_n Z_n\rho^{n-1}$$

$$=\Sigma(-1)^q S_n\{x;\ p,\ 1^q\}\rho^{n-1},$$

summed for all partitions of the form $(p,\ 1^q)$, where $p+q+1=n$, $Z_n=\Sigma x_r{}^n$.

Hence

$$\Sigma(-1)^m n g_{\lambda\mu\nu}\{\lambda\}\{\bar{\mu}\}=(-1)^q S_k \quad \text{if} \quad (\nu)=(p,\ 1^q), \tag{14}$$

$$=0 \quad \text{otherwise},$$

where (λ) is a partition of n, (μ) a partition of m, and (ν) a partition of

$$k=m+n=p+q.$$

5. *The generalized determinants of Naegelsbach and Aitken.*

In the expression of the S-function as a determinant,

$$\{\lambda\}=|h_{\lambda_s-s+t}|,$$

the suffixes of the h's increase by unity in passing from one column to the

next, but they decrease by arbitrary steps, depending on the particular S-function chosen, in passing from one row to the next.

The more general determinant in which the suffixes in the columns differ by arbitrary leaps, as well as in the rows, has received considerable attention from Naegelsbach* and others, and Aitken† has given a theorem of duality connecting such determinants with the corresponding determinants involving the a_r's.

We now consider the theory of these determinants, obtaining a generating function for them, the expression for them as a sum of S-functions, and another proof of Aitken's theorem.

The determinant may be written

$$\{\lambda/\mu\} = |h_{\lambda_s-\mu_t-s+t}|,$$

where (λ) denotes the partition $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and (μ) the partition

$$(\mu_1, \mu_2, \ldots, \mu_n), \quad (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n, \quad \mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n),$$

the determinant having n rows and columns.

Consider the determinant with n rows and columns,

$$\phi = \left| \frac{x_s^p f(x_s) - y_t^p f(y_t)}{x_s - y_t} \right|.$$

This is an alternating function of $x_1, x_2, \ldots, x_n$ and also of $y_1, y_2, \ldots, y_n$. The coefficient of $\Pi x_r^{\lambda_r+n-r} y_r^{a_r+n-r}$ is clearly

$$|h_{\lambda_s+a_t+2n-s-t-p+1}|,$$

or, if we put $\mu_t = q - a_{n-t}$,

$$|h_{\lambda_s-\mu_t-s+t+n+q-p+1}| = |h_{\lambda_s-\mu_t-s+t}|,$$

if $q = p-n-1$.

This result is unaffected if all the numbers λ_r and μ_r are increased or decreased by the same amount. Hence

THEOREM VII.

The coefficient of $\Pi x_r^{\lambda_r+n-r-k} y_r^{p-1-\mu_r-n+r+k}$ *in*

$$\phi = \left| \frac{x_s^p f(x_s) - y_t^p f(y_t)}{x_s - y_t} \right|$$

is $|h_{\lambda_s-\mu_t-s+t}| = \{\lambda/\mu\}$.

* See Muir, *Theory of determinants*, 3 (1920), 144.

† Aitken, *Proc. Edinburgh Math. Soc.* (2), 1 (1929), 55, and (2), 2 (1931), 164.

This may be regarded as a generating function for the quantities $\{\lambda/\mu\}$. Now the coefficient of $\Pi y_r^{p-1-\mu_r-n+r}$ is clearly

$$|x_s^{u_t+n-t} f(x_s)|,$$

which is a generating function for the quantities $\{\lambda/\mu\}$ when (μ) is fixed.

$$\begin{aligned}|x_s^{u_t+n-t} f(x_s)| &= \Pi f(x_r)\,|x_s^{u_t+n-t}| \\ &= [1+\Sigma\{\nu\}\{x;\ \nu\}]\{x;\ \mu\}\,\Delta(x_1, \ldots, x_n) \\ &= \Sigma g_{\nu\mu\lambda}\{\nu\}\{x;\ \lambda\}\,\Delta(x_1, \ldots, x_n) \\ &= \Sigma g_{\nu\mu\lambda}\{\nu\}\,\Pi x_r^{\lambda_r+n-r},\end{aligned}$$

where $g_{\nu\mu\lambda}$ is the coefficient of $\{\lambda\}$ in the product $\{\nu\}\{\mu\}$.

Hence the coefficient of $\Pi x_r^{\lambda_r+n-r} y_r^{p-1-\mu_r-n+r}$ in ϕ is $\Sigma g_{\nu\mu\lambda}\{\nu\}$. Using Theorem VII, we obtain

THEOREM VIII.

$$\{\lambda/\mu\} = |h_{\lambda_s-\mu_t-s+t}| = \Sigma g_{\nu\mu\lambda}\{\nu\}.$$

Zia-ud-din* gives a method of expressing this determinant as a sum of bi-alternants, *i.e.*, as a sum of S-functions. His formula for finding the coefficient is a trifle involved, but Theorem VIII may be deduced fairly easily from his result.

As an example of Theorem VIII, consider the determinant

$$\begin{vmatrix} h_4, & h_5, & h_7 \\ h_2, & h_3, & h_5 \\ 1, & h_1, & h_3 \end{vmatrix}.$$

This may be expressed as $|h_{\lambda_s-\mu_t-s+t}|$ if $(\lambda) = (5, 4, 3)$ and $(\mu) = (1^2)$.

Now $\{5, 4, 3\}$ appears in the product $\{\nu\}\{1^2\}$ with coefficient unity if either

$$\{\nu\} = \{4, 3^2\},\ \{4^2, 2\},\ \text{or}\ \{5, 3, 2\}.$$

Hence
$$\begin{vmatrix} h_4, & h_5, & h_7 \\ h_2, & h_3, & h_5 \\ 1, & h_1, & h_3 \end{vmatrix} = \{4, 3^2\}+\{4^2, 2\}+\{5, 3, 2\}.$$

* Zia-ud-Din, *Proc. Edinburgh Math. Soc.* (2), 4 (1934), 47.

Aitken's theorem may now be deduced immediately. The conjugate S-functions, being the S-functions of the series $1/f(-x)$, obviously obey the same multiplication law as the S-functions themselves. Hence $g_{\nu\mu\lambda}$ is also the coefficient of $\{\bar{\lambda}\}$ in the product $\{\bar{\nu}\}\{\bar{\mu}\}$. Thus, denoting conjugate partitions and S-functions by a bar, we have

THEOREM IX.

$$\{\bar{\lambda}/\bar{\mu}\} = \Sigma g_{\nu\mu\lambda}\{\bar{\nu}\} = |a_{\lambda_s-\mu_t-s+t}|.$$

This, expressed differently, is Aitken's theorem.

6. *A third type of generating function for S-functions.*

We now obtain a generating function for all S-functions which correspond to partitions in which not more than m parts are greater than n in magnitude.

The simplest case, when $m = n = 1$, presents no difficulty.

THEOREM X.

$$\frac{f(x)-f(y)}{(x-y)f(y)} = \Sigma(-1)^\beta\{1+\alpha,\ 1^\beta\}x^\alpha y^\beta.$$

Clearly,

$$[f(x)-f(y)]/(x-y) = \Sigma h_r(x^{r-1}+x^{r-2}y+\ldots+y^{r-1}).$$

Hence the coefficient of $x^\alpha y^\beta$ in

$$[f(x)-f(y)]/(x-y)f(y) = [\Sigma h_r(x^{r-1}+x^{r-2}y+\ldots+y^{r-1})]\,[1+\Sigma(-1)^i a_i y^i]$$

is $$(-1)^\beta[h_{\alpha+1}a_\beta - h_{\alpha+2}a_{\beta-1}+\ldots\pm h_{\alpha+\beta+1}] = (-1)^\beta\{1+\alpha,\ 1^\beta\};$$

this proves the theorem.

We next consider the general case when $m = n$. We require $2m$ variables

$$x_1,\ x_2,\ \ldots,\ x_m;\quad y_1,\ y_2,\ \ldots,\ y_m,$$

and obtain a generating function for

$$\Sigma\pm\{1+\lambda_1,\ 2+\lambda_2,\ \ldots,\ m+\lambda_m,\ m^{\mu_m}(m-1)^{\mu_{m-1}-\mu_m}\ldots 1^{\mu_1-\mu_2}\} \times\Pi x_r^{\lambda_r}y_r^{\mu_r}. \tag{15}$$

Expressing the S-function, given as the coefficient of the typical term in (15), as a determinant in terms of the h_r's, we consider the Laplace development in terms of the first m rows.

The minor obtained from the first m rows and the i_1-th, i_2-th, ..., i_m-th columns is

$$|h_{\lambda_s+i_t}|.$$

The conjugate minor is a determinant of the form

$$|h_{\alpha_s-\beta_t-s+t}|,$$

where β_t is the number of terms i_j-j that are greater than or equal to t, and α_s is the number of terms μ_j-m+j greater than or equal to t.

From Theorem IX (Aitken's theorem),

$$|h_{\alpha_s-\beta_t-s+t}| = |a_{\mu_s-i_t+1}|.$$

Hence

$$\{1+\lambda_1, 2+\lambda_2, \ldots, m+\lambda_m, m^{\mu_m}(m-1)^{\mu_{m-1}-\mu_m-1}\ldots 1^{\mu_1-\mu_2-1}\}$$
$$= \Sigma\pm|h_{\lambda_s+i_t}||a_{\mu_s-i_t+1}|.$$

Now consider the determinant

$$\left|\frac{f(x_s)-f(y_t)}{x_s-y_t}\right|.$$

The coefficient of $\Pi x_r^{\lambda_r} y_r^{i_r-1}$ is $|h_{\lambda_s+i_t}|$.

Hence the coefficient of $\Pi x_r^{\lambda_r} y_r^{\mu_r}$ in

$$\left|\frac{f(x_s)-f(y_t)}{x_s-y_i}\right| \Big/ \Pi f(y_r)$$

is $\Sigma\pm|h_{\lambda_s+i_t}||a_{\mu_s-i_t+1}| = \pm\{1+\lambda_1', \ldots, m+\lambda_m, m^{\mu_m}\ldots 1^{\mu_1-\mu_2-1}\}$.

THEOREM XI.

$$\left|\frac{f(x_s)-f(y_t)}{x_s-y_t}\right| \Big/ \Pi f(y_r) = \Sigma\pm\{1+\lambda_1, \ldots, m+\lambda_m, m^{\mu_m}\ldots 1^{\mu_1-\mu_2-1}\}\,\Pi x_r^{\lambda_r} y_r^{\mu_r}.$$

Throughout these equations when alternative signs are indicated, the factor -1 is introduced for a negative permutation of the suffixes of the x's, or of the y's, and again if the total degree in the y's $-\frac{1}{2}n(n-1)$ is odd.

The left-hand side of the equation of Theorem XI may be written

$$\left|\frac{f(x_s)-f(y_t)}{(x_s-y_t)f(y_t)}\right|,$$

which by Theorem X is equal to

$$|\Sigma(-1)^\beta\{1+\alpha, 1^\beta\}x_s^\alpha y_t^\beta|.$$

Picking out the coefficient of $\Pi x_r^{\lambda_r} y_r^{\mu_r}$, and equating to the corresponding coefficient on the right-hand side of Theorem XI, we obtain

THEOREM XII.

$$\{1+\lambda_1,\ 2+\lambda_2,\ \ldots,\ m+\lambda_m,\ m^{\mu_m}(m-1)^{\mu_{m-1}-\mu_m}\ldots 1^{\mu_1-\mu_2}\} = |\{1+\lambda_s,\ 1^{\mu_t}\}|.$$

This expresses any S-function as a determinant in which the elements are S-functions of the form $\{p,\ 1^q\}$, and may be compared with the formulae expressing S-functions as determinants in which the elements are respectively of the form h_r or $\{r\}$, and a_r or $\{1^r\}$.

For example,

$$(16)\qquad \{4,\ 3,\ 2^2,\ 1\} = \begin{vmatrix} h_4, & h_5, & h_6, & h_7, & h_8 \\ h_2, & h_3, & h_4, & h_5, & h_6 \\ 1, & h_1, & h_2, & h_3, & h_4 \\ & 1, & h_1, & h_2, & h_3 \\ & & & 1, & h_1 \end{vmatrix} = \begin{vmatrix} a_5, & a_6, & a_7, & a_8 \\ a_3, & a_4, & a_5, & a_6 \\ 1, & a_1, & a_2, & a_3 \\ & & 1, & a_1 \end{vmatrix}$$

$$= \begin{vmatrix} \{5,\ 1^2\}, & \{6,\ 1^2\} \\ \{5,\ 1\}, & \{6,\ 1\} \end{vmatrix}.$$

A further generalisation is obtained later.

It is of interest to substitute in Theorem XII the known S-functions of the function $f(x) = (1-x)^{-m}$. The factors involving m, $m+1$, $m-1$, etc., may be taken out of the columns and rows, and correspond exactly on the two sides of the equation, but we are led to a formula for χ_0, the degree of the character corresponding to the S-function.

$$\frac{\chi_0}{n!} = \left| \frac{1}{\lambda_s!\,\mu_t!(\lambda_s+\mu_t+1)} \right|,$$

$$(17)\qquad \chi_0 = \frac{n!}{\mu_1!\,\mu_2!\ldots\mu_m!\,\lambda_1!\,\lambda_2!\ldots\lambda_m!} \left| \frac{1}{\lambda_s+\mu_t+1} \right|$$

$$\times \frac{n!\,\Delta(\lambda_1,\ \ldots,\ \lambda_m)\,\Delta(\mu 1,\ \ldots,\ \mu_m)}{\mu_1!\,\mu_2!\ldots\mu_m!\,\lambda_1!\,\lambda_2!\ldots\lambda_m!\,\Pi(\lambda_r+\mu_p+1)}.$$

This formula was obtained otherwise by Frobenius*.

The case still remains to be considered of the generating function for S-functions which correspond to partitions in which not more than m parts are greater than n in magnitude, and $m \neq n$.

---

* Frobenius, *Berliner Sitzungsberichte* (1900), 516.

Consider first $m > n$, say $m = n+\alpha$. We take $n+m$ variables

$$x_1,\ x_2,\ \ldots,\ x_m;\ y_1,\ y_2,\ \ldots,\ y_n.$$

Denote by Δ_{mn} the determinant

$$\Delta_{mn} = \left| x_s^{\alpha-t'} f(x_s) \;\middle|\; \frac{f(x_s)-f(y_{t''})}{x_s-y_{t''}} \right|,$$

in which s runs from 1 up to m, t' from 1 up to α, and t'' from 1 up to n.

The coefficient of $\Pi x_r^{\lambda_r} y_r^{i_r}$ is clearly

$$|h_{\lambda_s-\alpha+t'} \mid h_{\lambda_s+i_{t''}}|.$$

Thus the coefficient of $\Pi x_s^{\lambda_s} y_t^{\mu_t}$ in the quotient

$$\Delta_{mn}/\Pi f(y_r)$$

is

$$\Sigma \pm |h_{\lambda_s-\alpha+t'} \mid h_{\lambda_s+i_{t''}}||a_{\mu_s-i_t+1}|$$
$$= \{\lambda_1-\alpha+1,\ \lambda_2-\alpha+2,\ \ldots,\ \lambda_m+n,\ n^{\mu_n}(n-1)^{\mu_{n-1}-\mu_n-1}\ldots 1^{\mu_1-\mu_2-1}\}.$$

Hence we have

THEOREM XIII.

$$\left| x_s^{\alpha-t'} f(x_s) \;\middle|\; \frac{f(x_s)-f(y_{t''})}{x_s-y_{t''}} \right| \Big/ \Pi f(y_r)$$
$$= \Sigma \pm \{\lambda_1-\alpha+1,\ \lambda_2-\alpha+2,\ \ldots,\ \lambda_m+n,\ n^{\mu_n},\ \ldots,\ 1^{\mu_1-\mu_2-1}\} \Pi x_r^{\lambda_r} y_r^{\mu_r}.$$

The case in which $n > m$ can be obtained from this by substituting $1/f(x)$ for $f(x)$. Put $\alpha = n-m$.

THEOREM XIV.

$$\left| y_s^{\alpha-t'} \;\middle|\; \frac{f(x_{t''})-f(y_s)}{x_{t''}-y_s} \right| \Big/ \Pi f(y_r)$$
$$= \Sigma \pm \{\lambda_1+\alpha+1,\ \lambda_2+\alpha+2,\ \ldots,\ \lambda_m+n,\ n^{\mu_n},\ \ldots,\ 1^{\mu_1-\mu_2-1}\} \Pi x_r^{\lambda_r} y_r^{\mu_r}.$$

Theorems XIII and XIV are a generalization of Theorems III, IV, X, and XI, which may be obtained from these by putting respectively

$n = 0$ in Theorem XIII,

$m = 0$ in Theorem XIV,

$m = n = 1$ in either theorem,

$m = n$ in either theorem.

In the same manner in which Theorem XII is deduced from Theorem XI we obtain

THEOREM XV.

$$\{\lambda_1-\alpha+1,\ \lambda_2-\alpha+2,\ \ldots,\ \lambda_m+n,\ n^{\mu_n}(n-1)^{\mu_{n-1}-\mu_n-1}\ldots 1^{\mu_1-\mu_2-1}\}$$
$$=|\{\lambda_s-\alpha+t'\}|\{1+\lambda_s,\ 1^{\mu_{t'}}\}|.$$

THEOREM XVI.

$$\{\lambda_1+\alpha+1,\ \lambda_2+\alpha+2,\ \ldots,\ \lambda_m+n,\ n^{\mu_n}(n-1)^{\mu_{n-1}-\mu_n-1}\ldots 1^{\mu_1-\mu_2-1}\}$$
$$=|\{1^{\mu_s-\alpha+t'}\}|\{1+\lambda_{t'},\ 1^{\mu_s}\}|.$$

As an example, besides the determinant forms given in (16), $\{4, 3, 2^2, 1\}$ may be expressed

$$\{4, 3, 2^2, 1\}=\begin{vmatrix} a_5, & a_6, & \{2, 1^6\} \\ a_3, & a_4, & \{2, 1^4\} \\ 1, & a_1, & \{2, 1\} \end{vmatrix}=\begin{vmatrix} h_4, & h_5, & h_6, & \{7, 1\} \\ h_2, & h_3, & h_4, & \{5, 1\} \\ 1, & h_1, & h_2, & \{3, 1\} \\ & 1, & h_1, & \{2, 1\} \end{vmatrix}.$$

By considering the S-functions associated with $(1-x)^{-m}$, we obtain a generalization of Frobenius' formula (17).

THEOREM XVII. *If a partition (ζ) of w is expressed in any way in the form*

$$\{\lambda_1+\alpha+1,\ \lambda_2+\alpha+2,\ \ldots,\ \lambda_m+n,\ n^{\mu_n}(n-1)^{\mu_{n-1}-\mu_n-1}\ldots 1^{\mu_1-\mu_2-1}\},$$

where $\alpha=n-m$ is a positive, zero, or negative integer, then

$$\chi_0^{(\zeta)}=\frac{w!\,\Delta(\lambda_1,\ \ldots,\ \lambda_m)\,\Delta(\mu_1,\ \ldots,\ \mu_n)}{\Pi\lambda_r!\,\Pi\mu_r!\,\Pi(\lambda_r+\mu_s+1)}.$$

7. *Extension to rational fractions of the formula for the S-function as a quotient of determinants.*

It is known that, if $f(x)=1\Big/\prod_{r=1}^{q}(1-\alpha_r x)$, then the corresponding S-functions $\{\lambda\}$ satisfy

$$\{\lambda\}=\frac{|\alpha_s^{\lambda_t+q-t}|}{|\alpha_s^{q-t}|}.$$

Again, if $g(x) = \prod_{r=1}^{p} (1-\beta_r x)$, and $(\bar{\mu})$ is the conjugate partition of (μ), then the corresponding S-functions satisfy

$$\{\mu\} = \frac{|\beta_s^{\bar{\mu}_t+p-t}|}{|\beta_s^{p-t}|}.$$

We seek an extension of these results to the case when

$$F(x) = \prod_{1}^{p} (1-\beta_r x) \Big/ \prod_{1}^{q} (1-\alpha_r x).$$

Denote S-functions of $F(x)$, $f(x)$, and $g(x)$ respectively by $\{\lambda\}$, $\{\lambda\}'$, and $\{\lambda\}''$. Then, from Theorem VI, we have

$$\{\nu\} = \Sigma g_{\lambda\mu\nu} \{\lambda\}' \{\mu\}'',$$

whence

$$\{\nu\} \Delta(\alpha_1, \ldots, \alpha_q) \Delta(\beta_1, \ldots, \beta_p) = \Sigma g_{\lambda\mu\nu} |\alpha_s^{\lambda_t+q-t}| |\beta_s^{\bar{\mu}_t+p-t}|.$$

Consider first the particular case when $p = q = 2$, and $\{\nu\} = \{2^2\}$. We have

$$\begin{aligned}\{\nu\}(\alpha_1-\alpha_2)(\beta_1-\beta_2) = &\begin{vmatrix} \alpha_1^3, & \alpha_1^2 \\ \alpha_2^3, & \alpha_2^2 \end{vmatrix} \begin{vmatrix} \beta_1, & 1 \\ \beta_2, & 1 \end{vmatrix} - \begin{vmatrix} \alpha_1^3, & \alpha_1 \\ \alpha_2^3, & \alpha_2 \end{vmatrix} \begin{vmatrix} \beta_1^2, & 1 \\ \beta_2^2, & 1 \end{vmatrix} \\ &+ \begin{vmatrix} \alpha_1^3, & 1 \\ \alpha_2^3, & 1 \end{vmatrix} \begin{vmatrix} \beta_1^2, & \beta_1 \\ \beta_2^2, & \beta_2 \end{vmatrix} + \begin{vmatrix} \alpha_1^2, & \alpha_1 \\ \alpha_2^2, & \alpha_2 \end{vmatrix} \begin{vmatrix} \beta_1^3, & 1 \\ \beta_2^3, & 1 \end{vmatrix} \\ &- \begin{vmatrix} \alpha_1^2, & 1 \\ \alpha_2^2, & 1 \end{vmatrix} \begin{vmatrix} \beta_1^3, & \beta_1 \\ \beta_2^3, & \beta_2 \end{vmatrix} + \begin{vmatrix} \alpha_1, & 1 \\ \alpha_2, & 1 \end{vmatrix} \begin{vmatrix} \beta_1^3, & \beta_1^2 \\ \beta_2^3, & \beta_2^2 \end{vmatrix} \\ = &\begin{vmatrix} \alpha_1^3, & \alpha_1^2, & \alpha_1, & 1 \\ \alpha_2^3, & \alpha_2^2, & \alpha_2, & 1 \\ \beta_1^3, & \beta_1^2, & \beta_1, & 1 \\ \beta_2^3, & \beta_2^2, & \beta_2, & 1 \end{vmatrix}.\end{aligned}$$

Thus $$\{\nu\} = \Delta(\alpha_1, \alpha_2, \beta_1, \beta_2) / \Delta(\alpha_1, \alpha_2) \Delta(\beta_1, \beta_2).$$

This result is easily extended to the case

$$F(x) = \prod_{1}^{p} (1-\beta_r x) \Big/ \prod_{1}^{q} (1-\alpha_r x),$$

for the S-function $\{p^q\}$.

Clearly $\{p^q\} = \Sigma \{\lambda_1, \lambda_2, \ldots, \lambda_q\}' \{p-\lambda_q, p-\lambda_{q-1}, \ldots, p-\lambda_1\}''$.

Hence

$$\{p^q\}\Delta(\alpha_1, \ldots, \alpha_q)\Delta(\beta_1, \ldots, \beta_p) = \Sigma(-1)^{pq+\lambda_1+\ldots+\lambda_q}|\alpha_s^{\lambda_t+q-t}||\beta_s^{\mu_t+q-t}|,$$

where (μ) is the partition conjugate to $(p-\lambda_q, p-\lambda_{q-1}, \ldots, p-\lambda_1)$. (μ) may be expressed in the form

$$\{q^{p-\lambda_1}, (q-1)^{\lambda_1-\lambda_2}, (q-2)^{\lambda_2-\lambda_3}, \ldots\},$$

whence it is clear that the indices in the second determinant are the indices omitted from the first determinant. Hence

THEOREM XVII. *For the function*

$$F(x) = \prod_1^p (1-\beta_r x) \Big/ \prod_1^q (1-\alpha_r x),$$

$$\{p^q\}\Delta(\alpha_1, \ldots, \alpha_q)\Delta(\beta_1, \ldots, \beta_p) = \Delta(\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p).$$

Another proof of this theorem could be obtained by showing that, if the partition (λ) has $q+1$ parts greater than or equal to $p+1$ in magnitude, and

$$F(x) = \prod_1^p (1-\beta_r x) \Big/ \prod_1^q (1-\alpha_r x),$$

then $$\{\lambda\} = 0,$$

for $$\{\lambda\} = \Sigma g_{\nu\mu\lambda}\{\nu\}'\{\mu\}'',$$

and, since an S-function of p quantities corresponding to a partition into more than p parts is zero*, for each term in the sum either $\{\nu\}' = 0$ or $\{\mu\}'' = 0$. Hence, if $\alpha_r = \beta_s$, then $\{p^q\}$ must be zero, and $\{p^q\}$ contains a factor $\Pi(\alpha_r - \beta_s)$. Elementary considerations complete the proof of the theorem.

More generally, it has been found possible to express any S-function corresponding to a partition into equal parts, of a rational fraction, as a quotient of determinants, as the following three theorems will show.

Consider first the case when $\{\nu\} = \{p^q\}$, and

$$F(x) = \prod_{r=1}^{p+i} (1-\beta_r x) \Big/ \prod_{r=1}^{q+i} (1-\alpha_r x).$$

An exact repetition of the reasoning which leads to Theorem XVII enables us to show that

$$\{p^q\}\Delta(\alpha_1, \ldots, \alpha_{q+1})\Delta(\beta_1, \ldots, \beta_{p+i}) = \Sigma(-1)^{pq+\lambda_1+\ldots+\lambda_q}|\alpha_s^{\lambda_t+q-t}||\beta_s^{\mu_t+q-t}|.$$

In the product of determinants on the right, the indices are taken from the numbers

$$p+q+i-1,\ p+q+i-2,\ \ldots,\ 3,\ 2,\ 1,\ 0.$$

---

* D. E. Littlewood and A. R. Richardson, *Phil. Trans. Royal Soc.* (A), 233 (1934), 111.

The first $p+q$ numbers appear as indices in one of the determinants only, and the last i numbers appear as indices in both determinants. Hence we have

THEOREM XVIII. *If*

$$F(x) = \prod_{r=1}^{p+i}(1-\beta_r x) \Big/ \prod_{r=1}^{q+i}(1-\alpha_r x),$$

then the corresponding S-functions satisfy

$$\{p^q\}\Delta(\alpha_1, \ldots, \alpha_{q+i})\Delta(\beta_1, \ldots, \beta_{p+i}) = \begin{vmatrix} \alpha_s^{p+q+i-t} & \alpha_s^{i-t'} & 0 \\ \beta_s^{p+q+i-t} & 0 & \beta_s^{i-t''} \end{vmatrix},$$

in which t runs from 1 to $p+q$, and t' and t'' run from 1 to i.

There remains the case when the difference between the number of poles and the number of zeros of the rational fraction is not equal to the difference between the number of parts and the magnitude of each part in the partition. The results for this case are easily deduced from Theorem XVIII.

First put $\beta_{p+i} = 0$. The last row contains only unity in the last column and zeros elsewhere. The determinant is equal to the minor of this element in the last row, last column. The factors $\beta_1, \beta_2, \ldots, \beta_{p+i-1}$, which will no longer appear in the product of the differences of the β's if β_{p+i} is omitted, may be taken out as a factor from the corresponding rows. This process may be repeated if more of the quantities β_r are zero. Hence

THEOREM XIX. *If*

$$F(x) = \prod_{r=1}^{p+i-j}(1-\beta_r x) \Big/ \prod_{r=1}^{q+i}(1-\alpha_r x),$$

the value of $\{p^q\}\Delta(\alpha_1, \ldots, \alpha_{q+i})\Delta(\beta_1, \ldots, \beta_{p+i-j})$

is given by the determinant in Theorem XVIII if the rows involving $\beta_{p+i-j+1}, \ldots, \beta_{p+i}$ are struck out, and all the indices of the β's are decreased by j, the columns with negative indices being struck out.

In an exactly similar manner we may obtain

THEOREM XX. *If*

$$F(x) = \prod_{r=1}^{p+i}(1-\beta_r x) \Big/ \prod_{r=1}^{q+i-j}(1-\alpha_r x),$$

then the value of

$$\{p^q\}\Delta(\alpha_1, \ldots, \alpha_{q+i-j})\Delta(\beta_1, \ldots, \beta_{p+i})$$

is given by the determinant in Theorem XVIII if the rows involving $a_{q+i-j+1}, \ldots, a_{q+i}$ *are struck out, and all the indices of the* a*'s are decreased by* j*, the columns with negative indices being struck out.*

Example. If $F(x)=(1-\beta_1 x)(1-\beta_2 x)/(1-a_1 x)(1-a_2 x)(1-a_3 x)$, then we have

$$\{2^3\}=\Pi(a_r-\beta_s).$$

$$\{3^3\}\Delta(a_1, a_2, a_3)\Delta(\beta_1, \beta_2)=\begin{vmatrix} a_1^5, & a_1^4, & a_1^3, & a_1^2, & a_1 \\ a_2^5, & a_2^4, & a_2^3, & a_2^2, & a_2 \\ a_3^5, & a_3^4, & a_3^3, & a_3^2, & a_3 \\ \beta_1^4, & \beta_1^3, & \beta_1^2, & \beta_1, & 1 \\ \beta_2^4, & \beta_2^3, & \beta_2^2, & \beta_2, & 1 \end{vmatrix},$$

$$\{2^2\}\Delta(a_1, a_2, a_3)\Delta(\beta_1, \beta_2)=\begin{vmatrix} a_1^4, & a_1^3, & a_1^2, & a_1, & 1 \\ a_2^4, & a_2^3, & a_2^2, & a_2, & 1 \\ a_3^4, & a_3^3, & a_3^2, & a_3, & 1 \\ \beta_1^3, & \beta_1^2, & \beta_1, & 1, & \\ \beta_2^3, & \beta_2^2, & \beta_2, & 1, & \end{vmatrix},$$

$$\{1^2\}\Delta(a_1, a_2, a_3)\Delta(\beta_1, \beta_2)=\begin{vmatrix} a_1^3, & a_1^2, & a_1, & 1, & \\ a_2^3, & a_2^2, & a_2, & 1, & \\ a_3^3, & a_3^2, & a_3, & 1, & \\ \beta_1^3, & \beta_1^2, & \beta_1, & & 1 \\ \beta_2^3, & \beta_2^2, & \beta_2, & & 1 \end{vmatrix},$$

$$\{1\}\Delta(a_1, a_2, a_3)\Delta(\beta_1, \beta_2)=\begin{vmatrix} a_1^3, & a_1^2, & a_1, & 1, & \\ a_2^3, & a_2^2, & a_2, & 1, & \\ a_3^3, & a_3^2, & a_3, & 1, & \\ \beta_1^2, & \beta_1, & & & 1 \\ \beta_2^2, & \beta_2, & & & 1 \end{vmatrix}.$$

For partitions into unequal parts no corresponding simple result has been found, even for the most simple S-function $\{2, 1\}$.

University College,
Swansea.

ON THE INTEGRAL REPRESENTATIONS OF THE LOMMEL FUNCTIONS

By PIOTR SZYMAŃSKI.

[Received 23 May, 1934.—Read 14 June, 1934.]

1. *Introduction.*

In Dr. Goldstein's paper "On the vortex theory of screw propellers"*, the following boundary problem is solved.

Let r, θ be the polar coordinates of a point in the plane. The single-valued and continuous function of position ϕ has to be found, which is to satisfy the differential equation

$$\left(r\frac{\partial}{\partial r}\right)^2\phi+(1+r^2)\frac{\partial^2\phi}{\partial\theta^2}=0$$

and the boundary conditions

$$\frac{\partial\phi}{\partial\theta}=-\frac{r^2}{1+r^2} \quad \text{for} \quad \theta=0, \text{ or } \pi, \text{ and } 0\leqslant r\leqslant r_0,$$

$$\phi=0 \quad \text{for} \quad \theta=0, \text{ or } \pi, \text{ and } r>r_0;$$

the derivatives $\partial\phi/\partial r$ and $\partial\phi/\partial\theta$ must vanish when r is infinite.

Dr. Goldstein finds the solution, for $0\leqslant\theta\leqslant\pi$ and $0\leqslant r\leqslant r_0$, in the form of a trigonometrical expansion

$$\phi=\sum_{m=0}^{\infty}\left[\frac{4}{\pi}\frac{T_{1,2m+1}(\sqrt{2m+1}\,r)}{(2m+1)^2}+a_m I_{2m+1}(\sqrt{2m+1}\,r)\right]\cos(2m+1)\theta,$$

---

* *Proc. Royal Soc.* (A), 123 (1929), 440–465.

where the I_{2m+1} are Bessel functions of imaginary argument, and $T_{1,\nu}(z)$ satisfies the differential equation

$$z^2\frac{d^2T}{dz^2}+z\frac{dT}{dz}-(\nu^2+z^2)\,T=-z^2,$$

which is the Lommel equation for imaginary argument*.

In order to have a function which has no singularity at $z=0$, Goldstein puts (for $\nu=2m+1$)

$$T_{1,\nu}(z)=\frac{\nu\pi}{2\sin\frac{1}{2}\nu\pi}\,I_\nu(z)-t_{1,\nu}(z),$$

where

$$t_{1,\nu}(z)=\frac{z^2}{2^2-\nu^2}+\frac{z^4}{(2^2-\nu^2)(4^2-\nu^2)}+\frac{z^6}{(2^2-\nu^2)(4^2-\nu^2)(6^2-\nu^2)}+\cdots.$$

The function $T_{1,\nu}(z)$ differs from the Lommel function $S_{1,\nu}(iz)$, defined in Watson's *Bessel functions*, by the term $\nu e^{\frac{1}{2}\nu\pi i}K_\nu(z)$, which is infinite for $z=0$.

In the present paper I give the integral representation of the function $T_{1,2m+1}(z)$ and of the more general function $T_{\mu,\nu}(z)$ which differs from the general Lommel function $S_{\mu,\nu}(iz)$ by the term in $K_\nu(z)$.

With the help of this expression I find the sum of Goldstein's expansion

$$\sum_{m=1}^{\infty}\frac{T_{1,2m+1}(\overline{2m+1}\,r)}{(2m+1)^2}\cos(2m+1)\theta$$

in finite form.

I give further the analogous representations of the Lommel functions $S_{\mu,\nu}(z)$ and $S_{\mu,\nu}(iz)$ by contour integrals. All these representations are valid only when $\nu-\mu$ is an even positive integer. The case $\nu=\mu$ is considered separately.

2. *Integral representations of the Lommel functions of imaginary argument.*

2.1. *The function* $t_{\mu,\nu}(z)$. Let us consider first the Lommel equation for imaginary argument:

$$z^2\frac{d^2T}{dz^2}+z\frac{dT}{dz}-(\nu^2+z^2)\,T=z^{\mu+1}. \tag{1}$$

* See G. N. Watson, *A treatise on the theory of Bessel functions* (1922), 347, § 10.71.

When $\nu-\mu$ is an even positive integer, this equation has a solution in series proceeding in ascending powers of z, viz.

$$t_{\mu,\nu}(z) = z^{\mu+1}\left[\frac{1}{(\mu+1)^2-\nu^2} + \frac{z^2}{[(\mu+1)^2-\nu^2][(\mu+3)^2-\nu^2]} + \cdots\right]. \quad (2)$$

Hence the general solution is of the form

$$T(z) = t_{\mu,\nu}(z) + aI_\nu(z) + bK_\nu(z),$$

where $I_\nu(z)$ and $K_\nu(z)$ denote the Bessel functions of imaginary argument of the first and the second kind respectively and a and b are constants.

We shall examine whether the definite integral

$$t(z) = z^{\mu+1}\int_0^1 \cosh(zt)\, W(t)\, dt. \quad (3)$$

can be the solution of (1) by suitable choice of the function $W(t)$.

The result of operating on the integral with Bessel's differential operator

$$\bar{\nabla}_\nu \equiv z^2\frac{d^2}{dz^2} + z\frac{d}{dz} - (\nu^2+z^2) \quad (4)$$

is

$$z^{\mu+1}[(2\mu+1)\, W(1)\cosh z - W'(0)]$$

$$-z^{\mu+1}\int_0^1 \cosh(zt)\left\{\frac{d^2}{dt^2}[W(t)(1-t^2)] + (2\mu+3)\frac{d}{dt}[tW(t)] + [\nu^2-(\mu+1)^2]\, W(t)\right\} dt$$

after a partial integration. Thus, if the function $W(t)$ is chosen so that

$$(1-t^2)\frac{d^2W}{dt^2} + (2\mu-1)\, t\frac{dW}{dt} + (\nu^2-\mu^2)\, W = 0 \quad (5)$$

and

$$W(1) = 0, \quad (6)$$

$$W'(0) = -1, \quad (7)$$

we obtain the solution of (1).

The equation (5) and the conditions (6) and (7) are satisfied by the function

$$W_{\mu,\nu}(t) = -(1-t^2)^{\mu+\frac{1}{2}}\frac{C^{\mu+1}_{\nu-\mu-1}(t)}{[C^{\mu+1}_{\nu-\mu-1}(t)]'_{t=0}}, \quad (8)$$

where $C_n^k(t)$ denotes Gegenbauer's function*, which for integral values of n is defined to be the coefficient of h^n in the expansion of $(1-2ht+h^2)^{-k}$ in ascending powers of h, *i.e.*

$$(1-2ht+h^2)^{-k} = \sum_{n=0}^{\infty} C_n^k(t)\, h^n.$$

When $\nu-\mu-1$ is an odd positive integer†, we have

$$[C^{\mu+1}_{\nu-\mu-1}(t)]'_{t=0} = 2(-1)^{\frac{1}{2}(\nu-\mu)+1} \frac{\Gamma\{\frac{1}{2}(\nu+\mu)+1\}}{\Gamma(\mu+1)\,\Gamma\{\frac{1}{2}(\nu-\mu)\}},$$

and thus we get

$$W_{\mu,\nu}(t) = (-1)^{\frac{1}{2}(\nu-\mu)} \frac{\Gamma(\mu+1)\,\Gamma\{\frac{1}{2}(\nu-\mu)\}}{2\Gamma\{\frac{1}{2}(\nu+\mu)+1\}} (1-t^2)^{\mu+\frac{1}{2}} C^{\mu+1}_{\nu-\mu-1}(t). \tag{9}$$

Hence the integral (3), where the function $W(t)$ is defined by the formula (9), is a solution of the Lommel differential equation (1).

The integral involved is evidently finite for $z=0$, so that this solution is of the form $O(z^{\mu+1})$ for $z=0$. But, integrating both sides of the equation (5), we easily obtain, by partial integrations,

$$\left.\frac{t(z)}{z^{\mu+1}}\right|_{z\to 0} = \int_0^1 W_{\mu,\nu}(t)\,dt = \frac{1}{(\mu+1)^2-\nu^2}.$$

Hence we get, by means of (2),

$$t_{\mu,\nu}(z) = z^{\mu+1} \int_0^1 \cosh(zt)\, W_{\mu,\nu}(t)\,dt. \tag{10}$$

2.2. *The function* $\mathbf{T}_{\mu,\nu}(z)$. Consider the well-known formula

$$I_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})} \int_0^1 \cosh(zt)(1-t^2)^{\nu-\frac{1}{2}}\,dt. \tag{1}$$

Suppose that $\mu \geqslant 0$ and let $\nu-\mu-1$ be an odd positive integer. On integrating $\nu-\mu-1$ times by parts and observing that any derivative of the function $(1-t^2)^{\nu-\frac{1}{2}}$, the order of which is an odd integer, vanishes for $t=0$, we get

$$I_\nu(z) = -\frac{z^{\mu+1}}{2^{\nu-1}\,\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})} \int_0^1 \sinh(zt)\, \frac{d^{\nu-\mu-1}}{dt^{\nu-\mu-1}}\,[(1-t^2)^{\nu-\frac{1}{2}}]\,dt. \tag{2}$$

---

* See, *e.g.*, E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 329, § 15.8.

† When $\nu-\mu-1$ is an even positive integer, $[C^{\mu+1}_{\nu-\mu-1}(t)]'_{t=0} = 0$.

Using the formula*

$$(1-x^2)^{k-\frac{1}{2}} C_n^k(x) = \frac{(-2)^n \Gamma(n+k)\Gamma(n+2k)}{\Gamma(n+1)\Gamma(k)\Gamma(2n+2k)} \frac{d^n}{dx^n}[(1-x^2)^{n+k-\frac{1}{2}}], \quad (3)$$

which is valid when n is an integer, we obtain now, from § 2.1 (9),

$$W_{\mu,\nu}(t) = (-1)^{\frac{1}{2}(\nu-\mu)-1} \frac{\Gamma(\nu)\Gamma(\nu+\mu+1)}{2^{\mu-\nu+2}\Gamma(\nu-\mu)\Gamma(2\nu)} \frac{\Gamma\{\frac{1}{2}(\nu-\mu)\}}{\Gamma\{\frac{1}{2}(\nu+\mu)+1\}} \frac{d^{\nu-\mu-1}}{dt^{\nu-\mu-1}}[(1-t^2)^{\nu-\frac{1}{2}}]$$

and, by the well-known duplication formula

$$\Gamma(2u) = \frac{2^{2u-1}}{\sqrt{\pi}} \Gamma(u)\Gamma(u+\tfrac{1}{2}), \quad (4)$$

we finally get

$$W_{\mu,\nu}(t) = (-1)^{\frac{1}{2}(\nu-\mu)-1} \sqrt{\pi} \frac{\Gamma\{\frac{1}{2}(\nu+\mu+1)\}}{2^{\nu-\mu}\Gamma(\nu+\frac{1}{2})\Gamma\{\frac{1}{2}(\nu-\mu+1)\}} \frac{d^{\nu-\mu-1}}{dt^{\nu-\mu-1}}[(1-t^2)^{\nu-\frac{1}{2}}]. \quad (5)$$

By this result we obtain, from (5),

$$2^{\mu-1}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right) I_\nu(z) = z^{\mu+1}\int_0^1 \sinh(zt)\, W_{\mu,\nu}(t)\, dt \quad (6)$$

after using the known formula

$$\Gamma(u)\Gamma(1-u) = \frac{\pi}{\sin \pi u}. \quad (7)$$

On comparing the equations (6) and § 2.1 (10) we are naturally led to define the function

$$\mathbf{T}_{\mu,\nu}(z) = t_{\mu,\nu}(z) - 2^{\mu-1}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right) I_\nu(z), \quad (8)$$

which is the solution of the Lommel equation § 2.1 (1).

Thus we have

$$\mathbf{T}_{\mu,\nu}(z) = z^{\mu+1}\int_0^1 e^{-zt} W_{\mu,\nu}(t)\, dt. \quad (9)$$

2.3. *The sum of Goldstein's expansion.* In the particular case $\mu = 1$, $\nu = 2m+1$, the function $\mathbf{T}_{\mu,\nu}(z)$ defined above, § 2.2 (8), differs only in sign from the function $T_{1,2m+1}(z)$ used by Goldstein.

* *Modern analysis*, 329, 15.8.

Thus we get formally, by § 2.2 (9) and § 2.1 (9) and by the definition of Gegenbauer's coefficients $C_n^k(t)$,

$$\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\mathbf{T}_{1,2m+1}(\overline{2m+1}\,r)}{(2m+1)^2}\cos(2m+1)\theta$$

$$=-\frac{r^2}{\pi}R\left\{\int_0^{\frac{1}{2}\pi}\left[-ih\sin 2\phi+i\left(h+\frac{1}{h}\right)\log\frac{1-h^2e^{2i\phi}}{1-h^2e^{-2i\phi}}\right]\cos\phi\,d\phi\right\}, \quad (1)$$

where $$h=e^{i\theta-r\sin\phi}, \quad (2)$$

and the symbol $R(z)$ denotes the real part of the complex variable z.

To justify the term-by-term integration after expanding the integrand in series of ascending powers of h, use the expression of $W_{1,2m+1}(\sin\phi)$:

$$W_{1,2m+1}(\sin\phi)=-\tfrac{1}{4}\left[\frac{\sin 2m\phi}{m}+\frac{\sin 2(m+1)\phi}{m+1}\right], \quad (3)$$

which can be easily verified.

3. *The Lommel functions $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$.*

3.1. *The function $s_{\mu,\nu}(z)$.* When $\nu-\mu-1$ is an odd positive integer, the Lommel equation

$$z^2\frac{d^2S}{dz^2}+z\frac{dS}{dz}-(\nu^2-z^2)S=z^{\mu+1} \quad (1)$$

is satisfied by the function

$$s_{\mu,\nu}(z)=z^{\mu+1}\left\{\frac{1}{(\mu+1)^2-\nu^2}-\frac{z^2}{[(\mu+1)^2-\nu^2][(\mu+3)^2-\nu^2]}+\ldots\right\}. \quad (2)$$

As in the case of the function $t_{\mu,\nu}(z)$ we endeavour to represent this solution by a definite integral.

Thus we write

$$S=z^{\mu+1}\int_0^1\cos(zt)\,W(t)\,dt, \quad (3)$$

and, substituting on the left-hand side of the equation (1), we obtain the same conditions as in § 2.1, viz. §§ 2.1 (5), 2.1 (6), and 2.1 (7). In the manner of § 2.1 we prove that the integral (3) represents the function $s_{\mu,\nu}(z)$. Thus we have

$$s_{\mu,\nu}(z)=z^{\mu+1}\int_0^1\cos(zt)\,W_{\mu,\nu}(t)\,dt. \quad (4)$$

3.2. *The function* $\mathbf{S}_{\mu,\nu}(z)$. The other standard solution of Lommel equation is of the form*

$$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + 2^{\mu-1}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right) \times\left[\sin\frac{\pi(\mu-\nu)}{2}\,J_\nu(z) - \cos\frac{\pi(\mu-\nu)}{2}\,Y_\nu(z)\right], \quad (1)$$

where $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions of the first and the second kind respectively.

On introducing Hankel's function of the third kind,

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z), \quad (2)$$

we transform the right-hand side of (1) into

$$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + 2^{\mu-1}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right) \times\left[e^{\frac{1}{2}\pi i(\mu-\nu-1)}\,J_\nu(z) + i\cos\frac{\pi(\mu-\nu)}{2}\,H_\nu^{(1)}(z)\right]. \quad (3)$$

We define the function $\mathbf{S}_{\mu,\nu}(z)$ by

$$\mathbf{S}_{\mu,\nu}(z) = s_{\mu,\nu}(z) + 2^{\mu-1}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)e^{\frac{1}{2}\pi i(\mu-\nu-1)}\,J_\nu(z). \quad (4)$$

The function $\mathbf{S}_{\mu,\nu}(z)$ is an analogue of the function $\mathbf{T}_{\mu,\nu}(z)$ of § 2.2 and has an analogous integral representation.

For, integrate $\nu-\mu-1$ times by parts the well-known integral representing $J_\nu(z)$, viz.

$$J_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}\int_0^1 \cos(zt)(1-t^2)^{\nu-\frac{1}{2}}\,dt; \quad (5)$$

we get

$$J_\nu(z) = \frac{(-1)^{\frac{1}{2}(\nu-\mu)}z^{\mu+1}}{2^{\nu-1}\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}\int_0^1 \sin(zt)\,\frac{d^{\nu-\mu-1}}{dt^{\nu-\mu-1}}\left[(1-t^2)^{\nu-\frac{1}{2}}\right]dt,$$

and, by means of the formulae §§ 2.2 (5), 3.1 (3), and 3.2 (4), we obtain

$$\mathbf{S}_{\mu,\nu}(z) = z^{\mu+1}\int_0^1 e^{izt}\,W_{\mu,\nu}(t)\,dt. \quad (6)$$

3.3. *The function* $S_{\mu,\nu}(z)$. The results of §§ 2.1, 2.2, 3.1, and 3.2 are obviously valid when z is a complex variable. But in order to avoid

* G. N. Watson, *Bessel functions*, 347, § 10.71 (3).

confusion, it is necessary to make definite the many-valued function $W_{\mu,\nu}(t)$. The formulae §§ 2.2 (5), 2.1 (7), and 2.1 (8) show that the phase of $W_{\mu,\nu}(t)$ is determined by the phase of $1-t^2$ and this last must be put equal to zero at $t=0$.

With this restriction the formulae §§ 2.1 (10), 2.2 (9), 3.1 (4), and 3.2 (7) remain true for any complex value of z, and also for any finite path of integration, which goes from 0 to $+1$ and is reconcilable with the simple straight path.

By suitable choice of the path of integration the function $S_{\mu,\nu}(z)$ can be represented also by a contour integral. For that purpose consider the integral*

$$H_\nu^{(1)}(z) = \frac{\Gamma(\frac{1}{2}-\nu)(\frac{1}{2}z)^\nu}{\pi i\,\Gamma(\frac{1}{2})} \int_{1+i\infty}^{(1+)} e^{izt}(t^2-1)^{\nu-\frac{1}{2}}\,dt, \tag{1}$$

where the contour of integration is that in Fig. 1. The phase of t^2-1 at the point B is π and $|\arg z| < \frac{1}{2}\pi$.

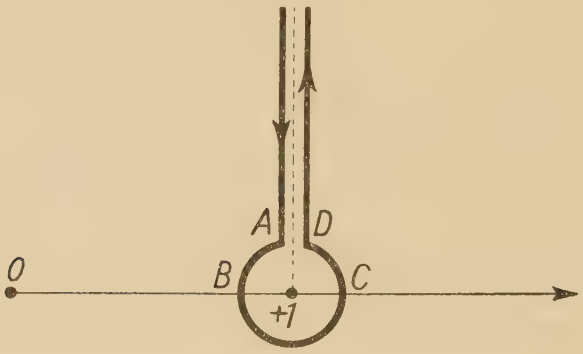


Fig. 1.

If the phase of $1-t^2$ is supposed to be 0 at the point B, the function $e^{\pi i(\nu-\frac{1}{2})}(1-t^2)^{\nu-\frac{1}{2}}$ has the same values everywhere as the function $(t^2-1)^{\nu-\frac{1}{2}}$ in (1) with the phase of t^2-1 stated above. Thus the integral (1) can be written

$$H_\nu^{(1)}(z) = \frac{\Gamma(\frac{1}{2}-\nu)(\frac{1}{2}z)^\nu\, e^{\pi i(\nu-\frac{1}{2})}}{\pi i\,\Gamma(\frac{1}{2})} \int_{1+i\infty}^{(1+)} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt.$$

On replacing this contour integral by the straight-line integral from point D to $1+i\infty$, we have

$$H_\nu^{(1)}(z) = \frac{\Gamma(\frac{1}{2}-\nu)(\frac{1}{2}z)^\nu}{\pi i\,\Gamma(\frac{1}{2})}\, e^{\pi i(\nu-\frac{1}{2})}[1-e^{-2\pi i(\nu-\frac{1}{2})}] \int_{(D)}^{(1+i\infty)} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt,$$

* G. N. Watson, *Bessel functions*, 166, § 6.11 (4).

where the value of $(1-t^2)^{\nu-\frac{1}{2}}$ at point D is obtained from the value at B by analytic continuation along the arc BCD.

We then have, by the formula § 2.2 (7),

$$H_\nu^{(1)}(z) = -\frac{2(\frac{1}{2}z)^\nu}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}\int_{(D)}^{(1+i\infty)} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt. \tag{2}$$

Suppose that $\mu \geqslant 0$ and let $\nu-\mu-1$ be an odd positive integer. By partial integration $\nu-\mu-1$ times repeated we get

$$H_\nu^{(1)}(z) = -\frac{z^{\mu+1}i^{\nu-\mu-1}}{2^{\nu-1}\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}\int_{(D)}^{(1+i\infty)} e^{izt}\frac{d^{\nu-\mu-1}}{dt^{\nu-\mu-1}}\left[(1-t^2)^{\nu-\frac{1}{2}}\right]dt.$$

Now, by means of §§ 2.2 (5) and 2.2 (7),

$$2^{\mu-1}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right) i\cos\frac{\nu-\mu}{2}\pi\, H_\nu^{(1)}(z) = z^{\mu+1}\int_{(D)}^{(1+i\infty)} e^{izt}W_{\mu,\nu}(t)\,dt, \tag{3}$$

and, finally, by the formulae §§ 3.2 (3), 3.2 (4), and 3.2 (6), we get

$$S_{\mu,\nu}(z) = z^{\mu+1}\int_{(0,\,+1,\,1+i\infty)} e^{izt}W_{\mu,\nu}(t)\,dt, \tag{4}$$

where the contour of integration is that in Fig. 2, and the phase of $1-t^2$ is supposed to be 0 at point B. The formula is valid when $|\arg z| < \frac{1}{2}\pi$.

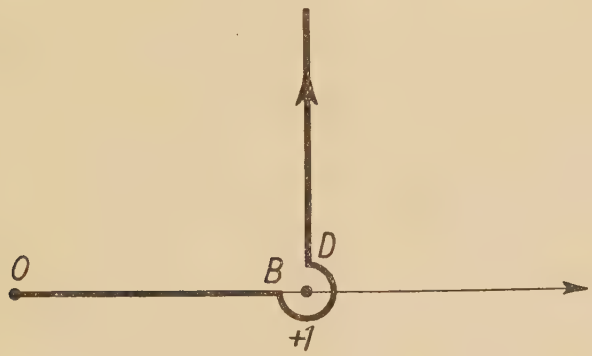


Fig. 2.

An analogous but more general formula can be obtained to represent the function $S_{\mu,\nu}(z)$ for other complex values of z by using the modified contour integral*

$$H_\nu^{(1)}(z) = \frac{\Gamma(\frac{1}{2}-\nu)(\frac{1}{2}z)^\nu}{\pi i\,\Gamma(\frac{1}{2})}\int_{\infty\exp(i\omega)}^{(1+)} e^{izt}(t^2-1)^{\nu-\frac{1}{2}}\,dt \tag{5}$$

* G. N. Watson, *Bessel functions*, 167, § 6.12 (6).

if $-\pi < \omega < \pi$ and $-\omega < \arg z < \pi - \omega$. The phase of t^2-1 is chosen as in (1) and the path of integration as in Fig. 3.

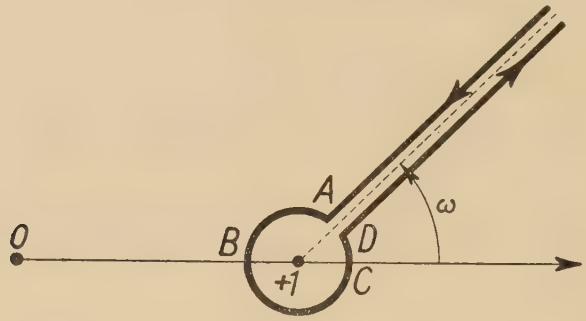


Fig. 3.

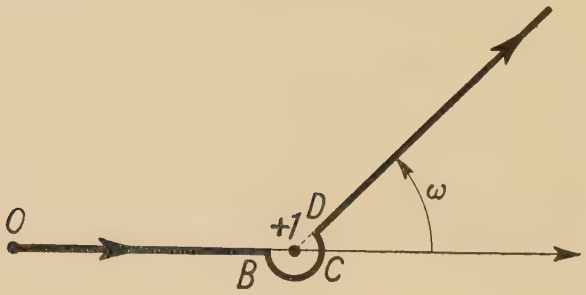


Fig. 4.

By transformations as above we get

$$2^{\mu-1}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right) i \cos\frac{\mu-\nu}{2}\pi H_\nu^{(1)}(z) = z^{\mu+1}\int_{(D)}^{\infty \exp(i\omega)} e^{izt} W_{\mu,\nu}(t)\,dt, \tag{6}$$

and, by §§ 3.2 (3), 3.2 (4), and 3.2 (6),

$$S_{\mu,\nu}(z) = z^{\mu+1}\int_{[0,\,+1,\,\infty\exp(i\omega)]} e^{izt} W_{\mu,\nu}(t)\,dt, \tag{7}$$

where $-\pi < \omega < \pi$, $-\omega < \arg z < \pi - \omega$, and the phase of $1-t^2$ is 0 at B. The path of integration is shown in Fig. 4.

3.4. *The functions* $s_{\mu,\nu}(z)$, $\mathbf{S}_{\mu,\nu}(z)$, *and* $S_{\mu,\nu}(z)$ *for imaginary argument.* Putting $z = ue^{\frac{1}{2}\pi i}$ in the formula § 3.1 (2), we obtain

$$s_{\mu,\nu}(ue^{\frac{1}{2}\pi i}) = e^{\frac{1}{2}\pi i(\mu+1)} t_{\mu,\nu}(u), \tag{1}$$

and substituting in § 3 . 2 (4), and using the known formula*

$$J_\nu(ue^{\frac{1}{2}\pi i}) = e^{\frac{1}{2}\nu\pi i} I_\nu(z), \tag{2}$$

we get

$$\mathbf{S}_{\mu,\nu}(ue^{\frac{1}{2}\pi i}) = e^{\frac{1}{2}\pi i(\mu+1)} \mathbf{T}_{\mu,\nu}(u). \tag{3}$$

We naturally define the function $T_{\mu,\nu}(u)$ by the equation

$$S_{\mu,\nu}(ue^{\frac{1}{2}\pi i}) = e^{\frac{1}{2}\pi i(\mu+1)} T_{\mu,\nu}(u). \tag{4}$$

Thus we have

$$T_{\mu,\nu}(u) = t_{\mu,\nu}(u) - 2^{\mu-1}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right) \times\left[I_\nu(u)+\frac{2}{\pi}\cos\frac{\mu-\nu}{2}\pi e^{-\frac{1}{2}\pi i(\mu+\nu-1)}K_\nu(u)\right] \tag{5}$$

by means of the formula†

$$iH_\nu^{(1)}(iu) = \frac{2}{\pi} e^{-\frac{1}{2}\nu\pi i} K_\nu(u). \tag{6}$$

The integral representation of the function $T_{\mu,\nu}(u)$ can be obtained from § 3 . 3 (7). In particular, when $|\arg u| < \frac{1}{2}\pi$, by putting $z = ue^{\frac{1}{2}\pi i}$ and $\omega = 0$, we get from § 3 . 3 (7)

$$T_{\mu,\nu}(u) = u^{\mu+1}\int_{(0,\,+1,\,\infty)} e^{-ut} W_{\mu,\nu}(t)\,dt, \tag{7}$$

where the contour of integration is that in Fig. 4 in the case of $\omega = 0$.

4. *The case* $\nu = \mu$.

All the results of the preceding sections are valid only in the case in which $\nu-\mu$ is an even *positive* integer. We next proceed to examine the case $\nu-\mu = 0$.

The integral $z^\nu\int_a^b e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt$ satisfies Bessel's equation when the function $e^{izt}(1-t^2)^{\nu-\frac{1}{2}}$ vanishes at $t = a$ and $t = b$. In the general case the result of operating on this integral with Bessel's differential operator

$$\nabla_\nu \equiv z^2\frac{d^2}{dz^2} + z\frac{d}{dz} - (\nu^2 - z^2) \tag{1}$$

is

$$-iz^{\nu+1}\left[e^{izt}(1-t^2)^{\nu+\frac{1}{2}}\right]_a^b.$$

* G. N. Watson, *Bessel functions*, § 3.7 (2) and following.

† *Loc. cit.*, § 3.7 (8).

Thus, if we put $a = 0$ and the above function vanishes at b, the integral

$$S(z) = -iz^{\nu} \int_0^b e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\, dt \tag{2}$$

satisfies the equation

$$z^2 \frac{d^2S}{dz^2} + z\frac{dS}{dz} - (\nu^2 - z^2)\, S = z^{\nu+1}, \tag{3}$$

which is the Lommel equation in the case $\nu = \mu$.

Consider first the integral

$$S(z) = -iz^{\nu} \int_0^1 e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\, dt;$$

we then have

$$S(z) = z^{\nu} \int_0^1 \sin(zt)(1-t^2)^{\nu-\frac{1}{2}}\, dt - iz^{\nu} \int_0^1 \cos(zt)(1-t^2)^{\nu-\frac{1}{2}}\, dt.$$

The former of these integrals represents the function $s_{\nu,\nu}(z)$. For, denoting this integral by $S_1(z)$, we have

$$\frac{S_1(z)}{z^{\nu+1}}\bigg|_{z\to 0} = \int_0^1 t(1-t^2)^{\nu-\frac{1}{2}}\, dt = \frac{1}{2\nu+1}.$$

Since the only solution of (3) which has this property is $s_{\nu,\nu}(z)$, we can thus write

$$s_{\nu,\nu}(z) = z^{\nu} \int_0^1 \sin(zt)(1-t^2)^{\nu-\frac{1}{2}}\, dt. \tag{4}$$

The second integral involved is equal to

$$-i\, 2^{\nu-1}\, \Gamma(\nu+\tfrac{1}{2})\, \Gamma(\tfrac{1}{2})\, J_{\nu}(z);$$

thus, by the formula §3.2 (4),

$$\mathbf{S}_{\nu,\nu}(z) = -iz^{\nu} \int_0^1 e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\, dt. \tag{5}$$

Analogous formulae can be obtained to represent the functions $t_{\nu,\nu}(z)$, $\mathbf{T}_{\nu,\nu}(z)$, $S_{\nu,\nu}(z)$, and $T_{\nu,\nu}(z)$.

Aerodynamical Institute,
Warsaw.

SINGULAR MODULI (3)†

By G. N. WATSON.

[Read 15 June, 1933.—Received 14 August, 1934.]

1. *Introduction.*

The study of singular moduli dates from the remark made by Abel‡ that they satisfy algebraic equations which are soluble by radicals. Investigations prompted by this remark are divisible into two distinct categories; on the one hand we have the theoretical researches, initiated by Kronecker and Dedekind, of which an admirable account is contained in the third volume of Weber's *Algebra*; on the other hand we have the calculations and computations which have been required in the practical problem of determining the values of particular singular moduli.

There are three investigations of the latter type which it is relevant to mention here. A paper by Weber§ contains the values, in the form of surds, of the sixty-five singular moduli corresponding to the sixty-five negative determinants of quadratic forms which (according to the classification of Gauss) have one class in each genus, the number of genera being I, II, IV, VIII, or XVI; a paper by Berwick‖ contains the values of the fifty-four Dedekind invariants $j(\omega)$ which are algebraic integers of the second and third degrees; and, lastly, the note-books of Ramanujan contain evidence that he had embarked on the task of calculating systematically the singular moduli corresponding to negative determinants

† The papers numbered 1 and 2 in this series (which dealt with results enunciated by Ramanujan) were published in the *Quart. J. of Math.* (Oxford series), 3 (1932), 81–98 and 189–212.

After I communicated the present paper to the Society in June, 1933, I decided to amplify it by the insertion of the solutions by radicals of all relevant non-trivial Abelian equations of degrees 5, 7, 9, and 15. The construction of these solutions was a task which I should not have ventured to attempt if I had not been in possession of the mass of computations which I had carried out in the course of constructing the equations themselves.

‡ N. H. Abel, *Journal für Math.*, 3 (1828), 184.

§ H. Weber, *Math. Annalen*, 33 (1889), 390–410.

‖ W. E. H. Berwick, *Proc. London Math. Soc.* (2), 28 (1928), 53–69.

of quadratic forms which (according to the classification of Gauss) have two classes in each genus†.

My reason for mentioning these investigations is that my methods of obtaining the results contained in the present paper are of the same general character; there are, of course, a number of other papers (such as those by Russell and Greenhill) devoted to the determination of the values of singular moduli by means of modular equations. The complexity of all modular equations apart from a few of low order is an effective limitation on the scope of this method; the methods (based on the theory of quadratic forms) which are used in the present paper are much more powerful.

I here take those negative determinants, eighty in all, which (according to the classification of Gauss) have class-numbers 1, 3, 5, 7, 9, 11, 13, 15, with generic classification I in each case, and I give the equations satisfied by the corresponding singular moduli together with data sufficient to enable the reader to write down immediately the solutions of all of these equations with the exception of the four which are of degree 11 and the six which are of degree 13.

n	h	n	h	n	h	n	h
1	I.1	131	I.15	367	I.9	823	I.9
2	I.1	139	I.9	379	I.9	883	I.9
3	I.1	151	I.7	439	I.15	907	I.9
4	I.1	163	I.3	443	I.15	947	I.15
7	I.1	167	I.11	463	I.7	967	I.11
11	I.3	179	I.15	487	I.7	1051	I.15
19	I.3	191	I.13	499	I.9	1087	I.9
23	I.3	199	I.9	523	I.15	1123	I.15
27	I.3	211	I.9	547	I.9	1303	I.11
31	I.3	223	I.7	571	I.15	1327	I.15
43	I.3	227	I.15	607	I.13	1423	I.9
47	I.5	239	I.15	619	I.15	1567	I.15
59	I.9	243	I.9	631	I.13	1723	I.15
67	I.3	263	I.13	643	I.9	1747	I.15
71	I.7	271	I.11	683	I.15	1867	I.15
79	I.5	283	I.9	691	I.15	2143	I.13
83	I.9	307	I.9	727	I.13	2203	I.15
103	I.5	331	I.9	739	I.15	2347	I.15
107	I.9	343	I.7	751	I.15	2647	I.15
127	I.5	347	I.15	787	I.15	2683	I.15

† The number of such singular moduli appears to be 231; I hope to publish their values in a subsequent paper.

The appended Table (constructed by means of the Tables of class-numbers computed by Gauss†) gives the eighty values of n for which the negative determinant $-n$ has one of the class-numbers (h) just specified.

It is to be observed that, with some trivial exceptions under the heading G.I.1, the values of n are all of the forms $8m-1$ and $8m+3$, and that the class-numbers which correspond to the latter are multiples of 3.

The Tables constructed by Gauss give class-numbers for values of n up to 3000 (with some isolated centuries beyond), and it is, of course, possible that there exist values of n beyond 3000 with a class-number not exceeding 15 and generic classification I. A search which I made up to 5000 proved to be fruitless, though my mind would have been more at ease if I had been able to examine the Table constructed by Kronecker‡, which extends up to 10000 and which apparently has never been published.

Some of the results which I give have been obtained by earlier writers, as will be described in detail later. The omission of any of these results would have detracted from the completeness of the present paper; in this connection I remark that, while it is a matter of elementary arithmetic to obtain the values of seventeen of Berwick's invariants from results given below, the converse process is much more tedious.

2. *Definitions and notation.*

For the most part I use the ordinary notation of elliptic functions, amplified by the notation used by Weber§.

Let $$q = e^{i\pi\tau} \quad [I(\tau) > 0, \ |q| < 1],$$

$$f(\tau) = q^{-\frac{1}{24}} \prod_{m=1}^{\infty} (1+q^{2m-1}), \quad f_1(\tau) = q^{-\frac{1}{24}} \prod_{m=1}^{\infty} (1-q^{2m-1}),$$

$$f_2(\tau) = 2^{\frac{1}{2}} q^{\frac{1}{12}} \prod_{m=1}^{\infty} (1+q^{2m}) = 2^{\frac{1}{2}} q^{\frac{1}{12}} \prod_{m=1}^{\infty} (1-q^{4m-2})^{-1}.$$

These functions are connected by the relations

$$f^8(\tau) = f_1{}^8(\tau) + f_2{}^8(\tau),$$

$$f(\tau) f_1(\tau) f_2(\tau) = \sqrt{2}.$$

† C. F. Gauss, *Werke*, 2 (1863), 449–476.

‡ See L. E. Dickson, *History of the theory of numbers*, 3 (1923), 109.

§ H. Weber, *Lehrbuch der Algebra*, 3 (1908), *passim*.

If k and k' are the moduli of elliptic functions for which τ is the quotient of the periods, then

$$f(\tau)=\left(\frac{4}{kk'}\right)^{\frac{1}{12}}, \quad f_1(\tau)=\left(\frac{4k'^2}{k}\right)^{\frac{1}{12}}, \quad f_2(\tau)=\left(\frac{4k^2}{k'}\right)^{\frac{1}{12}}.$$

When n is of the form $8m+3$ and is not a multiple of 3, the simplest class-invariant is† $f(\sqrt{-n})$, for which I shall frequently write simply f_n or even f.

When n is of the form $8m-1$ and is not a multiple of 3, the simplest class-invariant is $2^{-\frac{1}{4}}f(\sqrt{-n})$. This class-invariant is needed so frequently in the sequel that it is desirable to have a special notation for it, and I therefore write

$$\frac{f(\sqrt{-n})}{\sqrt{2}}=F_n=F.$$

In this paper it is hardly ever necessary to use Ramanujan's class-invariants

$$G_n=2^{-\frac{1}{4}}f(\sqrt{-n}) \quad (n \text{ odd}),$$

$$g_n=2^{-\frac{1}{4}}f_1(\sqrt{-n}) \quad (n \text{ even}).$$

In terms of k, Dedekind's invariant $j(\tau)$ is defined by the formula

$$j(\tau)=256\,\frac{(1-k^2+k^4)^3}{k^4(1-k^2)^2},$$

so that $j(\tau)$ is equal to each of the three expressions

$$\frac{\{f^{24}(\tau)-16\}^3}{f^{24}(\tau)}, \quad \frac{\{f_1^{24}(\tau)+16\}^3}{f_1^{24}(\tau)}, \quad \frac{\{f_2^{24}(\tau)+16\}^3}{f_2^{24}(\tau)},$$

and therefore $f^{24}(\tau)$, $-f_1^{24}(\tau)$, and $-f_2^{24}(\tau)$ are the roots of the equation

$$(x-16)^3-xj(\tau)=0.$$

It is sometimes convenient to work with Weber's function $\gamma_2(\tau)$ defined as $\sqrt[3]{j(\tau)}$; we then have $f^8(\tau)$, $-f_1^8(\tau)$, $-f_2^8(\tau)$ for the roots of the equation

$$x^3-x\gamma_2(\tau)-16=0.$$

We also have‡

$$-\gamma_2\left(\frac{-3+\tau}{2}\right)=\frac{f^{24}(\tau)-256}{f^{16}(\tau)}.$$

† H. Weber, *ibid.*, 473.

‡ H. Weber, *Lehrbuch der Algebra*, 3 (1908), 179, 457–460.

It is convenient to use† the Weierstrassian symbol * to indicate the absence of terms in an equation.

My thanks are due to Prof. Berwick for the trouble which he took in computing the fundamental cubic equations with discriminants -1087, -1327, -1423, -1567, -2647; these are outside the range of a table which he computed‡ some years ago. In this connection, I mention that the discriminant of a cubic is here understood to mean the product of the squares of the differences of its roots, so that, for instance, the discriminant of the cubic

$$x^3+3Hx+G=0$$

is taken to be $$-27(G^2+4H^3).$$

Throughout the paper I use Δ to denote the discriminant of a cubic, and, to save unnecessary calculations, I usually append it to each cubic for which its value seems to be of interest.

3. *The equations satisfied by class-invariants.*

I now explain the method by which the equations contained in this paper were constructed, taking the case in which n is of the form $8m-1$ and using the value $n=71$ for purposes of numerical illustration. The procedure when n is of the form $8m+3$ is much the same, but the details, where they differ, are slightly simpler in the latter case.

Let h be the class-number of the negative determinant $-n$ and let

$$a_r x^2+b_r xy+c_r y^2 \quad (r=1, 2, \ldots, h)$$

be a "complete set" of quadratic forms§ for which

$$b_r{}^2-4a_r c_r=-n.$$

Let the h values of τ (with positive imaginary parts) which satisfy the equations

$$a_r\tau^2+b_r\tau+c_r=0$$

be called $\tau_1, \tau_2, \ldots, \tau_h$.

Then the equation

$$\prod_{r=1}^{h}[x-j(\tau_r)]=0,$$

† This usage is really as old as Descartes.

‡ G. B. Mathews *Proc. London Math. Soc.* (2), 10 (1912), 137–138.

§ To construct such a set, take $b=\pm1, \pm3, \ldots, [|b|<\sqrt{(\frac{1}{3}n)}]$, and then express $\frac{1}{4}(b^2+n)$ as a product of positive factors a and c in all possible ways with $|b|\leqslant a\leqslant c$. *Cf.* L. E. Dickson, *History of the theory of numbers*, 3 (1923), 6.

when the left-hand side is multiplied out, is an equation in x of degree h with integral coefficients†.

It is evident from this result that the coefficients in the equation

$$\prod_{r=1}^{h} [(x-16)^3-xj(\tau_r)] = 0,$$

that is to say

$$\prod_{r=1}^{h} [\{x-f^{24}(\tau_r)\}\{x+f_1^{24}(\tau_r)\}\{x+f_2^{24}(\tau_r)\}] = 0,$$

are integers.

Observe first that one of the τ_r is a root of the equation

$$\tau^2-\tau+2m = 0,$$

and is therefore equal to

$$\frac{1+i\sqrt{n}}{2}.$$

Weber's formula

$$f(\tau)f_2(\tfrac{1}{2}+\tfrac{1}{2}\tau) = e^{\frac{1}{24}\pi i}\sqrt{2}$$

consequently shows that $x = F_{\frac{1}{2}}^{-24}$ is one of the roots of the equation of degree $3h$.

Again, another of the τ_r is a root of the equation

$$2\tau^2-\tau+m = 0,$$

and is therefore equal to

$$\frac{1+i\sqrt{n}}{4}.$$

Weber's formula

$$f_1(\tau)f_2(\tfrac{1}{2}\tau) = \sqrt{2}$$

consequently shows that the equation of degree $3h$ has a pair of roots, x_1 and x_2 say, connected by the relation

$$x_1x_2 = 2^{12}.$$

If it were not the case that the equation possessed h pairs of roots with the members of each pair connected by this relation, the equation would be reducible to an equation of degree other than h or $2h$. Hence the equation must possess these h pairs of roots‡, and, when these $2h$ roots are

† H. Weber, *Lehrbuch der Algebra*, 3 (1908), 418–423.

‡ In numerical examples it is a trivial matter to detect these h pairs of roots.

discarded, we are left with an equation of degree h, one of whose roots (the only real root) is F_n^{-24}.

This equation, say

$$\prod_{r=1}^{h}(x-\alpha_r)=0,$$

with integral coefficients, now has to be reduced to an equation, of degree h with integral coefficients, whose only real root is F_n^{-1}; this reduction is effected in stages, the successive equations having as their only real roots F_n^{-8}, F_n^{-4}, F_n^{-2}, F^{-1} respectively.

The first stage of the reduction gives the equation

$$\prod_{r=1}^{h}(x-\sqrt[3]{\alpha_r})=0,$$

where one of the $\sqrt[3]{\alpha_r}$ is F_n^{-8} and the others are pairs of conjugate complex numbers selected from the various cube roots in such a way that the equation has integral coefficients; *in particular*,

$$\sum_{r=1}^{h}\sqrt[3]{\alpha_r}$$

must be an integer. This criterion proves to be adequate for the purpose of determining the correct selection of the cube roots in all the examples investigated in this paper. When one term of the sum is F_n^{-8} and the rest are conjugate pairs of complex numbers, the sum may assume $3^{\frac{1}{2}(h-1)}$ values; and the selection of that value which is an integer involves a computation which can be carried out in about three hours when h is as large as 19; for larger values of h, not only would this computation take longer, but working to ten significant figures (as I did) would not always be adequate for determining the correct selection with certainty, and additional criteria would be required. These statements give an indication of the limitations of the method†.

The following stages of the reduction (to obtain in succession equations whose only real roots are F_n^{-4}, F_n^{-2}, and F_n^{-1}) are of the same general character, except that, at each stage, the number of possibilities which are open, from which the correct selection has to be made, is only $2^{\frac{1}{2}(h-1)}$ instead of $3^{\frac{1}{2}(h-1)}$.

It is thus seen that the degree of the equation obtained for F_n is equal to the class-number h. If, when all the coefficients in this equation have

† [*Added July* 26, 1935. Since this was written, I have dealt successfully with $n=479$ and $n=599$; for each of these values of n, $h=25$ and the selection has to be made from $3^{12}=5\,31441$ values. Each of the corresponding computations required twelve hours' work.]

been determined, the process of reduction is reversed, we ultimately obtain two equations with integral coefficients which are satisfied by $\gamma_2(-\frac{3}{2}+\frac{1}{2}i\sqrt{n})$ and $j(-\frac{3}{2}+\frac{1}{2}i\sqrt{n})$ respectively; these equations are also of degree h. The situation is altered when n is of the form $8m+3$; if m is not a multiple of 3, the degree of the equation obtained for f_n is equal to the class-number h, but the reversal of the process of reduction leads to two equations with integral coefficients which are satisfied by $\gamma_2(-\frac{3}{2}+\frac{1}{2}i\sqrt{n})$ and $j(-\frac{3}{2}+\frac{1}{2}i\sqrt{n})$ respectively, the degrees of these equations being $\frac{1}{3}h$ only and not h. In like manner, when n is of the form $24m+3$, the degree of the equation obtained for ${f_n}^3$ is equal to h, while the degree of the equation satisfied by $j(-\frac{3}{2}+\frac{1}{2}i\sqrt{n})$ is again only $\frac{1}{3}h$.

Thus, while f_{19} and F_{23} are both cubic irrationalities, the same is not true of both $\gamma_2(-\frac{3}{2}+\frac{1}{2}i\sqrt{19})$ and $\gamma_2(-\frac{3}{2}+\frac{1}{2}i\sqrt{23})$; the latter is a cubic irrationality, but the former is a rational integer.

I now give an abstract of the calculations in the case $n=71$. The seven quadratic forms are

$$x^2-xy+18y^2 \quad (\tau_1),$$

$$2x^2\mp xy+9y^2 \quad (\tau_2, \tau_3),$$

$$3x^2\mp xy+6y^2 \quad (\tau_4, \tau_5)$$

$$4x^2\mp 3xy+5y^2 \quad (\tau_6, \tau_7).$$

The logarithms (to base 10) of the moduli and the phases (in sexagesimal measure) of ${f^8}(\tau_r)$, ${f_1}^8(\tau_r)$, and ${f_2}^8(\tau_r)$ are given in the following Table:

	Logarithm.	Phase.	
$f^8(\tau_1)$	1·91607 32507	−29° 59′ 57″·0536	(i)
${f_1}^8(\tau_1)$	1·91607 32507	−30° 0′ 2″·9464	(ii)
${f_2}^8(\tau_1)$	−2·62802 65188	+60°	—
$f^8(\tau_2)$, $f^8(\tau_3)$	0·96131 94553	∓14° 34′ 2″·3111	(iii), (iv)
${f_1}^8(\tau_2)$, ${f_1}^8(\tau_3)$	0·95475 37955	∓15° 26′ 0″·6354	—
${f_2}^8(\tau_2)$, ${f_2}^8(\tau_3)$	−0·71195 32682	±30° 0′ 2″·9465	(ii), (i)
$f^8(\tau_4)$, $f^8(\tau_5)$	0·67506 65142	∓7° 14′ 51″·7552	(v), (vi)
${f_1}^8(\tau_4)$, ${f_1}^8(\tau_5)$	0·60205 99913	∓12° 48′ 38″·5893	(vii), (vii)
${f_2}^8(\tau_4)$, ${f_2}^8(\tau_5)$	−0·07300 65229	±20° 3′ 30″·3445	—
$f^8(\tau_6)$, $f^8(\tau_7)$	0·52905 34685	∓7° 14′ 51″·7552	(vi), (v)
${f_1}^8(\tau_6)$, ${f_1}^8(\tau_7)$	0·43226 59868	∓38° 11′ 5″·9337	—
${f_2}^8(\tau_6)$, ${f_2}^8(\tau_7)$	0·24280 05273	±45° 25′ 57″·6889	(iv), (iii)

This Table was constructed directly by the use of the appropriate q-products. The roman numerals in the last column indicate the seven pairs of roots which are discarded in consequence of the 21-ic equation satisfied by F_{71}^{-24} having seven pairs of roots whose products are equal to 2^{12}.

The next step is to determine the septimic equation with integral coefficients whose real root is F_{71}^{-8}. When the process of selecting the proper cube roots of

$$-f_2^{24}(\tau_1),\quad -f_1^{24}(\tau_2),\quad -f_1^{24}(\tau_3),\quad -f_2^{24}(\tau_4),\quad -f_2^{24}(\tau_5),\quad -f_1^{24}(\tau_6),\quad -f_1^{24}(\tau_7)$$

is carried out, it is found that the moduli and phases (save for multiples of 360°) of the seven roots are

$$\begin{aligned}
&0{\cdot}00235\,49 && 0°,\\
&9{\cdot}01060\,17 && \mp 315°\ 26'\ 0''{\cdot}6354,\\
&0{\cdot}84526\,61 && \pm 200°\ 3'\ 30''{\cdot}3445,\\
&2{\cdot}70561\,49 && \mp 218°\ 11'\ 5''{\cdot}9337.
\end{aligned}$$

The corresponding functions of the roots of the septimic equation whose real root is F_{71}^{-4} are then found to be

$$\begin{aligned}
&0{\cdot}04852\,74 && 0°,\\
&3{\cdot}00176\,64 && \mp 337°\ 43'\ 0''{\cdot}3177,\\
&0{\cdot}91938\,36 && \pm 280°\ 1'\ 45''{\cdot}1722,\\
&1{\cdot}64487\,54 && \mp 289°\ 5'\ 32''{\cdot}9668.
\end{aligned}$$

The corresponding functions of the roots of the septimic equation whose real root is F_{71}^{-2} are then found to be

$$\begin{aligned}
&0{\cdot}22028\,93 && 0°,\\
&1{\cdot}73256\,07 && \mp 348°\ 51'\ 30''{\cdot}1588,\\
&0{\cdot}95884\,49 && \pm 320°\ 0'\ 52''{\cdot}5861,\\
&1{\cdot}28252\,69 && \mp 144°\ 32'\ 46''{\cdot}4834.
\end{aligned}$$

Finally the corresponding functions of the roots of the septimic equation satisfied by F_{71}^{-1} are found to be

$$\begin{aligned}
&0{\cdot}46934\,99 && 0°,\\
&1{\cdot}31626\,77 && \mp 174°\ 25'\ 45''{\cdot}0794,\\
&0{\cdot}97920\,63 && \pm 340°\ 0'\ 26''{\cdot}2930,\\
&1{\cdot}13248\,71 && \mp 252°\ 16'\ 23''{\cdot}2417.
\end{aligned}$$

The roots of this septimic equation are therefore

$$+0{\cdot}46934\,99,$$

$$-1{\cdot}31005\,09\mp i\times 0{\cdot}12777\,84,$$

$$+0{\cdot}92019\,56\mp i\times 0{\cdot}33478\,77,$$

$$-0{\cdot}34481\,96\pm i\times 1{\cdot}07871\,48.$$

All that now remains to be done is to construct the septimic equation satisfied by F_{71}^{-1} by multiplying together the four expressions

$$x-0{\cdot}46934\,99,$$

$$x^2+2{\cdot}62010\,18x+1{\cdot}73256\,07,$$

$$x^2-1{\cdot}84039\,12x+0{\cdot}95884\,49,$$

$$x^2+0{\cdot}68963\,92x+1{\cdot}28252\,69.$$

It is found that the product is

$$x^7+x^6-x^5-x^4-x^3+x^2+2x-1,$$

so that

$$F_{71}^7-2F_{71}^6-F_{71}^5+F_{71}^4+F_{71}^3+F_{71}^2-F_{71}-1=0.$$

The remainder of the paper consists of statements of results concerning the eighty singular moduli enumerated in § 1, arranged in order of magnitude of the corresponding class-numbers. Even if it serves no other purpose, it should demonstrate the indispensability of a calculating machine for numerical work on singular moduli.

4. *Discriminants with Gaussian classification* G.I, *class-number* 1.

There are five discriminants in this set, namely 1, 2, 3, 4, and 7. The values of the corresponding class-invariants are as follows:

n = 1: $f(\sqrt{-1})=\sqrt[4]{2}, \quad G_1=1.$

n = 2: $f_1(\sqrt{-2})=\sqrt[4]{2}, \quad g_2=1.$

n = 3: $f^3(\sqrt{-3})=2.$

n = 4: $f_1^4(\sqrt{-4})=2\sqrt{2}.$

n = 7: $f(\sqrt{-7})=\sqrt{2}, \quad F_7=1.$

Results equivalent to the first four of these formulae were obtained by Legendre†; the last is due to Joubert‡.

It is to be noted that the value of $f_1(\sqrt{-4})$ is misprinted in Weber's list§ as $\sqrt[3]{8}$.

5. *Discriminants with Gaussian classification* G.I, *class-number* 3.

There are eight discriminants in this set; six of them, namely 11, 19, 27, 43, 67, and 163, are of the form $8m+3$; the other two, namely 23 and 31, are of the form $8m-1$.

The equations satisfied by the class-invariants corresponding to members of the first group are as follows:

$\mathbf{n=11}$: $f^3-2f^2+2f-2=0$ $(\Delta=-44)$.

$\mathbf{n=19}$: $f^3-*-2f-2=0$ $(\Delta=-76)$.

$\mathbf{n=27}$: $f^9-6f^6-12f^3-8=0$ $(\Delta=-6912)$,

$f^3-*-f\sqrt[3]{24}-2=0$ $(\Delta=-12)$,

$3f=\sqrt[3]{18}+\sqrt[3]{36}$.

$\mathbf{n=43}$: $f^3-2f^2-*-2=0$ $(\Delta=-172)$.

$\mathbf{n=67}$: $f^3-2f^2-2f-2=0$ $(\Delta=-268)$.

$\mathbf{n=163}$: $f^3-6f^2+4f-2=0$ $(\Delta=-652)$.

The equations satisfied by the class-invariants corresponding to members of the second group are as follows:

$\mathbf{n=23}$: $F^3-*-F-1=0$ $(\Delta=-23)$.

$\mathbf{n=31}$: $F^3-F^2-*-1=0$ $(\Delta=-31)$.

These eight cubic equations were constructed by Weber‖; the cubic equations satisfied by F_{23}^{24} and F_{31}^{24} had, however, been published nearly thirty years before by Joubert¶.

† A. M. Legendre, *Traité des fonctions elliptiques*, 1 (1825), 35, 107, 58–60, 105.

‡ P. Joubert, *Comptes rendus*, 50 (1860), 907–912.

§ H. Weber, *Lehrbuch der Algebra*, 3 (1908), 721.

‖ H. Weber, *Acta Math.*, 11 (1888), 333–390.

¶ P. Joubert, *Comptes rendus*, 50 (1860), 907–912.

6. *Discriminants with Gaussian classification* G.I, *class-number* 5.

There are four discriminants in this set, namely 47, 79, 103, and 127. The equations satisfied by the corresponding class-invariants are as follows

n = 47: $F^5 - * - F^3 - 2F^2 - 2F - 1 = 0.$

n = 79: $F^5 - 3F^4 + 2F^3 - F^2 + F - 1 = 0.$

n = 103: $F^5 - F^4 - 3F^3 - 3F^2 - 2F - 1 = 0.$

n = 127: $F^5 - 3F^4 - F^3 + 2F^2 + F - 1 = 0.$

The first of these equations was obtained by Weber†, the corresponding equation for F_{47}^2 being published at about the same time by Greenhill‡; the second and third equations were obtained by Russell§ and the fourth by Hanna‖.

The canonical solution of an Abelian quintic equation is given by the formula¶

$$10x_r = a + u_1\omega_r + u_2\omega_r^2 + u_3\omega_r^3 + u_4\omega_r^4,$$

where

$$\omega_r = \exp\frac{2r\pi i}{5}$$

and r takes the values 0, 1, 2, 3, 4 in succession; u_1^5, u_2^5, u_3^5, u_4^5 are the roots of an Abelian quartic equation; for all quintic equations contained in this paper, the real values of u_1, u_2, u_3, u_4, and other fifth roots are to be taken.

It appears that the canonical solutions of the quintics satisfied by F_n ($n = 47, 79, 103, 127$) involve rather larger numbers than the canonical solutions of the corresponding quintics satisfied by F_n^{-1} (a similar remark applies to equations satisfied by F_n for other values of n); accordingly I state the canonical solutions of the latter equations instead of the former. The results are as follows††:

n = 47. The roots of the equation

$$x^5 + 2x^4 + 2x^3 + x^2 - * - 1 = 0$$

† H. Weber, *Acta Math.*, 11 (1888), 333–390.

‡ A. G. Greenhill, *Proc. London Math. Soc.* (1), 19 (1889), 301–364.

§ R. Russell, *Proc. London Math. Soc.* (1), 21 (1891), 351–395.

‖ M. Hanna, *Proc. London Math. Soc.* (2), 28 (1928), 46–52.

¶ For references to the literature, see *Encyklopädie der Math. Wiss.*, 1 (1904), 516.

†† It is supposed, both here and elsewhere, that the upper signs in ambiguous expressions are associated with the first expressions on the left and the lower signs with the second; this discrimination is negligible so far as the real root of the quintic is concerned, but disregard of it would, of course, give erroneous values of the complex roots.

There is a certain amount of repetition in the statements of both these and other results, but such repetition seemed advisable in order to avoid risk of misinterpretation and confusion.

are derived from the following results:

$$a = -4,$$

$$u_1^5,\ u_4^5 = 4776-1720\sqrt{5} \pm (920-360\sqrt{5})\sqrt{47(5+2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 4776+1720\sqrt{5} \pm (920+360\sqrt{5})\sqrt{47(5-2\sqrt{5})}.$$

n = 79. The roots of the equation

$$x^5-x^4+x^3-2x^2+3x-1 = 0$$

are derived from the following results:

$$a = 2,$$

$$u_1^5,\ u_4^5 = -13168-6400\sqrt{5} \pm (2160+960\sqrt{5})\sqrt{79(5-2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = -13168+6400\sqrt{5} \pm (2160-960\sqrt{5})\sqrt{79(5+2\sqrt{5})}.$$

n = 103. The roots of the equation

$$x^5+2x^4+3x^3+3x^2+x-1 = 0$$

are derived from the following results:

$$a = -4,$$

$$u_1^5,\ u_4^5 = 8176+3840\sqrt{5} \pm (1200+480\sqrt{5})\sqrt{103(5-2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 8176-3840\sqrt{5} \pm (1200-480\sqrt{5})\sqrt{103(5+2\sqrt{5})}.$$

n = 127. The roots of the equation

$$x^5-x^4-2x^3+x^2+3x-1 = 0$$

are derived from the following results:

$$a = 2,$$

$$u_1^5,\ u_4^5 = -7768+6280\sqrt{5} \pm (360-120\sqrt{5})\sqrt{127(5-2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = -7768-6280\sqrt{5} \pm (360+120\sqrt{5})\sqrt{127(5+2\sqrt{5})}.$$

It is to be observed that the expressions following the ambiguous signs are all of the simple forms

$$(A+B\sqrt{5})\sqrt{n(5+2\sqrt{5})},\quad (A'+B'\sqrt{5})\sqrt{n(5-2\sqrt{5})}$$

first noted by Cayley† in connection with the quintic satisfied by F_{47}^{-2}. It

† A. Cayley, *American J. of Math.*, 13 (1891), 53–58.

must also be pointed out that the expressions

$$(A+B\sqrt{5})\sqrt{n(5+2\sqrt{5})}, \quad (A'+B'\sqrt{5})\sqrt{n(5-2\sqrt{5})}$$

are equal when

$$A' = 2A+5B, \quad B' = A+2B,$$

and that then

$$A'^2-5B'^2 = -(A^2-5B^2).$$

The reason for adopting one expression rather than the other in the formulae given above is the desirability of having all expressions following ambiguous signs written as positive numbers.

The solutions of the four quintic equations have not been published previously in the forms which have just been given. Canonical solutions of the corresponding quintic equations satisfied by F_{47}^{-2}, F_{79}^{2}, and F_{127}^{-2} have, however, been constructed by Young†, Watson‡, and Mitra§ respectively.

7. *Discriminants with Gaussian classification* G.I, *class-number* 7.

There are six discriminants in this set, namely 71, 151, 223, 343, 463, and 487. The equations satisfied by the corresponding class-invariants are as follows:

n = 71: $F^7-2F^6-F^5+F^4+F^3+F^2-F-1=0.$

n = 151: $F^7-3F^6-F^5-3F^4-*-F^2-F-1=0.$

n = 223: $F^7-5F^6+*+F^4-4F^3-F^2-*-1=0.$

n = 343: $F^7-7F^6-7F^5-7F^4-*-*-*-1=0.$

n = 463: $F^7-11F^6-9F^5-8F^4-7F^3-7F^2-3F-1=0.$

n = 487: $F^7-13F^6+4F^5-4F^4+7F^3-4F^2+F-1=0.$

Of these six equations, the first was obtained by Weber‖, the second and third by Hanna¶, and the remaining three are new.

I now state the numerical results by means of which the solutions by radicals of these six septimic equations can be immediately obtained. As

† G. P. Young, *American J. of Math.*, 10 (1888), 99–130.
‡ G. N. Watson, *J. Indian Math. Soc.*, 18 (1930), *273–275*.
§ S. C. Mitra, *Indian Math.-Phys. J.*, 2 (1931), 7–10.
‖ H. Weber, *Acta Math.*, 11 (1888), 333–390.
¶ M. Hanna, *Proc. London Math. Soc.* (2), 28 (1928), 46–52.

usual, I give the canonical form of the solutions of the equations whose roots are the reciprocals of the roots of the equations satisfied by F_n ($n = 71, 151, 223, 343, 463, 487$); no solutions of these equations or of equations derived from them have, so far as I know, been published previously.

The canonical solution of an Abelian septimic equation is given by the formula

$$7x_r = a + b_1\epsilon_r + b_2\epsilon_r^2 + b_3\epsilon_r^4 + c_1\epsilon_r^6 + c_2\epsilon_r^5 + c_3\epsilon_r^3,$$

where

$$\epsilon_r = \exp\frac{2r\pi i}{7}$$

and r takes the values 0, 1, 2, 3, 4, 5, 6 in succession. In each of the cases under consideration, the triads

$$b_1c_1, \quad b_2c_2, \quad b_3c_3;$$

$$\frac{b_2^2}{b_3}, \quad \frac{b_3^2}{b_1}, \quad \frac{b_1^2}{b_2};$$

$$\frac{c_2^2}{c_3}, \quad \frac{c_3^2}{c_1}, \quad \frac{c_1^2}{c_2};$$

are roots of irreducible cubic equations, the coefficients in the first equation being integers, and corresponding coefficients in the second and third equations being conjugate quadratic surds†.

The values of a and the three cubics in the respective cases are as follows:

n = 71: $a = -1,$

$$x^3 - 10x^2 + 24x - 8 = 0 \qquad (\Delta = +56^2),$$

$$x^3 + (45 \pm 3\sqrt{497})x^2 - \tfrac{1}{2}(1 \pm 9\sqrt{497})x + \tfrac{1}{2}(23 \pm \sqrt{497}) = 0$$

$$\left[\Delta = +\left(\frac{12285 \pm 469\sqrt{497}}{2}\right)^2\right].$$

n = 151: $a = -1,$

$$x^3 + 4x^2 - 39x - 169 = 0 \qquad (\Delta = +91^2),$$

$$x^3 - \frac{475 \pm 7\sqrt{1057}}{26}x^2 + \frac{29 \pm 35\sqrt{1057}}{26}x - 13 = 0$$

$$\left[\Delta = +\left(\frac{99463 \pm 70\sqrt{1057}}{169}\right)^2\right].$$

† Except in the case **n = 343**; they are then rational.

n = 223: $a = 0,$

$$x^3+7x^2-28x-203=0 \qquad (\Delta = +91^2),$$

$$x^3-\frac{483\pm 3\sqrt{1561}}{58}x^2-\frac{1169\pm 35\sqrt{1561}}{58}x+(42\pm\sqrt{1561})=0$$

$$\left[\Delta = +\left(\frac{5\,14850\pm 12179\sqrt{1561}}{841}\right)^2\right].$$

n = 343: $a = 0,$

$$x^3-*-343x+2401=0 \qquad (\Delta = +2401^2),$$

$$x^3-*-49x+49=0 \qquad (\Delta = +637^2),$$

$$x^3+7x^2-*-49=0 \qquad (\Delta = +49^2).$$

n = 463: $a = -3,$

$$x^3+22x^2-65x-169=0 \qquad (\Delta = +3731^2),$$

$$x^3-\frac{711\pm 7\sqrt{3241}}{26}x^2-\frac{6235\pm 49\sqrt{3241}}{26}x+13=0$$

$$\left[\Delta = +\left(\frac{19\,74021\pm 24262\sqrt{3241}}{169}\right)^2\right].$$

n = 487: $a = 1,$

$$x^3+25x^2+87x-841=0 \qquad (\Delta = +1624^2),$$

$$x^3-\frac{1903\pm 21\sqrt{3409}}{58}x^2-\frac{1583\pm 63\sqrt{3409}}{58}x-29=0$$

$$\left[\Delta = +\left(\frac{10\,22819\pm 18186\sqrt{3409}}{841}\right)^2\right].$$

When the solutions of the second and third cubics in these triads are known, the values of b_r and c_r $(r = 1, 2, 3)$ are obtainable from formulae of the type

$$b_1^7 = \left(\frac{b_1^2}{b_2}\right)^4\left(\frac{b_2^2}{b_3}\right)^2\left(\frac{b_3^2}{b_1}\right), \quad c_1^7 = \left(\frac{c_1^2}{c_2}\right)^4\left(\frac{c_2^2}{c_3}\right)^2\left(\frac{c_3^2}{c_1}\right).$$

The solutions of all of the eighteen cubics are expressible in a real form in terms of radicals involving rational numbers and the roots of a single irreducible cubic. The fundamental cubic which I adopted for this purpose is

$$8x^3-*-42x-7=0 \qquad \left[\Delta = \left(\frac{189}{8}\right)^2\right],$$

the roots of this cubic being

$$\sigma_r = \tfrac{1}{2} + 3\cos\frac{2^r\pi}{7} \quad (r = 1, 2, 3).$$

Let

$$\tau_1 = \sigma_3 - \sigma_2, \quad \tau_2 = \sigma_1 - \sigma_3, \quad \tau_3 = \sigma_2 - \sigma_1,$$

and let $3\mu_1$ be the negative acute angle such that

$$\cos 3\mu_1 = \frac{1}{2\sqrt{7}}.$$

Define μ_2, μ_3, ... by the formula

$$\mu_{r+1} = \mu_r + \frac{2\pi}{3}.$$

Then it is easy to verify that

$$\cos\mu_r = \frac{\sigma_r}{\sqrt{7}}, \quad \sin\mu_r = \frac{\tau_r}{\sqrt{21}},$$

and hence the solutions of the cubics which will now be given can easily be connected with Tartaglia's solutions. In writing out these solutions, it is convenient to define b_r and c_r for all integral values of r by the convention that they are periodic functions of r with period 3.

The roots of the eighteen cubics have the following values:

n = 71:

$$3b_r c_r = 10 + 4\sigma_r,$$

$$3b_r^2/b_{r+1} = -(45+3\sqrt{497}) + 6\sigma_r - (27+\sqrt{497})\tau_r,$$

$$3c_r^2/c_{r+1} = -(45-3\sqrt{497}) + 6\sigma_r - (27-\sqrt{497})\tau_r.$$

n = 151:

$$3b_r c_r = -4 + 8\sigma_r - 2\tau_r.$$

$$39b_r^2/b_{r+1} = \tfrac{1}{2}(475+7\sqrt{1057}) + (56+5\sqrt{1057})\sigma_r - 57\tau_r,$$

$$39c_r^2/c_{r+1} = \tfrac{1}{2}(475-7\sqrt{1057}) + (56-5\sqrt{1057})\sigma_r - 57\tau_r.$$

n = 223:

$$3b_r c_r = -7 - 7\sigma_r - 3\tau_r.$$

$$609\,b_r^2/b_{r+1} = \tfrac{1}{2}(3381+21\sqrt{1561}) + (1050+33\sqrt{1561})\sigma_r + (133-8\sqrt{1561})\tau_r,$$

$$609\,c_r^2/c_{r+1} = \tfrac{1}{2}(3381-21\sqrt{1561}) + (1050-33\sqrt{1561})\sigma_r + (133+8\sqrt{1561})\tau_r.$$

$\mathbf{n = 343}$:

$$3b_r c_r = -14\tau_r,$$

$$3b_r^2/b_{r+1} = -3\sigma_r - 5\tau_r,$$

$$3c_r^2/c_{r+1} = -7 - 4\sigma_r - 2\tau_r, \quad 2c_r^2/c_{r+1} = 21/\tau_{r-1}.$$

$\mathbf{n = 463}$:

$$3b_r c_r = -22 - 19\sigma_r + 3\tau_r,$$

$$39b_r^2/b_{r+1} = \tfrac{1}{2}(711 + 7\sqrt{3241}) + (210 + 5\sqrt{3241})\sigma_r + (11 - 2\sqrt{3241})\tau_r,$$

$$39c_r^2/c_{r+1} = \tfrac{1}{2}(711 - 7\sqrt{3241}) + (210 - 5\sqrt{3241})\sigma_r + (11 + 2\sqrt{3241})\tau_r.$$

$\mathbf{n = 487}$:

$$3b_r c_r = -25 + 14\sigma_r - 2\tau_r,$$

$$87b_r^2/b_{r+1} = \tfrac{1}{2}(1903 + 21\sqrt{3409}) - (496 + 9\sqrt{3409})\sigma_r - (169 + 4\sqrt{3409})\tau_r,$$

$$87c_r^2/c_{r+1} = \tfrac{1}{2}(1903 - 21\sqrt{3409}) - (496 - 9\sqrt{3409})\sigma_r - (169 - 4\sqrt{3409})\tau_r.$$

8. *Discriminants with Gaussian classification* G.I, *class-number* 9.

There are twenty discriminants in this set; fifteen of them, namely 59, 83, 107, 139, 211, 243, 283, 307, 331, 379, 499, 547, 643, 883, and 907, are of the form $8m+3$; the remaining five, namely 199, 367, 823, 1087, and 1423, are of the form $8m-1$. I consider these two groups separately, postponing the discussion of 243 (the only one of these discriminants which is not a prime) to the end of the first group. The equations satisfied by the class-invariants corresponding to the remaining members of the first group are as follows:

$\mathbf{n = 59}$:

$$f^9 - 4f^8 + 4f^7 - 2f^6 + 4f^5 - 8f^4 + 4f^3 - 8f^2 + 16f - 8 = 0.$$

$\mathbf{n = 83}$:

$$f^9 - 6f^8 + 14f^7 - 22f^6 + 20f^5 - 8f^4 - 4f^3 + * + 8f - 8 = 0.$$

$\mathbf{n = 107}$:

$$f^9 - * - 8f^7 - 18f^6 - 24f^5 - 32f^4 - 36f^3 - 32f^2 - 16f - 8 = 0.$$

$\mathbf{n = 139}$:

$$f^9 - 2f^8 - 10f^7 - 10f^6 - 4f^5 - 16f^4 - 28f^3 - 16f^2 - 8f - 8 = 0.$$

$\mathbf{n = 211}$:

$$f^9 - 6f^8 - 2f^7 - 14f^6 - 20f^5 - 32f^4 - 20f^3 - 32f^2 - 24f - 8 = 0.$$

n = 283:

$$f^9-8f^8-8f^7-14f^6+8f^5+8f^4+4f^3-32f^2-32f-8=0.$$

n = 307:

$$f^9-10f^8+2f^7-10f^6-8f^5-16f^4-12f^3-24f^2+24f-8=0.$$

n = 331:

$$f^9-14f^8+42f^7-94f^6+140f^5-152f^4+132f^3-80f^2+24f-8=0.$$

n = 379:

$$f^9-12f^8-10f^7-2f^6+20f^5-32f^4+20f^3-56f^2+40f-8=0.$$

n = 499:

$$f^9-16f^8-48f^7-18f^6+84f^5+24f^4-108f^3-24f^2+32f-8=0.$$

n = 547:

$$f^9-22f^8+16f^7-50f^6+16f^5-88f^4+60f^3-104f^2+32f-8=0.$$

n = 643:

$$f^9-26f^8-44f^7-34f^6-80f^5-184f^4-260f^3-184f^2-48f-8=0.$$

n = 883:

$$f^9-46f^8-136f^7-270f^6-368f^5-408f^4-268f^3-72f^2+48f-8=0.$$

n = 907:

$$f^9-58f^8+352f^7-998f^6+1640f^5-1656f^4+1004f^3-328f^2+48f-8=0.$$

These fourteen equations of the ninth degree are reducible to cubic equations of the form

$$f^3-2\alpha f^2+2\beta f-2=0$$

in which α and β are cubic irrationalities. In each case α and β are expressible rationally in terms of the cubic irrationality defined by Berwick's fundamental cubic† associated with the appropriate discriminant. In

† W. E. H. Berwick, *Proc. London Math. Soc.* (2), 28 (1928), 53–69 (62).

the following Table I quote the cubics constructed by Berwick and add the corresponding values of† α and β.

n		Δ	α	β
59	$\theta^3-2\theta^2-*-1=0$	-59	θ	$\frac{\theta+1}{\theta-1}$
83	$\theta^3-2\theta^2-2\theta-1=0$	-83	1	$1-\theta$
107	$\theta^3-\theta^2-3\theta-2=0$	-107	$\frac{\theta+2}{\theta}$	$-\frac{1}{\theta+1}$
139	$\theta^3-3\theta^2-5\theta-2=0$	-139	$\frac{2\theta+2}{\theta}$	$\frac{\theta}{\theta+1}$
211	$\theta^3-3\theta^2+\theta-2=0$	-211	θ	$-\theta$
283	$\theta^3-4\theta^2-*-1=0$	-283	θ	$-\theta$
307	$\theta^3-2\theta^2+4\theta-5=0$	-307	$\frac{1+\theta}{2-\theta}$	$\frac{6-2\theta}{3\theta-3}$
331	$\theta^3-4\theta^2-2\theta-1=0$	-331	$\theta+1$	1
379	$\theta^3-\theta^2+\theta-4=0$	-379	$\frac{\theta}{2-\theta}$	$\frac{\theta+2}{\theta-1}$
499	$\theta^3+*+4\theta-3=0$	-499	$\frac{4-\theta}{2\theta-1}$	$\frac{1+\theta}{1-\theta}$
547	$\theta^3-4\theta^2+2\theta-3=0$	-547	$\frac{2\theta}{\theta-3}$	θ
643	$\theta^3-10\theta^2-6\theta-1=0$	-643	$\frac{12\theta+4}{\theta-1}$	$-\frac{4\theta+1}{\theta}$
883	$\theta^3-5\theta^2-5\theta-2=0$	-883	$4\theta+1$	$\frac{6\theta+5}{\theta-1}$
907	$\theta^3-5\theta^2+\theta-2=0$	-907	$\frac{\theta-8}{\theta-5}$	2

In three instances ($n=307$, 379, 499) the cubic used by Berwick has not got $\Delta=-n$. In these cases I have replaced the cubics by other cubics constructed by Berwick in an earlier paper‡. The cubics of the latter paper are

n = 307: $\phi^3-5\phi^2-\phi-4=0$, $\phi=\frac{1+\theta}{2-\theta}$ $(\Delta=-2763)$.

† It is to be observed that I have given bilinear relations connecting α and β with θ instead of the more conventional expressions for α and β as quadratic functions of θ. It is thus made easier to construct the cubic equations satisfied by α and β; at the same time, it is a simple matter to derive the quadratic functions from the values given in the Table.

‡ G. B. Mathews, *Proc. London Math. Soc.* (2), 10 (1912), 128–138 (137–138). It may be mentioned that the cubic for which $\Delta=-107$ which is there given contains an error in the sign of one of the terms.

$\mathbf{n = 379}$: $\phi^3-6\phi^2-5\phi-2=0, \quad \phi=\dfrac{\theta}{2-\theta}$ $(\Delta=-1516)$.

$\mathbf{n = 499}$: $\phi^3-4\phi^2-5\phi-4=0, \quad \phi=\dfrac{1+\theta}{1-\theta}$ $(\Delta=-1996)$.

In five other instances ($n = 83, 139, 331, 643, 883$) the cubics used in the later paper differ from the cubics constructed in the earlier paper; the cubics of the earlier paper are obtained from the corresponding cubics quoted in the Table given above by substituting

$$\frac{1}{\phi-1}, \quad \frac{1-\phi}{\phi-2}, \quad \frac{1}{\phi-1}, \quad \frac{1}{\phi-3}, \quad \frac{1}{\phi-1}$$

for θ in the respective cases. The ϕ-cubics have $\Delta = -n$ except when $n = 883$; the discriminant of that cubic is $-883/16$.

The 9-ic equations for f_{59} and f_{83} were first constructed by Russell†; he also obtained equations of the type

$$f^3-2\alpha f^2+2\beta f-2=0$$

in the cases $n = 59, 83, 107, 211, 283, 331, 907$ together with the cubic equations satisfied by the invariant γ_2 in the cases $n = 59, 83, 107$. This invariant was expressed as a quadratic function of the appropriate θ for each of the fourteen values of n now under consideration by Berwick in the paper already cited.

I now proceed to discuss the class-invariant associated with 243. First take the modular equation of the third order of Schläfli's type, namely

$$\left(\frac{u}{v}\right)^6+\left(\frac{v}{u}\right)^6=(uv)^3-\frac{8}{(uv)^3},$$

where $$u=f(\tau), \quad v=f(3\tau).$$

By taking $\tau=\sqrt{-27}$ and writing

$$f^3(\sqrt{-27})=2(1+\sqrt[3]{2}+\sqrt[3]{4})=\phi,$$

for brevity, we see that $f(\sqrt{-243})$ is a solution of the equation

$$f^{12}-\phi^3f^9+8\phi f^3+\phi^4=0;$$

and, by taking $\tau=\sqrt{-3}$, we see that $f(\sqrt{-3})$ is a solution of the same equation. When we discard the factor thus indicated, namely f^3-2, from

† R. Russell, *Proc. London Math. Soc.* (1), 21 (1891), 351–395.

the 12-ic equation, we obtain the equation

$$f^9-6(\phi+1)^2f^6-12(\phi+1)^2f^3-8(3\phi^2+5\phi+3)=0$$

for the required class-invariant.

The irrationality which occurs in this equation is not given by Berwick's cubic associated with 243, namely

$$\theta^3-3=0.$$

Consequently we construct the rationalised form of the above cubic for $f^3(\sqrt{-243})$; the most direct method of doing this is to take the modular equation of order 9 of Schläfli's type, namely†

$$\left(\frac{u^{18}}{v^{18}}+\frac{v^{18}}{u^{18}}\right)+28\left(\frac{u^{15}}{v^{15}}+\frac{v^{15}}{u^{15}}\right)+298\left(\frac{u^{12}}{v^{12}}+\frac{v^{12}}{u^{12}}\right)+1548\left(\frac{u^9}{v^9}+\frac{v^9}{u^9}\right)$$

$$+4383\left(\frac{u^6}{v^6}+\frac{v^6}{u^6}\right)+7704\left(\frac{u^3}{v^3}+\frac{v^3}{u^3}\right)+9324$$

$$=\left(\frac{u^3}{v^3}+1+\frac{v^3}{u^3}\right)\left(u^{12}v^{12}+\frac{4096}{u^{12}v^{12}}\right),$$

where $$u=f(\tau), \quad v=f(9\tau),$$

and to take $\tau=\sqrt{-3}$; then write f in place of v.

Since $f^3(\sqrt{-3})=2$, we thus obtain an equation of degree 12 in f^3 with rational coefficients, and $f^3=f^3(\sqrt{-243})$ is one of its roots. By taking $\tau=\sqrt{-\frac{1}{3}}$ and remembering that

$$f^3(\sqrt{-\tfrac{1}{3}})=f^3(\sqrt{-3})=2,$$

we see that $f^3(\sqrt{-27})$ is also a root, so that the equation must contain the factor

$$f^9-6f^6-12f^3-8.$$

When we discard this factor, we obtain the equation for $f(\sqrt{-243})$ in the form

$$f^{27}-450f^{24}-2520f^{21}-10176f^{18}-24768f^{15}-44352f^{12}$$

$$-48000f^9-32256f^6-2304f^3-512=0.$$

It is easy to verify that, by means of Berwick's cubic

$$\theta^3-3=0 \qquad (\Delta=-243),$$

† G. N. Watson, *Journal für Math.*, 169 (1933), 238–251.

this 9-ic in f^3 is reducible to

$$f^9-(150+108\theta+72\theta^2)f^6-(12+8\theta)f^3-8=0,$$

a cubic in f^3 which obviously is not the same as the cubic previously obtained.

By applying the Euclidean algorithm to these two cubics, we can express $f^3(\sqrt{-243})$ rationally in terms of the numbers θ and ϕ. This expression is reducible to the form

$$3f^3(\sqrt{-243})=150+108\sqrt[3]{3}+72\sqrt[3]{9}+(120+84\sqrt[3]{3}+58\sqrt[3]{9})\sqrt[3]{2}$$
$$+(96+66\sqrt[3]{3}+46\sqrt[3]{9})\sqrt[3]{4}.$$

It is a matter of elementary arithmetic to verify that the result of eliminating the irrationality $\sqrt[3]{3}$ or the irrationality $\sqrt[3]{2}$ from this equation is the first or the second cubic satisfied by $f^3(\sqrt{-243})$ respectively.

When the last result is multiplied by $3-2\sqrt[3]{3}$, it is an easy matter to reduce it to the simple and beautiful form

$$f(\sqrt{-243})\sqrt[3]{\tfrac{3}{2}(3-2\sqrt[3]{3})}=\sqrt[3]{2}+\sqrt[3]{3}+\sqrt[3]{4}.$$

In effecting this last simplification, I found that the smallness of the difference between the numbers

$$\sqrt[3]{(3-2\sqrt[3]{3})},\quad \frac{3}{2(1+\sqrt[3]{9})}$$

(which are respectively equal to 0·48699 93765 and 0·48699 97331) was somewhat embarrassing. In this connection it may be noticed that

$$\frac{27}{8(1+\sqrt[3]{9})^3}-(3-2\sqrt[3]{3})=\frac{14299\sqrt[3]{3}+1377\sqrt[3]{9}-23487}{8000}=0\cdot00000\,02538.$$

I now turn to the five class-invariants of the second group, about which no results have been published previously. The equations satisfied by these class-invariants are as follows:

n = 199:

$$F^9-5F^8+3F^7-3F^6-*-*-3F^3-*-F-1=0.$$

By means of Berwick's cubic

$$\theta^3-\theta^2+4\theta-1=0 \qquad (\Delta=-199),$$

this is reducible to†

$$F^{-3}+\theta F^{-2}+(\theta^2+1)F^{-1}-\theta=0.$$

† As was to be anticipated from similar experiences with class-invariants already discussed, it proves to be simplest to write the cubic as a cubic in F^{-1} with leading coefficient equal to unity.

n = 367:

$$F^9-9F^8+3F^7-2F^6+*+2F^4-6F^3+F^2+2F-1=0.$$

By means of Berwick's cubic

$$\theta^3-2\theta^2+3\theta-5=0 \qquad (\Delta=-367),$$

this is reducible to

$$F^{-3}-\theta F^{-2}+(\theta^2-\theta)F^{-1}+(\theta-2)=0.$$

n = 823:

$$F^9-29F^8-36F^7-30F^6-33F^5-30F^4-12F^3-*-F-1=0.$$

By means of Berwick's cubic

$$\theta^3-3\theta^2-2\theta-3=0 \qquad (\Delta=-823),$$

this is reducible to

$$F^{-3}+(\theta^2-3\theta-1)F^{-2}+(\theta^2-2\theta-5)F^{-1}-(\theta^2-4\theta+1)=0.$$

n = 1087:

$$F^9-51F^8-100F^7-139F^6-124F^5-100F^4-61F^3-28F^2-8F-1=0.$$

By means of Berwick's cubic

$$\theta^3-2\theta^2+5\theta+3=0 \qquad (\Delta=-1087),$$

this is reducible to

$$F^{-3}-(\theta^2-\theta)F^{-2}-(\theta^2+2\theta-1)F^{-1}-(2\theta+1)=0.$$

n = 1423:

$$F^9-99F^8+38F^7-89F^6+42F^5-65F^4+37F^3-22F^2+5F-1=0.$$

By means of Berwick's cubic

$$\theta^3-\theta^2+6\theta-7=0 \qquad (\Delta=-1423),$$

this is reducible to

$$F^{-3}+(\theta-2)F^{-2}-(\theta-3)F^{-1}-(\theta-1)^2=0.$$

Although it is possible to exhibit the solutions of the five cubics which have just been constructed in a real form, and so to solve the five 9-ic equations by radicals in a real form, it seemed worth while to construct the canonical forms of the solutions of the 9-ic equations satisfied by

$$F_n^{-1} \quad (n=199,\ 367,\ 823,\ 1087,\ 1423).$$

The canonical solution of an Abelian 9-ic equation is given by the formula

$$9x_r = a + B\eta_r^3 + C\eta_r^6 + b_1\eta_r + b_2\eta_r^4 + b_3\eta_r^7 + c_1\eta_r^8 + c_2\eta_r^5 + c_3\eta_r^2,$$

where

$$\eta_r = \exp\frac{2r\pi i}{9}$$

and r takes the values 0, 1, 2, 3, 4, 5, 6, 7, 8 in succession. In each of the cases under consideration, the pair

$$B^3, \quad C^3$$

are the roots of a quadratic equation with integral coefficients, and the triads

$$b_1c_1, \quad b_2c_2, \quad b_3c_3;$$

$$b_1b_2^2, \quad b_2b_3^2, \quad b_3b_1^2;$$

$$c_1c_2^2, \quad c_2c_3^2, \quad c_3c_1^2$$

are roots of irreducible cubic equations, the coefficients in the first equation being integers, and corresponding coefficients in the second and third equations being conjugate quadratic surds which are "algebraic integers".

The values of a, the quadratics, and the three cubics in the respective cases are as follows:

n = 199:

$$a = -1,$$

$$x^2 - 7x - 1331 = 0,$$

$$x^3 - 15x^2 - 81x + 1367 = 0 \qquad (\Delta = +1224^2),$$

$$x^3 + 30x^2 - (6846 \pm 270\sqrt{597})x + \tfrac{1}{2}(3\,85049 \pm 16293\sqrt{597}) = 0$$

$$[\Delta = +\tfrac{1}{4}(16\,66359 \pm 70119\sqrt{597})^2].$$

n = 367:

$$a = 2,$$

$$x^2 - 97x - 125 = 0,$$

$$x^3 - 30x^2 - 249x + 1241 = 0 \qquad (\Delta = +19413^2),$$

$$x^3 - \tfrac{1}{2}(165 \pm 9\sqrt{1101})x^2 + \tfrac{1}{2}(2697 \pm 63\sqrt{1101})x$$

$$+ \tfrac{1}{2}(7\,16879 \pm 21765\sqrt{1101}) = 0$$

$$[\Delta = +\tfrac{1}{4}(39\,13515 \pm 1\,15047\sqrt{1101})^2].$$

$\mathbf{n = 823}$: $a = -1$,

$$x^2+389x-12167 = 0,$$

$$x^3-27x^2-246x+7451 = 0 \qquad (\Delta = +9063^2),$$

$$x^3+\tfrac{1}{2}(33\pm 9\sqrt{2469})\,x^2+\tfrac{1}{2}(4551\pm 99\sqrt{2469})\,x$$
$$+\tfrac{1}{2}(2\,82611\mp 26505\sqrt{2469}) = 0$$
$$(\Delta = +34\,20009^2).$$

$\mathbf{n = 1087}$: $a = -8$,

$$x^2-1199x-125 = 0,$$

$$x^3-9x^2-66x+233 = 0 \qquad (\Delta = +1791^2),$$

$$x^3+69x^2-\tfrac{1}{2}(4179\pm 63\sqrt{3261})x+\tfrac{1}{2}(38581\pm 687\sqrt{3261}) = 0$$
$$[\Delta = +(66393\pm 1728\sqrt{3261})^2].$$

$\mathbf{n = 1423}$: $a = 5$,

$$x^2+137x-4913 = 0,$$

$$x^3+81x^2+6x-1171 = 0 \qquad (\Delta = +49419^2),$$

$$x^3+(606\pm 9\sqrt{4269})\,x^2+\tfrac{1}{2}(35349\pm 567\sqrt{4269})\,x$$
$$-\tfrac{1}{2}(19\,72537\pm 30165\sqrt{4269}) = 0$$
$$[\Delta = +(579\,94326\pm 8\,88435\sqrt{4269})^2].$$

When the solutions of the last pair of cubics in each of these five sets are known, the values of b_r and c_r $(r = 1, 2, 3)$ are obtainable from formulae of the type

$$b_1{}^9 = \frac{(b_1{}^2 b_3)^4 (b_2{}^2 b_1)}{(b_3{}^2 b_2)^2}, \quad c_1{}^9 = \frac{(c_1{}^2 c_3)^4 (c_2{}^2 c_1)}{(c_3{}^2 c_2)^2}.$$

The solutions of all of the fifteen cubics are expressible in a real form in terms of radicals involving rational numbers and the roots of a single irreducible cubic. The fundamental cubic which I adopted for this purpose is the cubic whose roots are

$$\kappa_r = \cos\frac{2^{2r-1}\pi}{9} \quad (r = 1, 2, 3),$$

namely $8x^3-6x+1 = 0$ $[\Delta = +(\frac{9}{8})^2]$.

Let $$\lambda_1 = \kappa_3 - \kappa_2, \quad \lambda_2 = \kappa_1 - \kappa_3, \quad \lambda_3 = \kappa_2 - \kappa_1,$$

so that $$\frac{\lambda_r}{\sqrt{3}} = \sin \frac{2^{2r-1}\pi}{9} \quad (r = 1, 2, 3).$$

I now give solutions of the quadratics and cubics which make it possible to complete the solutions of the 9-ic equations. In writing out these solutions, it is convenient to define b_r, c_r, κ_r, λ_r for all integral values of r by the convention that they are periodic functions of r with period 3.

n = 199:

$$2B^3 = 7+3\sqrt{597}, \quad 2C^3 = 7-3\sqrt{597},$$

$$b_r c_r = 5-4\kappa_r+8\lambda_r,$$

$$b_r b_{r+1}^2 = -10-(69+2\sqrt{597})\kappa_r-(14+\sqrt{597})\lambda_r,$$

$$c_r c_{r+1}^2 = -10-(69-2\sqrt{597})\kappa_r-(14-\sqrt{597})\lambda_r.$$

n = 367:

$$2B^3 = 97+3\sqrt{1101}, \quad 2C^3 = 97-3\sqrt{1101},$$

$$b_r c_r = 10+12\kappa_r-14\lambda_r,$$

$$b_r b_{r+1}^2 = \tfrac{1}{2}(55+3\sqrt{1101})-(27+2\sqrt{1101})\kappa_r+(30+\sqrt{1101})\lambda_r,$$

$$c_r c_{r+1}^2 = \tfrac{1}{2}(55-3\sqrt{1101})-(27-2\sqrt{1101})\kappa_r+(30-\sqrt{1101})\lambda_r.$$

n = 823:

$$2B^3 = -389-9\sqrt{2469}, \quad 2C^3 = -389+9\sqrt{2469},$$

$$b_r c_r = 9+25\kappa_r-3\lambda_r,$$

$$b_r b_{r+1}^2 = -\tfrac{1}{2}(11+3\sqrt{2469})+39\kappa_r+77\lambda_r,$$

$$c_r c_{r+1}^2 = -\tfrac{1}{2}(11-3\sqrt{2469})+39\kappa_r+77\lambda_r.$$

n = 1087:

$$2B^3 = 1199+21\sqrt{3261}, \quad 2C^3 = 1199-21\sqrt{3261},$$

$$b_r c_r = 3-11\kappa_r-\lambda_r,$$

$$b_r b_{r+1}^2 = -23+(21+\sqrt{3261})\kappa_r-20\lambda_r,$$

$$c_r c_{r+1}^2 = -23+(21-\sqrt{3261})\kappa_r-20\lambda_r.$$

n = 1423:

$$2B^3 = -137+3\sqrt{4269}, \quad 2C^3 = -137-3\sqrt{4269},$$

$$b_r c_r = -27+49\kappa_r-13\lambda_r,$$

$$b_r b_{r+1}^2 = -(202+3\sqrt{4269})-(291+5\sqrt{4269})\kappa_r-(130+2\sqrt{4269})\lambda_r,$$

$$c_r c_{r+1}^2 = -(202-3\sqrt{4269})-(291-5\sqrt{4269})\kappa_r-(130-2\sqrt{4269})\lambda_r.$$

9. *Discriminants with Gaussian classification* G.I, *class-number* 11.

There are four discriminants in this set, namely 167, 271, 967, and 1303. The equations satisfied by the corresponding class-invariants are as follows:

n = 167:

$$F^{11}-2F^{10}-4F^9-9F^8-7F^7-11F^6-6F^5-10F^4-4F^3-5F^2-F-1=0.$$

n = 271:

$$F^{11}-5F^{10}-6F^9-5F^8+3F^7+6F^6+3F^5-3F^4-F^3-F^2-*-1=0.$$

n = 967:

$$F^{11}-43F^{10}+68F^9-122F^8+99F^7-81F^6+18F^5-6F^4-13F^3+F^2 -2F-1=0.$$

n = 1303:

$$F^{11}-79F^{10}-57F^9-33F^8-68F^7-73F^6-59F^5-67F^4-57F^3-30F^2 -8F-1=0.$$

The first of these four equations was obtained by Hanna†; the other three are new. I have not constructed the canonical solutions of any of the four equations, since the results to be attained would be pairs of irreducible quintic equations with all their roots real, and I decided that such objectives were not worth the labour necessary to obtain them.

10. *Discriminants with Gaussian classification* G.I, *class-number* 13.

There are six discriminants in this set, namely 191, 263, 607, 631, 727, and 2143. The equations satisfied by the corresponding class-invariants are as follows:

† M. Hanna, *Proc. London Math. Soc.* (2), 28 (1929), 46–52.

n = 191:

$$F^{13}-6F^{12}+10F^{11}-16F^{10}+22F^{9}-19F^{8}+11F^{7}-5F^{6}-F^{5}+5F^{4}-4F^{3}$$
$$+*+2F-1=0.$$

n = 263:

$$F^{13}-8F^{12}+16F^{11}-27F^{10}+38F^{9}-36F^{8}+22F^{7}-12F^{6}+13F^{5}-19F^{4}$$
$$+21F^{3}-15F^{2}+6F-1=0.$$

n = 607:

$$F^{13}-17F^{12}-15F^{11}+16F^{10}+34F^{9}+3F^{8}-21F^{7}-21F^{6}+9F^{5}+7F^{4}+F^{3}$$
$$-3F^{2}+2F-1=0.$$

n = 631:

$$F^{13}-17F^{12}-33F^{11}-69F^{10}-77F^{9}-71F^{8}-72F^{7}-54F^{6}-46F^{5}-37F^{4}$$
$$-21F^{3}-12F^{2}-4F-1=0.$$

n = 727:

$$F^{13}-25F^{12}+23F^{11}-41F^{10}+15F^{9}-30F^{8}-8F^{7}-4F^{6}-*-6F^{4}+5F^{3}$$
$$-8F^{2}+4F-1=0.$$

n = 2143:

$$F^{13}-299F^{12}-1147F^{11}-2058F^{10}-2161F^{9}-1292F^{8}-155F^{7}+396F^{6}$$
$$+268F^{5}+14F^{4}-43F^{3}-12F^{2}-*-1=0.$$

The first of these six equations was obtained by Hanna†; the other five are new. As in the case of the four equations of degree 11, I decided that it was not worth while to construct the canonical solutions of these equations. If I had used the same method as in the case of the septimic equations, the eight preliminary pieces of calculation which were then necessary to arrange the roots in the appropriate order would have been replaced by 3840 calculations‡; unless some more rapid procedure is devised, the construction of the canonical solutions does not seem to be worth the requisite expenditure of time and energy.

11. *Discriminants with Gaussian classification* G.I, *class-number* 15.

There are twenty-seven discriminants in this set; twenty-one of them, namely 131, 179, 227, 347, 443, 523, 571, 619, 683, 691, 739, 787, 947, 1051,

† M. Hanna, *Proc. London Math. Soc.* (2), 28 (1929), 46–52.

‡ The number necessary for an equation of degree $2r+1$ is $(r-1)!\times 2^{r-1}$.

1123, 1723, 1747, 1867, 2203, 2347, and 2683, are of the form $8m+3$; the remaining six, namely 239, 439, 751, 1327, 1567, and 2647, are of the form $8m-1$. I consider these two groups separately. The equations satisfied by the class-invariants corresponding to members of the first group are as follows:

$\mathbf{n=131}$:

$$f^{15}-4f^{14}-*-6f^{12}-8f^{11}-24f^{10}-24f^{9}-64f^{8}-80f^{7}-128f^{6}-128f^{5}-160f^{4}$$
$$-112f^{3}-128f^{2}-64f-32=0.$$

$\mathbf{n=179}$:

$$f^{15}-*-20f^{13}-62f^{12}-76f^{11}-32f^{10}+16f^{9}+8f^{8}-64f^{7}-160f^{6}-176f^{5}$$
$$-96f^{4}-16f^{3}+32f^{2}-*-32=0.$$

$\mathbf{n=227}$:

$$f^{15}-6f^{14}-2f^{13}-38f^{12}-48f^{11}-96f^{10}-120f^{9}-96f^{8}-112f^{7}-96f^{6}-96f^{5}$$
$$-160f^{4}-48f^{3}-96f^{2}+32f-32=0.$$

$\mathbf{n=347}$:

$$f^{15}-14f^{14}+30f^{13}-2f^{12}-88f^{11}-8f^{10}+128f^{9}-32f^{8}-208f^{7}+80f^{6}+64f^{5}$$
$$-256f^{4}+112f^{3}+32f^{2}-96f-32=0.$$

$\mathbf{n=443}$:

$$f^{15}-18f^{14}+46f^{13}-174f^{12}+236f^{11}-352f^{10}+312f^{9}-280f^{8}+288f^{7}-480f^{6}$$
$$+848f^{5}-928f^{4}+784f^{3}-576f^{2}+224f-32=0.$$

$\mathbf{n=523}$:

$$f^{15}-16f^{14}-70f^{13}-166f^{12}-244f^{11}-264f^{10}-216f^{9}-296f^{8}-528f^{7}-960f^{6}$$
$$-1168f^{5}-1152f^{4}-688f^{3}-352f^{2}-32f-32=0.$$

$\mathbf{n=571}$:

$$f^{15}-24f^{14}+32f^{13}-122f^{12}+72f^{11}-112f^{10}+16f^{9}+240f^{8}-144f^{7}+304f^{6}$$
$$-256f^{5}-192f^{4}-80f^{3}-256f^{2}+192f-32=0.$$

$\mathbf{n=619}$:

$$f^{15}-20f^{14}-136f^{13}-454f^{12}-896f^{11}-1112f^{10}-520f^{9}+704f^{8}+1664f^{7}$$
$$+1184f^{6}-224f^{5}-1376f^{4}-1264f^{3}-640f^{2}-128f-32=0.$$

n = 683:

$$f^{15}-30f^{14}-26f^{13}+242f^{12}-192f^{11}-1040f^{10}+3096f^{9}-3968f^{8}+2896f^{7}$$
$$-1728f^{6}+1632f^{5}-1696f^{4}+1168f^{3}-480f^{2}+32f-32=0.$$

n = 691:

$$f^{15}-30f^{14}-36f^{13}-54f^{12}-228f^{11}-264f^{10}+80f^{9}-264f^{8}-528f^{7}-320f^{6}$$
$$-48f^{5}-*-272f^{3}-384f^{2}+192f-32=0.$$

n = 739:

$$f^{15}-36f^{14}+32f^{13}-26f^{12}+48f^{11}-328f^{10}+808f^{9}-1344f^{8}+1728f^{7}-2096f^{6}$$
$$+2048f^{5}-1728f^{4}+1168f^{3}-832f^{2}+192f-32=0.$$

n = 787:

$$f^{15}-40f^{14}+30f^{13}-158f^{12}+80f^{11}-176f^{10}-248f^{9}+144f^{8}-752f^{7}-64f^{6}$$
$$-448f^{5}-864f^{4}-48f^{3}-576f^{2}-224f-32=0.$$

n = 947:

$$f^{15}-56f^{14}+4f^{13}-714f^{12}-1044f^{11}-432f^{10}+1792f^{9}+1160f^{8}-3344f^{7}$$
$$-9488f^{6}-10448f^{5}-6272f^{4}-1616f^{3}+96f^{2}+128f-32=0.$$

n = 1051:

$$f^{15}-72f^{14}+168f^{13}-358f^{12}+52f^{11}+584f^{10}-240f^{9}-1080f^{8}+96f^{7}+1024f^{6}$$
$$-272f^{5}-640f^{4}+1584f^{3}-544f^{2}-320f-32=0.$$

n = 1123:

$$f^{15}-80f^{14}-340f^{13}-866f^{12}-1252f^{11}-608f^{10}+1432f^{9}+3128f^{8}+2336f^{7}$$
$$-848f^{6}-3920f^{5}-4768f^{4}-3056f^{3}-736f^{2}+128f-32=0.$$

n = 1723:

$$f^{15}-224f^{14}-1126f^{13}-1870f^{12}-576f^{11}+1040f^{10}-1176f^{9}-5184f^{8}-4400f^{7}$$
$$+96f^{6}+288f^{5}-4576f^{4}-6448f^{3}-2368f^{2}+608f-32=0.$$

$\mathbf{n = 1747}$:

$$f^{15}-232f^{14}-1352f^{13}-3714f^{12}-6672f^{11}-8936f^{10}-10112f^{9}-12752f^{8}$$
$$-19856f^{7}-29328f^{6}-34976f^{5}-31456f^{4}-19024f^{3}-6272f^{2}-704f-32=0.$$

$\mathbf{n = 1867}$:

$$f^{15}-280f^{14}-1702f^{13}-4610f^{12}-6240f^{11}-2776f^{10}+6384f^{9}+12672f^{8}$$
$$+13136f^{7}+1968f^{6}-5408f^{5}-12480f^{4}-7248f^{3}-4416f^{2}+96f-32=0.$$

$\mathbf{n = 2203}$:

$$f^{15}-466f^{14}+52f^{13}+2174f^{12}-2272f^{11}-7408f^{10}+18528f^{9}-14944f^{8}$$
$$-8240f^{7}+27760f^{6}-21696f^{5}-3776f^{4}+14256f^{3}-5600f^{2}-896f-32=0.$$

$\mathbf{n = 2347}$:

$$f^{15}-566f^{14}-950f^{13}-2f^{12}+1676f^{11}+1688f^{10}-1216f^{9}-5080f^{8}-3520f^{7}$$
$$+1136f^{6}+3056f^{5}+5888f^{4}-3728f^{3}-9152f^{2}-224f-32=0.$$

$\mathbf{n = 2683}$:

$$f^{15}-882f^{14}+1474f^{13}-6162f^{12}+9456f^{11}-17240f^{10}+21104f^{9}-24960f^{8}$$
$$+31632f^{7}-38096f^{6}+46656f^{5}-44480f^{4}+29872f^{3}-16608f^{2}$$
$$+1376f-32=0.$$

The 15-ic equations satisfied by f_{131} and f_{179} were first constructed by Mitra†; no other results about these twenty-one class-invariants have been published.

Analogy with the class-invariants associated with the determinants with classification G.I, 9 suggests that the 15-ic equations are reducible to the form

$$f^{3}-2\alpha f^{2}+2\beta f-2=0,$$

where α and β are now quintic irrationalities. I proceed to state the

† S. C. Mitra, *Indian Phys. Math. Journal*, 2 (1931), 7–10 (8). Mitra has not explained how he constructed these equations.

quintic equations satisfied by α and β, and the corresponding solutions of these equations are taken to be

$$10\alpha_r = a+u_1\omega_r+u_2\omega_r^2+u_3\omega_r^3+u_4\omega_r^4,$$

$$10\beta_r = b+v_1\omega_r+v_2\omega_r^2+v_3\omega_r^3+v_4\omega_r^4,$$

where, as usual,

$$\omega_r = \exp\frac{2r\pi i}{5}$$

and r assumes the values 0, 1, 2, 3, 4 in succession. In accordance with the conventions used throughout the paper, the upper signs in ambiguities refer to u_1, u_2, v_1, v_2, and the lower signs to u_4, u_3, v_4, v_3.

I also give the expression† for α as a quartic polynomial in β with rational coefficients. The importance of this expression is due to the fact that the eighth powers of the roots of the equation

$$f^3-2\alpha f^2+2\beta f-2=0$$

satisfy the equation

$$f^{24}-[64\{2(\alpha^2-\beta)^2-(\beta^2-2\alpha)\}^2-32\{(\beta^2-2\alpha)^2-2(\alpha^2-\beta)\}]f^{16}$$
$$+[256\{(\beta^2-2\alpha)^2-2(\alpha^2-\beta)\}^2-256\{2(\alpha^2-\beta)^2-(\beta^2-2\alpha)\}]f^8-256=0;$$

and since, in each of the cases under consideration, we have

$$\{(\beta^2-2\alpha)^2-2(\alpha^2-\beta)\}^2-\{2(\alpha^2-\beta)^2-(\beta^2-2\alpha)\}=0,$$

$$64\{2(\alpha^2-\beta)^2-(\beta^2-2\alpha)\}^2-32\{(\beta^2-2\alpha)^2-2(\alpha^2-\beta)\}=-\gamma_2(-\tfrac{3}{2}+\tfrac{1}{2}\tau),$$

it follows that the class-invariant $-\gamma_2(-\frac{3}{2}+\frac{1}{2}\tau)$ is expressible as a rational function of the single quintic irrationality β. I did not originally intend to calculate this class-invariant for any of the twenty-one values of n under

† To this expression is appended the value of α as a quadratic fraction in β. This, being a (1, 2) relation, is from the aspect of function-theory more fundamental than the (1, 4) relation provided by the quartic polynomial, though the quartic polynomial is the more conventional expression in the theory of numbers. The coefficients in the quadratic fractions are, on the whole, much smaller than those occurring in the corresponding quartic polynomials.

consideration†; the data which I give make it a simple matter for anyone with the requisite perseverance to carry out the calculations.

The solution of the cubic

$$f^3-2\alpha f^2+2\beta f-2=0$$

with the values given for α and β, gives the values of $f(\tau)$ in terms of radicals. These values are not the canonical forms of the solutions of the 15-ic equations, and I have not worked out these canonical forms‡ since they are not required in the determination of the class-invariants $-\gamma_2(-\frac{3}{2}+\frac{1}{2}\tau)$.

n = 131. The quintics satisfied by α and β are

$$\alpha^5-2\alpha^4+\alpha^3-4\alpha^2+7\alpha-4=0,$$

$$\beta^5+2\beta^4+2\beta^3-3\beta^2-4\beta+4=0.$$

The roots of these quintics are derived from the following results:

$$a=4, \quad b=-4,$$

$$u_1^5,\ u_4^5 = 65424+29200\sqrt{5}\mp(1920+800\sqrt{5})\sqrt{131(5+2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 65424-29200\sqrt{5}\pm(1920-800\sqrt{5})\sqrt{131(5-2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = -72224+34880\sqrt{5}\pm(9120-4000\sqrt{5})\sqrt{131(5+2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = -72224-34880\sqrt{5}\mp(9120+4000\sqrt{5})\sqrt{131(5-2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$8\alpha=10-\beta-2\beta^2-4\beta^3-\beta^4,$$

$$\alpha=\frac{2+\beta-\beta^2}{2\beta+\beta^2}.$$

n = 179. The quintics satisfied by α and β are

$$\alpha^5-*-5\alpha^3-16\alpha^2-16\alpha-8=0,$$

$$\beta^5-*-\beta^3-16\beta^2-19\beta-14=0.$$

---

† It was suggested by a Referee that such a calculation might be given for one of the twenty-one class-invariants; accordingly I have inserted at the end of my results both the expression for $-\gamma_2(-\frac{3}{2}+\frac{1}{2}\tau)$ as a quartic polynomial in β and the quintic equation satisfied by $-\gamma_2(-\frac{3}{2}+\frac{1}{2}\tau)$ in the case $n=2683$.

‡ There was a similar omission for the same reason in § 8, in the discussion of the first fourteen discriminants with Gaussian classification G. I, 9.

The roots of these quintics are derived from the following results:

$$a = 0, \quad b = 0,$$

$$u_1^5,\ u_4^5 = 2\,20000 - 59200\sqrt{5} \mp (8720 + 80\sqrt{5})\sqrt{179(5-2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 2\,20000 + 59200\sqrt{5} \mp (8720 - 80\sqrt{5})\sqrt{179(5+2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = 3\,16000 - 58400\sqrt{5} \pm (14960 + 1840\sqrt{5})\sqrt{179(5-2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = 3\,16000 + 58400\sqrt{5} \pm (14960 - 1840\sqrt{5})\sqrt{179(5+2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$28\alpha = -28 - 16\beta + 7\beta^2 + \beta^3 + \beta^4,$$

$$\alpha = \frac{-10 - 6\beta + 2\beta^2}{9 - \beta - \beta^2}.$$

n = 227. The quintics satisfied by α and β are

$$\alpha^5 - 3\alpha^4 - \alpha^3 - 10\alpha^2 + 4\alpha - 8 = 0,$$

$$\beta^5 - \beta^4 - * - 9\beta^2 + 7\beta - 2 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 6, \quad b = 2,$$

$$u_1^5,\ u_4^5 = 1\,68176 - 14000\sqrt{5} \mp (9280 - 4000\sqrt{5})\sqrt{227(5+2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 1\,68176 + 14000\sqrt{5} \pm (9280 + 4000\sqrt{5})\sqrt{227(5-2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = 64832 + 14080\sqrt{5} \pm (2720 - 320\sqrt{5})\sqrt{227(5+2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = 64832 - 14080\sqrt{5} \mp (2720 + 320\sqrt{5})\sqrt{227(5-2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$5\alpha = 8 - 17\beta + 2\beta^2 - \beta^3 + 2\beta^4,$$

$$\alpha = \frac{2 - 4\beta}{1 + \beta - \beta^2}.$$

n = 347. The quintics satisfied by α and β are

$$\alpha^5 - 7\alpha^4 + 9\alpha^3 - * - * - 8 = 0,$$

$$\beta^5 + 3\beta^5 - 8\beta^3 - 3\beta^2 + \beta + 10 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 14, \quad b = -6,$$

$$u_1^5,\ u_4^5 = 4\,57024 + 2\,00640\sqrt{5} \mp (7120 + 2960\sqrt{5})\sqrt{347(5+2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 4\,57024 - 2\,00640\sqrt{5} \pm (7120 - 2960\sqrt{5})\sqrt{347(5-2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = -5\,18976 + 2\,28800\sqrt{5} \pm (30880 - 13760\sqrt{5})\sqrt{347(5+2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = -5\,18976 - 2\,28800\sqrt{5} \mp (30880 + 13760\sqrt{5})\sqrt{347(5-2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$16\alpha = -14 - 5\beta + 10\beta^2 + 6\beta^3 + \beta^4,$$

$$\alpha = \frac{-2 + 2\beta^2}{3 - \beta}.$$

n = 443. The quintics satisfied by α and β are

$$\alpha^5 - 9\alpha^4 + 8\alpha^3 - 13\alpha^2 + 5\alpha - 8 = 0,$$

$$\beta^5 - 7\beta^4 + 9\beta^3 - 7\beta^2 + 9\beta - 7 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 18, \quad b = 14,$$

$$u_1^5,\ u_4^5 = 17\,15968 + 86560\sqrt{5} \pm (34880 + 5440\sqrt{5})\sqrt{443(5-2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 17\,15968 - 86560\sqrt{5} \pm (34880 - 5440\sqrt{5})\sqrt{443(5+2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = 4\,25024 + 1\,64960\sqrt{5} \pm (7280 - 1680\sqrt{5})\sqrt{443(5+2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = 4\,25024 - 1\,64960\sqrt{5} \mp (7280 + 1680\sqrt{5})\sqrt{443(5-2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$10\alpha = -1 - \beta + 6\beta^2 + 5\beta^3 - \beta^4,$$

$$\alpha = \frac{-2\beta + 6\beta^2}{7 - 3\beta + \beta^2}.$$

n = 523. The quintics satisfied by α and β are

$$\alpha^5 - 8\alpha^4 - 17\alpha^3 - 34\alpha^2 - 9\alpha - 8 = 0,$$

$$\beta^5 + \beta^4 + 3\beta^3 - 5\beta^2 - \beta - 3 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 16, \quad b = -2,$$

$$u_1^5,\ u_4^5 = 69\,39776 - 13\,75360\sqrt{5} \pm (1\,52400 + 15600\sqrt{5})\sqrt{523(5-2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 69\,39776 + 13\,75360\sqrt{5} \pm (1\,52400 - 15600\sqrt{5})\sqrt{523(5+2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = 88368 - 33360\sqrt{5} \mp (1600 - 320\sqrt{5})\sqrt{523(5-2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = 88368 + 33360\sqrt{5} \mp (1600 + 320\sqrt{5})\sqrt{523(5+2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$2\alpha = 2 + \beta + 6\beta^2 + 2\beta^3 + \beta^4,$$

$$\alpha = \frac{-2\beta + 2\beta^2}{3 - \beta - \beta^2}.$$

n = 571. The quintics satisfied by α and β are

$$\alpha^5 - 12\alpha^4 + 5\alpha^3 - 20\alpha^2 + 25\alpha - 8 = 0,$$

$$\beta^5 - 6\beta^4 - 4\beta^3 - 7\beta^2 + 28\beta - 16 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 24, \quad b = 12,$$

$$u_1^5,\ u_4^5 = 95\,64624 + 10\,35600\sqrt{5} \mp (2\,23520 - 63680\sqrt{5})\sqrt{571(5+2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 95\,64624 - 10\,35600\sqrt{5} \pm (2\,23520 + 63680\sqrt{5})\sqrt{571(5-2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = 13\,43632 + 4\,02000\sqrt{5} \pm (21840 - 1680\sqrt{5})\sqrt{571(5+2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = 13\,43632 - 4\,02000\sqrt{5} \mp (21840 + 1680\sqrt{5})\sqrt{571(5-2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$28\alpha = 24 - 11\beta + 4\beta^2 - 6\beta^3 + \beta^4,$$

$$\alpha = \frac{20 - 9\beta + \beta^2}{16 + 11\beta - 2\beta^2}.$$

n = 619. The quintics satisfied by α and β are

$$\alpha^5 - 10\alpha^4 - 32\alpha^3 - 71\alpha^2 - 60\alpha - 16 = 0,$$

$$\beta^5 + 4\beta^4 + 10\beta^3 + 11\beta^2 + 6\beta + 4 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 20, \quad b = -8,$$

$$u_1^5,\ u_4^5 = 207\,10000 - 3\,09200\sqrt{5} \pm (5\,70000 - 1\,86000\sqrt{5})\sqrt{619(5+2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 207\,10000 + 3\,09200\sqrt{5} \mp (5\,70000 + 1\,86000\sqrt{5})\sqrt{619(5-2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = -13568 + 14240\sqrt{5} \mp (1440 - 480\sqrt{5})\sqrt{619(5-2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = -13568 - 14240\sqrt{5} \pm (1440 + 480\sqrt{5})\sqrt{619(5-2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$2\alpha = 8 + 7\beta + 20\beta^2 + 8\beta^3 + 3\beta^4,$$

$$\alpha = -\frac{2 + 2\beta + 2\beta^2}{2 + 3\beta + \beta^2}.$$

n = 683. The quintics satisfied by α and β are

$$\alpha^5 - 15\alpha^4 - 7\alpha^3 + 44\alpha^2 + 80\alpha - 128 = 0,$$

$$\beta^5 - \beta^4 - * - 9\beta^2 - 3\beta + 28 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 30, \quad b = 2,$$

$$u_1^5,\ u_4^5 = 437\,80000 - 147\,99200\sqrt{5} \mp (3\,70000 + 26000\sqrt{5})\sqrt{683(5-2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 437\,80000 + 147\,99200\sqrt{5} \mp (3\,70000 - 26000\sqrt{5})\sqrt{683(5+2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = -4\,31168 - 1\,97280\sqrt{5} \pm (2304 + 1024\sqrt{5})\sqrt{683(5-2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = -4\,31168 + 1\,97280\sqrt{5} \pm (2304 - 1024\sqrt{5})\sqrt{683(5+2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$5\alpha = 24-16\beta+6\beta^2-3\beta^3+\beta^4,$$

$$\alpha = \frac{4-\beta+\beta^2}{2+\beta}.$$

n = 691. The quintics satisfied by α and β are

$$\alpha^5-15\alpha^4-12\alpha^3-15\alpha^2+\alpha-24 = 0,$$

$$\beta^5-6\beta^4-3\beta^3-8\beta^2+11\beta-4 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 30, \quad b = 12,$$

$$u_1^5,\ u_4^5 = 434\,60000+98\,79200\mp(4\,20000+48000\sqrt{5})\sqrt{691(5+2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 434\,60000-98\,79200\pm(4\,20000-48000\sqrt{5})\sqrt{691(5-2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = 10\,89232+3\,21680\sqrt{5}\pm 18720\sqrt{691(5+2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = 10\,89232-3\,21680\sqrt{5}\mp 18720\sqrt{691(5-2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$18\alpha = 28-37\beta-11\beta^2-23\beta^3+4\beta^4,$$

$$\alpha = \frac{16-18\beta-4\beta^2}{1+17\beta-3\beta^2}.$$

n = 739. The quintics satisfied by α and β are

$$\alpha^5-18\alpha^4+5\alpha^3+4\alpha^2+103\alpha-104 = 0,$$

$$\beta^5-6\beta^4+8\beta^3-22\beta^2+79\beta-76 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 36, \quad b = 12,$$

$$u_1^5,\ u_4^5 = 758\,30576-192\,86160\sqrt{5}$$
$$\mp(9\,20640+1\,46400\sqrt{5})\sqrt{739(5-2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 758\,30576+192\,86160\sqrt{5}$$
$$\mp(9\,20640-1\,46400\sqrt{5})\sqrt{739(5+2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = 9\,84432-4\,13040\sqrt{5}\pm(12720-4240\sqrt{5})\sqrt{739(5-2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = 9\,84432+4\,13040\sqrt{5}\pm(12720+4240\sqrt{5})\sqrt{739(5+2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$16\alpha = 36-17\beta+7\beta^2-3\beta^3+\beta^4,$$

$$\alpha = \frac{16-6\beta+2\beta^2}{5+4\beta-\beta^2}.$$

n = 787. The quintics satisfied by α and β are

$$\alpha^5-20\alpha^4+11\alpha^3-20\alpha^2+49\alpha-30 = 0,$$

$$\beta^5+7\beta^4-2\beta^3+\beta^2-21\beta+18 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 40, \quad b = -14,$$

$$u_1^5,\ u_4^5 = 1010\,76000-131\,92000\sqrt{5}$$
$$\mp(13\,34640+4\,43760\sqrt{5})\sqrt{787(5-2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 1010\,76000+131\,92000\sqrt{5}$$
$$\mp(13\,34640-4\,43760\sqrt{5})\sqrt{787(5+2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = -24\,76624-10\,07440\sqrt{5}$$
$$\pm(1\,08720+47760\sqrt{5})\sqrt{787(5-2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = -24\,76624+10\,07440\sqrt{5}$$
$$\pm(1\,08720-47760\sqrt{5})\sqrt{787(5+2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$52\alpha = 58-15\beta+16\beta^2-22\beta^3-3\beta^4,$$

$$\alpha = \frac{12-\beta-\beta^2}{3+8\beta+\beta^2}.$$

n = 947. The quintics satisfied by α and β are

$$\alpha^5-28\alpha^4-\alpha^3-162\alpha^2+27\alpha-40 = 0,$$

$$\beta^5-4\beta^4-31\beta^3-95\beta^2-66\beta-29 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 56, \quad b = 8,$$

$$u_1^5,\ u_4^5 = 6838\,64576 + 193\,58240\sqrt{5} \pm (127\,68080 + 47\,85840\sqrt{5})\sqrt{947(5-2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 6838\,64576 - 193\,58240\sqrt{5} \pm (127\,68080 - 47\,85840\sqrt{5})\sqrt{947(5+2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = 68\,56368 - 7\,93360\sqrt{5}) \mp (1\,37120 + 39840\sqrt{5})\sqrt{947(5-2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = 68\,56368 + 7\,93360\sqrt{5} \mp (1\,37120 - 39840\sqrt{5})\sqrt{947(5+2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$200\alpha = -61 - 211\beta - 22\beta^2 - 13\beta^3 + 3\beta^4,$$

$$\alpha = -\frac{5 + 22\beta + \beta^2}{24 + 5\beta - \beta^2}.$$

n = 1051. The quintics satisfied by α and β are

$$\alpha^5 - 36\alpha^4 + 47\alpha^3 - 25\alpha^2 + 8\alpha - 27 = 0,$$

$$\beta^5 + 10\beta^4 - 19\beta^3 - 10\beta^2 + 13\beta + 42 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 72, \quad b = -20,$$

$$u_1^5,\ u_4^5 = 15791\,46432 + 1347\,71040\sqrt{5} \mp (45\,04560 + 2\,75120\sqrt{5})\sqrt{1051(5-2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 15791\,46432 - 1347\,71040\sqrt{5} \mp (45\,04560 - 2\,75120\sqrt{5})\sqrt{1051(5+2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = -126\,54000 + 41\,38800\sqrt{5} \mp (1\,22560 - 42240\sqrt{5})\sqrt{1051(5+2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = -126\,54000 - 41\,38800\sqrt{5} \pm (1\,22560 + 42240\sqrt{5})\sqrt{1051(5-2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$282\alpha = -48+37\beta+154\beta^2-28\beta^3-3\beta^4,$$

$$\alpha = \frac{-9+21\beta+87\beta^2}{157+43\beta+5\beta^2}.$$

n = 1123. The quintics satisfied by α and β are

$$\alpha^5-38\alpha^4-87\alpha^3-177\alpha^2-110\alpha-29 = 0,$$

$$\beta^5-4\beta^4-15\beta^3-42\beta^2-29\beta-10 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 76, \quad b = 8,$$

$$u_1^5,\ u_4^5 = 37772\,06576-338\,18640\sqrt{5}$$
$$\mp(328\,13760+120\,24960\sqrt{5})\sqrt{1123(5-2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 37772\,06576+338\,18640\sqrt{5}$$
$$\mp(328\,13760-120\,24960\sqrt{5})\sqrt{1123(5+2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = 21\,61968+1\,19280\sqrt{5}$$
$$\pm(55680+19520\sqrt{5})\sqrt{1123(5-2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = 21\,61968-1\,19280\sqrt{5}$$
$$\pm(55680-19520\sqrt{5})\sqrt{1123(5+2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$62\alpha = -38-34\beta+13\beta^2-\beta^3+\beta^4,$$

$$\alpha = -\frac{19-19\beta+5\beta^2}{31+3\beta-\beta^2}.$$

n = 1723. The quintics satisfied by α and β are

$$\alpha^5-112\alpha^4-291\alpha^3-373\alpha^2-238\alpha-75 = 0,$$

$$\beta^5-19\beta^4-38\beta^3-77\beta^2-63\beta-40 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 224, \quad b = 38,$$

$$u_1^5, u_4^5 = 65\,05317\,70224 + 7551\,15600\sqrt{5} \mp (4909\,44000 - 309\,38240\sqrt{5})\sqrt{1723(5+2\sqrt{5})},$$

$$u_2^5, u_3^5 = 65\,05317\,70224 - 7551\,15600 \pm (4909\,44000 + 309\,38240\sqrt{5})\sqrt{1723(5-2\sqrt{5})},$$

$$v_1^5, v_4^5 = 1653\,92368 + 2\,72560\sqrt{5} \pm (13\,48560 - 2\,42640\sqrt{5})\sqrt{1723\,(5+2\sqrt{5})},$$

$$v_2^5, v_3^5 = 1653\,92368 - 2\,72560\sqrt{5} \mp (13\,48560 + 2\,42640\sqrt{5})\sqrt{1723(5-2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$395\alpha = -445 - 389\beta - 108\beta^2 - 73\beta^3 + 4\beta^4,$$

$$\alpha = -\frac{5+21\beta+7\beta^2}{11+19\beta-\beta^2}.$$

n = 1747. The quintics satisfied by α and β are

$$\alpha^5 - 116\alpha^4 - 327\alpha^3 - 667\alpha^2 - 582\alpha - 325 = 0,$$

$$\beta^5 + 22\beta^4 + 80\beta^3 + 189\beta^2 + 216\beta + 140 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 232, \quad b = -44,$$

$$u_1^5, u_4^5 = 78\,35479\,87632 + 11434\,93680\sqrt{5} \pm (19982\,52480 - 8712\,67520\sqrt{5})\sqrt{1747(5+2\sqrt{5})},$$

$$u_2^5, u_3^5 = 78\,35479\,87632 - 11434\,93680\sqrt{5} \mp (19982\,52480 + 8712\,67520\sqrt{5})\sqrt{1747(5-2\sqrt{5})},$$

$$v_1^5, v_4^5 = -653\,88624 + 85\,80880\sqrt{5} \mp (12\,29520 - 5\,39760\sqrt{5})\sqrt{1747(5+2\sqrt{5})},$$

$$v_2^5, v_3^5 = -653\,88624 - 85\,80880\sqrt{5} \pm (12\,29520 + 5\,39760\sqrt{5})\sqrt{1747(5-2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$1764\alpha = -2030-1067\beta+184\beta^2-4\beta^3+\beta^4,$$

$$\alpha = \frac{-30-17\beta+2\beta^2}{26+\beta}.$$

n = 1867. The quintics satisfied by α and β are

$$\alpha^5-140\alpha^4-427\alpha^3-703\alpha^2-318\alpha-45 = 0,$$

$$\beta^5-3\beta^4-2\beta^3-27\beta^2+\beta-6 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 280, \quad b = 6,$$

$$u_1^5,\ u_4^5 = 197\,15169\,30000-3\,09045\,04400\sqrt{5}$$
$$\pm(31718\,91840+9965\,00160\sqrt{5})\sqrt{1867(5-2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 197\,15169\,30000+3\,09045\,04400\sqrt{5}$$
$$\pm(31718\,91840-9965\,00160\sqrt{5})\sqrt{1867(5+2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = 6\,80976+87760\sqrt{5}\mp(15600+5520\sqrt{5})\sqrt{1867(5-2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = 6\,80976-87760\sqrt{5}\mp(15600-5520\sqrt{5})\sqrt{1867(5+2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$6\alpha = \beta+8\beta^2+2\beta^3+\beta^4,$$

$$\alpha = \frac{-1+\beta+2\beta^2}{5-\beta}.$$

n = 2203. The quintics satisfied by α and β are

$$\alpha^5-233\alpha^4+27\alpha^3+494\alpha^2+204\alpha-592 = 0,$$

$$\beta^5+28\beta^4-58\beta^3-\beta^2+10\beta+36 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 466, \quad b = -56,$$

$$u_1^5,\ u_4^5 = 2193\,37609\,33776 + 54\,35363\,86800\sqrt{5} \mp (90920\,73440 - 21682\,40000\sqrt{5})\sqrt{2203(5+2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 2193\,37609\,33776 - 54\,35363\,86800\sqrt{5} \pm (90920\,73440 + 21682\,40000\sqrt{5})\sqrt{2203(5-2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = -8832\,72576 + 1330\,49280\sqrt{5} \mp (29\,65280 - 10\,32000\sqrt{5})\sqrt{2203(5+2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = -8832\,72576 - 1330\,49280\sqrt{5} \mp (29\,65280 + 10\,32000\sqrt{5})\sqrt{2203(5-2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$97\alpha = -2 + 83\beta + 170\beta^2 - 115\beta^3 - 4\beta^4,$$

$$\alpha = \frac{-46 + \beta + 5\beta^2}{17 - 30\beta - \beta^2}.$$

n = 2347. The quintics satisfied by α and β are

$$\alpha^5 - 283\alpha^4 - 234\alpha^3 - 58\alpha^2 + 89\alpha - 27 = 0,$$

$$\beta^5 + 7\beta^4 + 3\beta^3 + \beta^2 - \beta + 3 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 566, \quad b = -14,$$

$$u_1^5,\ u_4^5 = 5916\,97186\,80576 + 49\,86314\,34240\sqrt{5} \mp (2\,15160\,80640 - 84862\,64320\sqrt{5})\sqrt{2347(5+2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 5916\,97186\,80576 - 49\,86314\,34240\sqrt{5} \pm (2\,15160\,80640 + 84862\,64320\sqrt{5})\sqrt{2347(5-2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = -7\,92624 + 2\,99760\sqrt{5} \mp (4480 - 1760\sqrt{5})\sqrt{2347(5-2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = -7\,92624 - 2\,99760\sqrt{5} \mp (4480 + 1760\sqrt{5})\sqrt{2347(5+2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$\alpha = -\beta^3,$$

$$\alpha = \frac{3-\beta+\beta^2}{3+7\beta+\beta^2}.$$

n = 2683. The quintics satisfied by α and β are

$$\alpha^5-441\alpha^4+347\alpha^3-347\alpha^2+220\alpha-68 = 0,$$

$$\beta^5-43\beta^4+78\beta^3-87\beta^2+65\beta-50 = 0.$$

The roots of these quintics are derived from the following results:

$$a = 882, \quad b = 86,$$

$$u_1^5,\ u_4^5 = 52788\,65732\,74832+142\,59933\,11280\sqrt{5}$$
$$\mp(2\,96898\,88440-73159\,54920\sqrt{5})\sqrt{2683(5+2\sqrt{5})},$$

$$u_2^5,\ u_3^5 = 52788\,65732\,74832-142\,59933\,11280\sqrt{5}$$
$$\pm(2\,96898\,88440+73159\,54920\sqrt{5})\sqrt{2683(5-2\sqrt{5})},$$

$$v_1^5,\ v_4^5 = 36647\,82576+1953\,35280\sqrt{5}$$
$$\mp(44\,05840-8\,68880\sqrt{5})\sqrt{2683(5-2\sqrt{5})},$$

$$v_2^5,\ v_3^5 = 36647\,82576-1953\,35280\sqrt{5}$$
$$\mp(44\,05840+8\,68880\sqrt{5})\sqrt{2683(5+2\sqrt{5})}.$$

Equations which express α rationally in terms of β are

$$735\alpha = 575-693\beta+717\beta^2-177\beta^3+4\beta^4,$$

$$\alpha = \frac{95-80\beta+81\beta^2}{105+46\beta-\beta^2}.$$

[*Added* 26 *November*, 1934. As already stated, it is a simple matter to express the class-invariant $\gamma_2(-\frac{3}{2}+\frac{1}{2}\tau)$ as a quartic polynomial in β for each of the twenty-one values of n now under consideration. By using the

method already explained, I find that, for $n = 2683$,

$$-\tfrac{4}{3}\tfrac{9}{2}\gamma_2(-\tfrac{3}{2}+\tfrac{1}{2}i\sqrt{2683}) = 244\,59172\,55602\,13730 - 312\,02608\,62104\,49996\beta$$
$$+418\,00790\,68153\,63974\beta^2 - 371\,40622\,07892\,42294\beta^3$$
$$+201\,32435\,34810\,36873\,\beta^4,$$

where β is the real root of the quintic equation given above.

Further, the quintic equation with integral coefficients which has $-\gamma_2(-\frac{3}{2}+\frac{1}{2}i\sqrt{2683})$ for its real root is

$$x^5 - 3607\,26554\,51292\,77422\,30560\,x^4$$
$$+4\,25827\,57738\,34297\,09011\,96800\,x^3$$
$$-708\,44844\,07314\,82683\,25724\,16000\,x^2$$
$$+2471\,74352\,38657\,65863\,81721\,60000\,x$$
$$-73777\,64763\,01144\,54870\,42560\,00000 = 0.$$

This equation was obtained from the 15-ic equation satisfied by f_{2683} by constructing the 15-ic equation satisfied by f_{2683}^8 (by repeated squarings, after the manner of Graefe) and then making suitable rearrangements.]

The equations satisfied by the class-invariants corresponding to the six members of the second group are as follows:

n = 239:

$$F^{15} - 6F^{14} + 2F^{13} + 8F^{12} + 4F^{11} - 27F^{10} + 13F^9 + 15F^8 - 4F^7$$
$$-20F^6 + 13F^5 + 5F^4 - 4F^3 - 4F^2 + 4F - 1 = 0.$$

By means of Berwick's cubic

$$\theta^3 - 3\theta^2 + 2\theta - 3 = 0 \qquad (\Delta = -239)$$

this is reducible to

$$F^{-5} + (\theta^2 - \theta - 2)\,F^{-4} + 3F^{-3} - (\theta^2 - 4\theta + 2)\,F^{-2}$$
$$+(2\theta^2 - 6\theta + 2)\,F^{-1} - (\theta^2 - 3\theta + 1) = 0.$$

n = 439:

$$F^{15} - 13F^{14} + 23F^{13} - 7F^{12} - 24F^{11} + 20F^{10} + 13F^9 - 38F^8 + 29F^7$$
$$+F^6 - 17F^5 + 7F^{34} + 9F^3 - 11F^2 + 5F - 1 = 0.$$

By means of Berwick's cubic

$$\theta^3-2\theta^2-\theta-3=0 \qquad (\Delta=-439)$$

this is reducible to

$$F^{-5}-(\theta^2-2\theta+1)\,F^{-4}+(\theta^2-3)\,F^{-3}+(\theta^2-4\theta)\,F^{-2}$$
$$-(\theta-4)\,F^{-1}-(\theta^2-2\theta-2)=0.$$

n = 751:

$$F^{15}-25F^{14}-12F^{13}-53\,F^{12}+9F^{11}-21F^{10}+4F^9-2F^8+29F^7$$
$$-17F^6+12F^5-7F^4+6F^3-9F^2+5F-1=0.$$

By means of Berwick's cubic

$$\theta^3-\theta^2+6\theta-1=0 \qquad (\Delta=-751)$$

this is reducible to

$$F^{-5}+(\theta^2+2)\,F^{-4}-(2\theta-3)\,F^{-3}+(2\theta+1)\,F^{-2}+(\theta^2-2\theta+1)\,F^{-1}-\theta^2=0.$$

n = 1327:

$$F^{15}-83F^{14}-21F^{13}+18F^{12}+100F^{11}-13F^{10}-20F^9-52F^8+3F^7$$
$$-24F^6+F^5-38F^4-24F^3-16F^2-6F-1=0.$$

By means of Berwick's cubic

$$\theta^3+*+\theta-7=0 \qquad (\Delta=-1327)$$

this is reducible to

$$F^{-5}+2F^{-4}+(\theta^2-\theta+2)\,F^{-3}+2\theta F^{-2}+(\theta^2-1)\,F^{-1}-(\theta^2-3)=0.$$

n = 1567:

$$F^{15}-129F^{14}+401F^{13}-670F^{12}+650F^{11}-532F^{10}+522F^9-406F^8$$
$$+330F^7-317F^6+182F^5-76F^4+44F^3-13F^2+F-1=0.$$

By means of Berwick's cubic

$$\theta^3-\theta^2-2\theta-7=0 \qquad (\Delta=-1567)$$

this is reducible to

$$F^{-5}-(\theta^2-\theta-1)\,F^{-4}+(\theta+2)\,F^{-3}-(\theta^2-2\theta+1)\,F^{-2}$$
$$+(\theta^2-2\theta-1)\,F^{-1}+(\theta^2+\theta-10)=0.$$

n = 2647:

$$F^{15}-595F^{14}+167F^{13}+859F^{12}-140F^{11}-277F^{10}+242F^{9}-422F^{8}$$
$$-778F^{7}-332F^{6}-173F^{5}-193F^{4}-61F^{3}-2F^{2}+4F-1=0.$$

By means of Berwick's cubic

$$\theta^3-\theta^2+8\theta+3=0 \qquad (\Delta=-2647)$$

this is reducible to

$$F^{-5}+(2\theta-2)F^{-4}+(3\theta^2-\theta+4)\,F^{-3}+(3\theta^2-7\theta-7)\,F^{-2}$$
$$+(\theta^2-12\theta-3)\,F^{-1}-(\theta^2+6\theta+2)=0.$$

The 15-ic equation satisfied by F_{239} was first constructed by Hanna†; no other results about these six class-invariants have been published.

These quintic equations are not, of course, the end of the story. It must be possible to express their solutions by means of radicals, though an inspection of the quintics does not suggest any simple method of effecting this object. The clue to the labyrinth was indicated by the 15-ic equation associated with 751; the construction of the equation which is satisfied by F_{751}^{-1} was somewhat hampered by the fact that the fifteen roots could be grouped in five triads such that the sum of the members of each triad is 1. It was therefore to be expected that, when the five cubic equations whose roots were the several triads were constructed, the last two coefficients in the cubic equations would prove to be quintic irrationalities (each being expressible rationally in terms of the other, after the manner of the quintic irrationalities occurring in connection with 131, 179, ...). By applying the Euclidean algorithm to a quintic equation and the corresponding cubic equation, we should be able to obtain an expression for the roots of the 15-ic equation in the form of a rational function of a cubic irrationality and a quintic irrationality. Obviously such an expression can be written in the form

$$\alpha_r\theta_s^{\,2}-\beta_r\theta_s+\gamma_r \qquad \begin{matrix} r=0,\ 1,\ 2,\ 3,\ 4,\\ s=1,\ 2,\ 3,\end{matrix}$$

where θ_1, θ_2, θ_3 are the roots of Berwick's cubic and α_r, β_r, γ_r are quintic irrationalities.

There were no reasons for supposing that this method of solving a 15-ic equation was confined to the case of 751, and the only difference between

† M. Hanna, *Proc. London Math. Soc.* (2), 28 (1929), 46–52 (50).

this 15-ic equation and the others was that in the case of 751 I was spared the somewhat arduous task of arranging the roots in five suitable triads, since the arrangement in the case of 751 was obvious†.

I now state the solutions of the six 15-ic equations. The solutions are taken in the form

$$\alpha_r\theta_s^2-\beta_r\theta_s+\gamma_r,$$

where θ_1, θ_2, θ_3 are the roots of the appropriate cubic, the quintic equations satisfied by α_r, β_r, γ_r are given, and corresponding solutions of the quintic equations are taken to be

$$10\alpha_r = a+\omega_r u_1+\omega_r^2 u_2+\omega_r^3 u_3+\omega_r^4 u_4,$$

$$10\beta_r = b+\omega_r v_1+\omega_r^2 v_2+\omega_r^3 v_3+\omega_r^4 v_4,$$

$$10\gamma_r = c+\omega_r w_1+\omega_r^2 w_2+\omega_r^3 w_3+\omega_r^4 w_4,$$

where, as usual,

$$\omega_r = \exp\frac{2r\pi i}{5}$$

and r assumes the values 0, 1, 2, 3, 4 in succession. In accordance with the conventions used throughout the paper, the upper signs in ambiguous expressions refer to u_1, u_2, v_1, v_2, w_1, w_2 and the lower signs to u_4, u_3, v_4, v_3, w_4, w_3.

In each case I have constructed the quartic polynomials (and the equivalent quadratic fractions) which express the roots β_r and γ_r rationally in terms of the corresponding α_r.

The first equation to be solved is associated with $\mathbf{n = 239}$. The roots of this equation, namely

$$x^{15}-4x^{14}+4x^{13}+4x^{12}-5x^{11}-13x^{10}+20x^9+4x^8-15x^7$$
$$-13x^6+27x^5-4x^4-8x^3-2x^2+6x-1=0,$$

are given by the formula

$$x_{r,s} = \alpha_r\theta_s^2-\beta_r\theta_s+\gamma_r \quad \begin{cases} r = 0, 1, 2, 3, 4, \\ s = 1, 2, 3, \end{cases}$$

† The arrangement of the roots in three pentads which is effected by Berwick's cubic makes it feasible to group the roots in five triads, the number of preliminary calculations being reduced to 120 from $7!\times 2^7 = 645120$.

where $\theta_1, \theta_2, \theta_3$ are the roots of Berwick's cubic

$$\theta^3-3\theta^3+2\theta-3=0 \qquad (\Delta=-239)$$

and $\alpha_r, \beta_r, \gamma_r$ are the roots of the quintics

$$\alpha^5+\alpha^4+\frac{61\alpha^3}{239}-\frac{20\alpha^2}{239}-\frac{2793\alpha}{239^2}-\frac{361}{239^2}=0,$$

$$\beta^5+\beta^4+\frac{8\beta^3}{239}-\frac{95\beta^2}{239}-\frac{11507\beta}{239^2}-\frac{3431}{239^2}=0,$$

$$\gamma^5-2\gamma^4+\frac{396\gamma^3}{239}+\frac{17\gamma^2}{239}-\frac{33358\gamma}{239^2}+\frac{9299}{239^2}=0.$$

The roots of these quintics are derived from the following results:

$$a=-2, \quad b=-2, \quad c=4,$$

$$(u_1\sqrt{239})^5, \quad (u_4\sqrt{239})^5$$
$$=(12\,10128-51680\sqrt{5})\sqrt{239}\mp(157\,00960-55\,93520\sqrt{5})\sqrt{5+2\sqrt{5}},$$

$$(u_2\sqrt{239})^5, \quad (u_3\sqrt{239})^5$$
$$=(12\,10128+51680\sqrt{5})\sqrt{239}\pm(157\,00960+55\,93520\sqrt{5})\sqrt{5-2\sqrt{5}},$$

$$(v_1\sqrt{239})^5, \quad (v_4\sqrt{239})^5$$
$$=(192\,39528+40\,75400\sqrt{5})\sqrt{239}$$
$$\pm(2569\,02280+920\,89960\sqrt{5})\sqrt{5-2\sqrt{5}},$$

$$(v_2\sqrt{239})^5, \quad (v_3\sqrt{239})^5$$
$$=(192\,39528-40\,75400\sqrt{5})\sqrt{239}$$
$$\pm(2569\,02280-920\,89960\sqrt{5})\sqrt{5+2\sqrt{5}},$$

$$(w_1\sqrt{239})^5, \quad (w_4\sqrt{239})^5$$
$$=(-775\,61896+3\,48760\sqrt{5})\sqrt{239}$$
$$\pm(10189\,19640)+2802\,31960\sqrt{5})\sqrt{5-2\sqrt{5}},$$

$$(w_2\sqrt{239})^5, \quad (w_3\sqrt{239})^5$$
$$=(-775\,61896-3\,48760\sqrt{5})\sqrt{239}$$
$$\pm(10189\,19640-2802\,31960\sqrt{5})\sqrt{5+2\sqrt{5}}.$$

Equations which express β and γ rationally in terms of α are

$$991\,20853\beta = 1411\,38802 + 6548\,92484\alpha - 4454\,27885\alpha^2 - 37891\,21535\alpha^3 - 44175\,09656\alpha^4,$$

$$991\,20853\gamma = 5050\,40406 + 17794\,84278\alpha - 17131\,43874\alpha^2 - 1\,44972\,52679\alpha^3 - 1\,79257\,12220\alpha^4,$$

$$\beta = -\frac{11267 + 769\alpha - 1\,68811\alpha^2}{7391 + 61490\alpha - 1\,11613\alpha^2},$$

$$\gamma = -\frac{627 + 1561\alpha + 3282\alpha^2}{817 + 4785\alpha + 3107\alpha^2}.$$

The next equation to be solved is associated with $\mathbf{n = 439}$. The roots of this equation, namely

$$x^{15} - 5x^{14} + 11x^{13} - 9x^{12} - 7x^{11} + 17x^{10} - x^9 - 29x^8 + 38x^7 - 13x^6 - 20x^5 + 24x^4 + 7x^3 - 23x^2 + 13x - 1 = 0,$$

are given by the formula

$$x_{r,s} = \alpha_r \theta_s^2 - \beta_r \theta_s + \gamma_r \quad \begin{cases} r = 0, 1, 2, 3, 4, \\ s = 1, 2, 3, \end{cases}$$

where θ_1, θ_2, θ_3 are the roots of Berwick's cubic

$$\theta^3 - 2\theta^2 - \theta - 3 = 0 \qquad (\Delta = -439)$$

and α_r, β_r, γ_r are the roots of the quintics

$$\alpha^5 - \alpha^4 + \frac{265\alpha^3}{439} - \frac{87\alpha^2}{433} + \frac{5918\alpha}{439^2} - \frac{351}{439^2} = 0,$$

$$\beta^5 - 2\beta^4 + \frac{1107\beta^3}{439} - \frac{756\beta^2}{439} + \frac{1\,45276\beta}{439^2} - \frac{26351}{439^2} = 0,$$

$$\gamma^5 - \gamma^4 + \frac{180\gamma^3}{439} + \frac{207\gamma^2}{439} - \frac{1\,65581\gamma}{439^2} + \frac{1\,10357}{439^2} = 0.$$

The roots of these quintics are derived from the following results:

$$a = 2, \quad b = 4, \quad c = 2,$$

$$(u_1\sqrt{439})^5, \quad (u_4\sqrt{439})^5$$
$$= (9\,48872 + 3\,80920\sqrt{5})\sqrt{439} \pm (394\,34520 + 157\,17480\sqrt{5})\sqrt{5-2\sqrt{5}},$$

$$(u_2\sqrt{439})^5, \quad (u_3\sqrt{439})^5$$
$$= (9\,48872 - 3\,80920\sqrt{5})\sqrt{439} \pm (394\,34520 - 157\,17480\sqrt{5})\sqrt{5+2\sqrt{5}},$$

$$(v_1\sqrt{439})^5, \quad (v_4\sqrt{439})^5$$
$$= (-184\,59496 + 80\,77800\sqrt{5})\sqrt{439}$$
$$\pm (7424\,52840 - 3164\,93160\sqrt{5})\sqrt{5-2\sqrt{5}},$$

$$(v_2\sqrt{439})^5, \quad (v_3\sqrt{439})^5$$
$$= (-184\,59496 - 80\,77800\sqrt{5})\sqrt{439}$$
$$\pm (7424\,52840 + 3164\,93160\sqrt{5})\sqrt{5+2\sqrt{5}},$$

$$(w_1\sqrt{439})^5, \quad (w_4\sqrt{439})^5$$
$$= (-19976\,49128 - 8910\,64680\sqrt{5})\sqrt{439}$$
$$\mp (5\,75514\,15480 + 2\,57226\,46200\sqrt{5})\sqrt{5-2\sqrt{5}},$$

$$(w_2\sqrt{439})^5, \quad (w_3\sqrt{439})^5$$
$$= (-19976\,49128 + 8910\,64680\sqrt{5})\sqrt{439}$$
$$\mp (5\,75514\,15480 - 2\,57226\,46200\sqrt{5})\sqrt{5+2\sqrt{5}}.$$

Equations which express β and γ rationally in terms of α are

$$1839\,78561\beta = -8829\,31881 + 1\,04006\,93864\alpha - 3\,82950\,61354\alpha^2$$
$$+ 6\,86117\,59536\alpha^3 - 8\,10956\,47753\alpha^4,$$

$$1839\,78561\gamma = -12620\,71941 + 1\,69666\,59566\alpha - 7\,14149\,23357\alpha^2$$
$$+ 12\,87788\,70294\alpha^3 - 16\,27669\,53133\alpha^4,$$

$$\beta = \frac{43501 - 23083\alpha - 8\,99154\alpha^2}{1\,07834 - 7\,32534\alpha + 2\,19061\alpha^2},$$

$$\gamma = \frac{86164 - 5\,17724\alpha + 10\,58029\alpha^2}{1\,00420 - 4\,93557\alpha + 60143\alpha^2}.$$

The next equation to be solved is associated with $\mathbf{n = 751}$. The roots of this equation, namely

$$x^{15}-5x^{14}+9x^{13}-6x^{12}+7x^{11}-12x^{10}+17x^9-29x^8+2x^7$$
$$-4x^6+21x^5-9x^4+53x^3+12x^2+25x-1=0,$$

are given by the formula

$$x_{r,s}=\alpha_r\theta_s^2-\beta_r\theta_s+\gamma_r \quad \begin{cases} r=0,\ 1,\ 2,\ 3,\ 4, \\ s=1,\ 2,\ 3, \end{cases}$$

where θ_1, θ_2, θ_3 are the roots of Berwick's cubic

$$\theta^3-\theta^2+6\theta-1=0 \qquad (\Delta=-751)$$

and α_r, β_r, γ_r are the roots of the quintics

$$\alpha^5+\alpha^4+\frac{347\alpha^3}{751}+\frac{78\alpha^2}{751}+\frac{6831\alpha}{751^2}+\frac{243}{751^2}=0,$$

$$\beta^5+*+\frac{319\beta^3}{751}+\frac{220\beta^2}{751}-\frac{11292\beta}{751^2}+\frac{739}{751^2}=0,$$

$$\gamma^5+2\gamma^4+\frac{2097\gamma^3}{751}+\frac{1259\gamma^2}{751}+\frac{5\,86637\gamma}{751^2}+\frac{24441}{751^2}=0.$$

The roots of these quintics are derived from the following results:

$$a=-2, \quad b=0, \quad c=-4,$$

$$(u_1\sqrt{751})^5, \quad (u_4\sqrt{751})^5$$
$$=(31\,18368+13\,93200\sqrt{5})\sqrt{751}$$
$$\mp(289\,58320+119\,12160\sqrt{5})\sqrt{5+2\sqrt{5}},$$

$$(u_2\sqrt{751})^5, \quad (u_3\sqrt{751})^5$$
$$=(31\,18368-13\,93200\sqrt{5})\sqrt{751}$$
$$\pm(289\,58320-119\,12160\sqrt{5})\sqrt{5-2\sqrt{5}},$$

$$(v_1\sqrt{751})^5, \quad (v_4\sqrt{751})^5$$
$$=(-1824\,69000+34\,98600\sqrt{5})\sqrt{751}$$
$$\mp(44520\,54440+12332\,45560\sqrt{5})\sqrt{5-2\sqrt{5}},$$

$$(v_2\sqrt{751})^5, \quad (v_3\sqrt{751})^5$$
$$=(-1824\,69000-34\,98600\sqrt{5})\sqrt{751}$$
$$\mp(44520\,54440-12332\,45560\sqrt{5})\sqrt{5+2\sqrt{5}},$$

$$(w_1\sqrt{751})^5, \quad (w_4\sqrt{751})^5$$
$$= (32375\,52176+14542\,77600\sqrt{5})\sqrt{751}$$
$$\mp(2\,96576\,70240+1\,30026\,99120\sqrt{5})\sqrt{5+2\sqrt{5}}$$
$$(w_2\sqrt{751})^5, \quad (w_3\sqrt{751})^5$$
$$= (32375\,52176-14542\,77600\sqrt{5})\sqrt{751}$$
$$\pm(2\,96576\,70240-1\,30026\,99120\sqrt{5})\sqrt{5-2\sqrt{5}}$$

Equations which express β and γ rationally in terms of α are

$$1100\,51307\beta = -1584\,02016-26905\,07919\alpha-1\,81411\,90546\alpha^2$$
$$-4\,79846\,41079\alpha^3-5\,63702\,07947\alpha^4,$$
$$3301\,53921\gamma = -483\,50709-14799\,43542\alpha-1\,81411\,90546\alpha^2$$
$$-4\,79846\,41079\alpha^3-5\,63702\,07947\alpha^4,$$
$$\beta = -\frac{12510+35362\alpha+57137\alpha^2}{34731+2\,73394\alpha+6\,96177\alpha^2},$$
$$\gamma = \frac{16506+3\,90903\alpha+14\,55314\alpha^2}{42939+2\,75341\alpha-2\,32059\alpha^2}.$$

It is to be observed that, by the former pair of formulae,

$$3\gamma = 11\alpha+\beta+1.$$

Hence, by using Berwick's cubic, we have

$$\sum_{s=1}^{3}(\alpha_r\theta_s^2-\beta_r\theta_s+\gamma_r) = -11\alpha_r-\beta_r+3\gamma_r$$
$$= 1.$$

This is the result previously stated that the roots of the particular 15-ic equation (associated with 751) now under consideration can be grouped in five triads such that the sum of the roots forming each triad is unity.

The next equation to be solved is associated with $\mathbf{n = 1327}$. The roots of this equation, namely

$$x^{15}+6x^{14}+16x^{13}+24x^{12}+38x^{11}-x^{10}+24x^9-3x^8+52x^7$$
$$+20x^6+13x^5-100x^4-18x^3+21x^2+83x-1 = 0,$$

are given by the formula

$$x_{r,s} = \alpha_r\theta_s^2-\beta_r\theta_s+\gamma \quad \begin{cases} r = 0, 1, 2, 3, 4, \\ s = 1, 2, 3, \end{cases}$$

where θ_1, θ_2, θ_3 are the roots of Berwick's cubic

$$\theta^3+*+\theta-7=0 \qquad (\Delta=-1327)$$

and α_r, β_r, γ_r are the roots of the quintics

$$\alpha^5+*+\frac{125\alpha^3}{1327}+\frac{\alpha^2}{1327}+\frac{4601\alpha}{1327^2}+\frac{243}{1327^2}=0,$$

$$\beta^5+*+\frac{843\beta^3}{1327}-\frac{24\beta^2}{1327}+\frac{1\,85554\beta}{1327^2}-\frac{55101}{1327^2}=0,$$

$$\gamma^5+2\gamma^4+\frac{2857\gamma^3}{1327}-\frac{308\gamma^2}{1327}-\frac{10\,61412\gamma}{1327^2}-\frac{3\,74011}{1327^2}=0.$$

The roots of these quintics are derived from the following results:

$$a=0, \quad b=0, \quad c=-4,$$

$$(u_1\sqrt{1327})^5, \quad (u_4\sqrt{1327})^5$$
$$=(-68\,82000-30\,79200\sqrt{5})\sqrt{1327}$$
$$\pm(948\,59440+423\,47840\sqrt{5})\sqrt{5+2\sqrt{5}},$$

$$(u_2\sqrt{1327})^5, \quad (u_3\sqrt{1327})^5$$
$$=(-68\,82000+30\,79200\sqrt{5})\sqrt{1327}$$
$$\mp(948\,59440-423\,47840\sqrt{5})\sqrt{5-2\sqrt{5}},$$

$$(v_1\sqrt{1327})^5, \quad (v_4\sqrt{1327})^5$$
$$=(13143\,19000-5877\,85000\sqrt{5})\sqrt{1327}$$
$$\mp(1\,66357\,39560-74395\,23960\sqrt{5})\sqrt{5-2\sqrt{5}},$$

$$(v_2\sqrt{1327})^5, \quad (v_3\sqrt{1327})^5$$
$$=(13143\,19000+5877\,85000\sqrt{5})\sqrt{1327}$$
$$\mp(1\,66357\,39560+74395\,23960\sqrt{5})\sqrt{5+2\sqrt{5}},$$

$$(w_1\sqrt{1327})^5, \quad (w_4\sqrt{1327})^5$$
$$=(47830\,23704-3064\,32040\sqrt{5})\sqrt{1327}$$
$$\pm(19\,72306\,44840-6\,69125\,95800\sqrt{5})\sqrt{5+2\sqrt{5}},$$

$$(w_2\sqrt{1327})^5, \quad (w_3\sqrt{1327})^5$$
$$=(47830\,23704+3064\,32040\sqrt{5})\sqrt{1327}$$
$$\mp(19\,72306\,44840+6\,69125\,95800\sqrt{5})\sqrt{5-2\sqrt{5}}.$$

Equations which express β and γ rationally in terms of α are

$$6190\,91055\beta = 1998\,58779 + 10561\,30162\alpha - 14999\,09427\alpha^2 + 1\,71285\,56383\alpha^3 - 17\,04103\,82117\alpha^4,$$

$$1238\,18211\gamma = 931\,29516 + 6843\,08126\alpha + 45016\,63047\alpha^2 + 1\,23018\,49994\alpha^3 + 2\,22916\,00211\alpha^4,$$

$$\beta = \frac{49035 + 87224\alpha + 18\,01591\alpha^2}{2\,00992 - 1\,61596\alpha + 25\,43859\alpha^2},$$

$$\gamma = \frac{39215 + 1\,19515\alpha + 10\,61181\alpha^2}{54178 - 1\,85745\alpha + 4\,48526\alpha^2}.$$

The next equation to be solved is associated with $\mathbf{n = 1567}$. The roots of this equation, namely

$$x^{15} - x^{14} + 13x^{13} - 44x^{12} + 76x^{11} - 182x^{10} + 317x^9 - 330x^8 + 406x^7 - 522x^6 + 532x^5 - 650x^4 + 670x^3 - 401x^2 + 129x - 1 = 0,$$

are given by the formula

$$x_{r,s} = \alpha_r \theta_s^2 - \beta_r \theta_s + \gamma_r \qquad \begin{cases} r = 0, 1, 2, 3, 4, \\ s = 1, 2, 3, \end{cases}$$

where θ_1, θ_2, θ_3 are the roots of Berwick's cubic

$$\theta^3 - \theta^2 - 2\theta - 7 = 0 \qquad (\Delta = -1567)$$

and α_r, β_r, γ_r are the roots of the quintics

$$\alpha^5 - \alpha^4 + \frac{562\alpha^3}{1567} - \frac{74\alpha^2}{1567} - \frac{1935\alpha}{1567^2} + \frac{1175}{1567^2} = 0,$$

$$\beta^5 - \beta^4 + \frac{834\beta^3}{1567} + \frac{81\beta^2}{1567} - \frac{2\,56777\beta}{1567^2} + \frac{1\,18369}{1567^2} = 0,$$

$$\gamma^5 + \gamma^4 + \frac{5990\gamma^3}{1567} - \frac{601\gamma^2}{1567} - \frac{3\,07805\gamma}{1567^2} + \frac{21\,65419}{1567^2} = 0.$$

The roots of these quintics are derived from the following results:

$$a = 2, \quad b = 2, \quad c = -2,$$

$$(u_1\sqrt{1567})^5, \quad (u_4\sqrt{1567})^5 = (-11\,37152 - 51200\sqrt{5})\sqrt{1567} \mp (338\,99520 - 80\,40960\sqrt{5})\sqrt{5+2\sqrt{5}},$$

$$(u_2\sqrt{1567})^5, \quad (u_3\sqrt{1567})^5$$
$$= (-11\,37152+51200\sqrt{5})\sqrt{1567}$$
$$\pm(338\,99520+80\,40960\sqrt{5})\sqrt{5-2\sqrt{5}},$$
$$(v_1\sqrt{1567})^5, \quad (v_4\sqrt{1567})^5$$
$$= (-10977\,01752+4686\,88680\sqrt{5})\sqrt{1567}$$
$$\pm(1\,51103\,41080-56034\,89480\sqrt{5})\sqrt{5-2\sqrt{5}},$$
$$(v_2\sqrt{1567})^5, \quad (v_3\sqrt{1567})^5$$
$$= (-10977\,01752-4686\,88680\sqrt{5})\sqrt{1567}$$
$$\pm(1\,51103\,41080+56034\,89480\sqrt{5})\sqrt{5+2\sqrt{5}},$$
$$(w_1\sqrt{1567})^5, \quad (w_4\sqrt{1567})^5$$
$$= (-3\,26632\,99848-2\,44146\,06120\sqrt{5})\sqrt{1567}$$
$$\pm(260\,39158\,77160+97\,99790\,96520\sqrt{5})\sqrt{5-2\sqrt{5}},$$
$$(w_2\sqrt{1567})^5, \quad (w_3\sqrt{1567})^5$$
$$= (-3\,26632\,99848+2\,44146\,06120\sqrt{5})\sqrt{1567}$$
$$\pm(260\,39158\,77160-97\,99790\,96520\sqrt{5})\sqrt{5+2\sqrt{5}}.$$

Equations which express β and γ rationally in terms of α are

$$4\,45160\beta = 59\,72920+109\,28779\alpha-5707\,96987\alpha^2$$
$$+26838\,49477\alpha^3-37642\,64637\alpha^4,$$
$$4\,45160\gamma = 179\,63550+276\,62727\alpha-17157\,60011\alpha^2$$
$$+82234\,32661\alpha^3-1\,17838\,91711\alpha^4,$$
$$\beta = -\frac{10255-89460\alpha+1\,95081\alpha^2}{27165-1\,63224\alpha+2\,46019\alpha^2},$$
$$\gamma = -\frac{1\,14455-8\,88970\alpha+8\,63199\alpha^2}{2\,03145-13\,28984\alpha+22\,26707\alpha^2}.$$

The last equation to be solved is associated with $\mathbf{n = 2647}$. The roots of this equation, namely

$$x^{15}-4x^{14}+2x^{13}+61x^{12}+193x^{11}+173x^{10}+332x^9+778x^8+422x^7$$
$$-242x^6+277x^5+140x^4-859x^3-167x^2+595x-1=0,$$

are given by the formula

$$x_{r,s} = \alpha_r\theta_s^2-\beta_r\theta_s+\gamma_r \quad \begin{cases} r = 0,\ 1,\ 2,\ 3,\ 4, \\ s = 1,\ 2,\ 3, \end{cases}$$

where θ_1, θ_2, θ_3 are the roots of Berwick's cubic

$$\theta^3-\theta^2+8\theta+3=0 \qquad (\Delta=-2647)$$

and α_r, β_r, γ_r are the roots of the quintics

$$\alpha^5-*-\frac{51\alpha^3}{2647}-\frac{\alpha^2}{2647}+\frac{4065\alpha}{2647^2}-\frac{369}{2647^2}=0,$$

$$\beta^5-2\beta^4+\frac{7629\beta^3}{2647}-\frac{6197\beta^2}{2647}+\frac{78\,84535\beta}{2647^2}-\frac{12\,31281}{2647^2}=0,$$

$$\gamma^5-2\gamma^4+\frac{8557\gamma^3}{2647}-\frac{6319\gamma^2}{2647}+\frac{69\,34629\gamma}{2647^2}+\frac{9\,23977}{2647^2}=0.$$

The roots of these quintics are derived from the following results:

$$a=0,\quad b=4,\quad c=4,$$

$$(u_1\sqrt{2647})^5,\quad (u_4\sqrt{2647})^5$$
$$=(8\,24000+8\,82800\sqrt{5})\sqrt{2647}$$
$$\pm(612\,67920+185\,95680\sqrt{5})\sqrt{5-2\sqrt{5}},$$

$$(u_2\sqrt{2647})^5,\quad (u_3\sqrt{2647})^5$$
$$=(8\,24000-8\,82800\sqrt{5})\sqrt{2647}$$
$$\pm(612\,67920-185\,95680\sqrt{5})\sqrt{5+2\sqrt{5}},$$

$$(v_1\sqrt{2647})^5,\quad (v_4\sqrt{2647})^5$$
$$=(-1\,34564\,58784+60189\,40560\sqrt{5})\sqrt{2647}$$
$$\mp(124\,62904\,29120-55\,73406\,64080\sqrt{5})\sqrt{5+2\sqrt{5}},$$

$$(v_2\sqrt{2647})^5,\quad (v_3\sqrt{2647})^5$$
$$=(-1\,34564\,58784-60189\,40560\sqrt{5})\sqrt{2647}$$
$$\pm(124\,62904\,29120+55\,73406\,64080\sqrt{5})\sqrt{5-2\sqrt{5}},$$

$$(w_1\sqrt{2647})^5,\quad (w_4\sqrt{2647})^5$$
$$=(-4\,94775\,52384-2\,21221\,00240\sqrt{5})\sqrt{2647}$$
$$\pm(382\,31879\,90160+170\,96991\,71040\sqrt{5})\sqrt{5-2\sqrt{5}},$$

$$(w_2\sqrt{2647})^5,\quad (w_3\sqrt{2647})^5$$
$$=(-4\,94775\,52384+2\,21221\,00240\sqrt{5})\sqrt{2647}$$
$$\pm(382\,31879\,90160-170\,96991\,71040\sqrt{5})\sqrt{5+2\sqrt{5}}.$$

Equations which express β and γ rationally in terms of α are

$$4599\,28611\beta = -584\,66943 + 20010\,43110\alpha + 48527\,23924\alpha^2 - 24312\,93323\alpha^3 - 65\,13343\,72640\alpha^4,$$

$$4599\,28611\gamma = -4293\,79470 + 25426\,21260\alpha + 3\,22187\,99599\alpha^2 - 8\,15709\,41978\alpha^3 - 121\,50861\,32780\alpha^4,$$

$$\beta = \frac{24114 - 47099\alpha - 22\,07968\alpha^2}{19671 - 3\,51052\alpha - 15\,53789\alpha^2},$$

$$\gamma = \frac{-2\,67114 + 82400\alpha + 167\,87065\alpha^2}{1\,11093 + 89734\alpha + 161\,57288\alpha^2}.$$

The University,
Birmingham.

SPECIAL SCROLLS AND INVOLUTIONS ON CANONICAL CURVES

By W. G. WELCHMAN.

[Received 28 September, 1934.—Read 15 November, 1934.]

Introduction.

1.1. There is at present no satisfactory general theory of special ruled surfaces, or scrolls, and the main object of this paper is to develop the theory of certain fundamental scrolls from which all special scrolls may be obtained by projection. These fundamental scrolls are generated by the joins of pairs of points of a γ_2^1 on a canonical curve and were obtained by Segre*, whose argument and results are stated in §2.6. The general theory is developed in §§3 to 6, and in §§7 to 10 the fundamental scrolls of genus 2, 3, 4, and 5 are discussed in greater detail. In his classification of scrolls in [3] Edge† obtains his non-special scrolls by two methods, from the representation on a quadric in [5] and by projection from a normal scroll, but he has no general method of obtaining his special scrolls by projection. They are obtained from fundamental scrolls in §11, and it should be remarked that in classifying special scrolls the bisecant curves must be taken into account as well as the directrix curves‡. There are endless particularities of the fundamental scrolls which are not considered, though the important ones are discussed. For instance the γ_2^1 which gives a fundamental scroll may itself be compounded of a γ_2^1, which may be compounded of a γ_2^1, and so on. I have only considered the important particular case of this type in which the fundamental scroll is hyperelliptic.

A canonical curve which possesses a γ_2^1 is not a general curve of its genus, but must have its moduli restricted in some way. In the course of

---

* S 3.

† E. Exact references are given in §11.

‡ *Cp.* §11.5.

the general theory it is shown that the quadrics through such a canonical curve form an harmonic system*, and this leads, in § 12, to a sufficient geometrical condition for a canonical curve to possess a γ_2^1 of a prescribed genus. The argument involves a discussion of involutions on rational normal scrolls, which is interesting for its own sake, and which leads finally to a statement of all the cases in which the quadrics through a canonical curve, form an harmonic system.

It is convenient to explain in the first paragraph some of the essential ideas involved in the paper, though most of the contents of this paragraph are to be found in existing literature.

1.2. The use of the word "special" in the theory of ruled surfaces is an extension of its use in the theory of curves. A curve C_p^n in any space is said to be special and to have speciality i if a prime section is contained in i (>0) independent canonical sets. By the Riemann-Roch theorem, the prime sections are then contained in a complete series of freedom $n-p+i$, and the curve may be obtained as the projection of a curve of the same order in $[n-p+i]$, but not in space of higher dimensions. In fact, a curve C_p^n is normal in $[n-p]$ at least, and if it is actually normal in $[n-p+i]$, where $i>0$, the curve is special, the number i being the speciality. Clearly a curve C_p^n for which $n>2p-2$ cannot be special, nor can a curve which is rational, elliptic, or hyperelliptic† be special. If we project a normal curve from points of itself we obtain curves in lower spaces which are normal curves since their prime sections form complete series. The speciality is increased by j when we project from a secant space‡ $[k]$ meeting the normal curve in $k+1+j$ points, so we may state at once that a rational normal curve can have no secant space, that an elliptic normal curve can have no secant space other than a prime, that a normal curve of genus p and order greater than $2p$ can have no double point, that a normal curve of genus p and order greater than $2p+1$ can have no trisecant line, and so on§. The natural method of investigating special curves of genus p is to make use of the fact that, since the prime sections are contained in the canonical series, the curves are projections of a canonical curve K_p^{2p-2} in $[p-1]$. In fact, any C_p^n of speciality i is the projection of a normal curve of order n in $[n-p+i]$ which is the projection of a canonical curve K_p^{2p-2} in $[p-1]$

* See § 2.7.

† See the end of § 1.3.

‡ A "secant space" of a curve is a space $[k]$ which meets the curve in more than $k+1$ points, and does not contain the curve.

§ *Cp.* S 1, 215.

from $2p-2-n$ of its points lying in a $[2p-2-n-i]$. This would enable us to find all the special curves of a given genus should we wish to do so.

1.3. Similar ideas may be applied to ruled surfaces, and to loci generated by a single infinity of $[k]$. I shall use the name "scroll" for a ruled surface, "plane-scroll" for a locus generated by ∞^1 planes*, and $[k]$-scroll for the locus generated by ∞^1 $[k]$. I denote scrolls and plane-scrolls of order n and genus p by R_p^n and E_p^n respectively. Suppose then that on a scroll R_p^n in $[k]$ there is a bisecant curve C_π^ν, *i.e.* a simple curve meeting each generator in two points. Let C_π^ν degenerate into two directrix curves of orders ν_1 and ν_2 and genus necessarily p. These curves must have $\nu_1+\nu_2-n=\nu-n$ intersections, so from the formula for the genus of a degenerate curve we have

$$\pi = 2p+\nu-n-1,$$

so that

$$\nu-\pi = n-2p+1\dagger.$$

The above argument breaks down if C_π^ν has a double point which counts twice as the intersections of a generator of the scroll, for this double point will correspond to one of the intersections of $C_p^{\nu_1}$ and $C_p^{\nu_2}$. However, a double point of C_π^ν which is also a double point of R_p^n will correspond to a double point on one of the directrix curves $C_p^{\nu_1}$ and $C_p^{\nu_2}$, and will not affect the argument. We therefore define a "proper" bisecant curve to be one which has no double point counting twice as the intersections of a generator, and the formula is true for proper bisecant curves.

The formula may also be regarded as giving the order n of the scroll generated by the joins of pairs of points of an involution γ_2^1 of genus p on a curve of order ν and genus π. Now any scroll contains proper bisecant curves, for the section by any quadric primal which does not touch the scroll is such a curve. Suppose then that on a scroll R_p^n there is a proper bisecant C_π^ν of speciality i. C_π^ν is the projection of a normal curve $C_\pi'^\nu$ in $[\nu-\pi+i]$, and the γ_2^1 on C_π^ν is the projection of a γ_2' on C' whose joins generate a scroll of genus p and therefore of order n, by our formula. Thus R_p^n is the projection of a scroll of the same order in $[\nu-\pi+i]$, or $[n-2p+1+i]$. Also, if R_p^n is the projection of a normal scroll R' of order n, C_π^ν must be the projection of a bisecant curve of order ν and genus π on

* I have previously used the term "planar threefold." See W 3.

† *Cp.* S 2, 3. A similar degeneration method shows at once that, for a C_π^ν which lies on a $[k]$-scroll of order n and genus p and meets each $[k]$ in $k+1$ independent points, $\nu-\pi=n-(k+1)p+k$. The usual inductive correspondence proof of this formula is rather long.

R', which must lie in a space of $\nu-\pi+i$ dimensions at most, and this space contains R'. We see, therefore, that $R_p{}^n$ is actually normal in $[n-2p+1+i]$, and we are led to the following use of the word special.

Any scroll $R_p{}^n$ is normal in $[n-2p+1]$ at least. If it is normal in $[n-2p+1+i]$ and $i>0$, we say that the scroll is "special" and call i the speciality. All the proper bisecant curves have the same speciality as the scroll and are normal when the scroll is normal. When the speciality is zero we say that the scroll is non-special.

Incidentally, a rational scroll of order n cannot lie in a space of more than $n+1$ dimensions and cannot therefore be special. Consequently an hyperelliptic curve, which is a proper bisecant curve on the scroll given by its $g_2{}^1$, cannot be special.

1.4. Segre obtained his theorems about special scrolls by a discussion of directrix curves*. Clearly the generators of a scroll establish a (1, 1) correspondence between any two directrix curves, so any scroll of order n and genus p may be obtained by joining corresponding points of two curves $C_p{}^m$ and $C_p{}^{m'}$ in (1, 1) correspondence with $m+m'-n$ common self-corresponding points. Segre proved that on a normal scroll the directrix curves of sufficiently low order are normal. Suppose then that on a normal $R_p{}^n$ of speciality i in $[n-2p+1+i]$ there are two normal directrix curves, $C_p{}^{m'}$ and $C_p{}^{m''}$, of speciality i' and i'' and lying accordingly in two spaces $[m'-p+i']$ and $[m''-p+i'']$ which we suppose to meet in an

$$[m'+m''-n-1-\delta]$$

containing the $m'+m''-n$ intersections of $C_p{}^{m'}$ and $C_p{}^{m''}$. Then, since the space containing the scroll is that determined by the spaces of $C_p{}^{m'}$ and $C_p{}^{m''}$, we must have

$$i=i'+i''+\delta.$$

In particular, if the two directrix curves have no intersections,

$$i=i'+i''.$$

This result will be useful later. Segre only considers special scrolls whose prime sections are non-special curves, and by taking sections by primes through as many generators as possible he obtains theorems about the special directrix curves which lie on special scrolls†. These theorems apply readily to scrolls of high speciality, but are inadequate as a general

* See § 2, 15.

† Note that the symbol i used by Segre does not represent the speciality.

theory. *In fact the discussion of directrix curves, which is sufficient for the theory of rational scrolls, is no longer sufficient for dealing with special scrolls, and we are led to consider bisecant curves.*

1.5. The projection of a normal scroll R from a simple point P of itself is another normal scroll R', for any proper bisecant curve of R which passes through P is normal and therefore projects into a normal proper bisecant curve of R', wherefore R' is normal. (A bisecant curve of R not passing through P would give an improper bisecant curve of R'.) Thus the projection of R from h of its points which lie in an $[h-1-j]$ is a normal scroll R' whose speciality is greater than the speciality of R by j. Now on a special R_p^n of speciality i any proper bisecant C_π^ν has speciality i and is therefore the projection of a canonical curve of genus π, $K_\pi^{2\pi-2}$, in $[\pi-1]$ from $2\pi-2-\nu$ of its points lying in a $[2\pi-2-\nu-i]$ and from certain other points if C_π^ν and R_p^n are not normal*. The γ_2^1 on C_π^ν cut out by the generators of R_p^n is the projection of a γ_2^1 on $K_\pi^{2\pi-2}$, whose joins generate a scroll of genus p and order $2p+\pi-3$ (using the formula of § 1.3), which is normal and which projects into R_p^n. I call a scroll, given in this manner by a γ_2^1 on a canonical curve, a *fundamental scroll*, and we have shown that *any special scroll is the projection of a fundamental scroll.* Naturally one would choose a bisecant curve on R_p^n of the lowest possible genus in order to obtain R_p^n as the projection of a fundamental scroll of the lowest possible order. Actually in some cases it is necessary to consider an improper bisecant curve to obtain the simplest possible projection. *In fact, just as in dealing with non-special scrolls it is best to attack the normal ones from which they may all be obtained by projection, so in dealing with special scrolls it is best to develop a general theory of the fundamental scrolls, and this we proceed to do.*

1.6. Throughout the paper when I refer to an involution I shall mean an involution of order 2. The calculations and the printing are greatly simplified by using the dimension of the space containing a canonical curve, *i.e.* the freedom of the canonical series, rather than the genus of the curve, and I hope that this will cause no confusion. I shall use Clarendon type for a fundamental scroll, and $\mathbf{R}(n, r)$ will denote the fundamental scroll in $[n]$ given by a γ_2^1 of genus $r+1$ on a canonical curve of genus $n+1$, which is denoted by K_{n+1}^{2n} or by $K(n)$. When there is no ambiguity I shall use simply $\mathbf{R}$ and K. A symbol such as (A) is used to denote a set of points of which A or A_λ is a typical member. When dealing with rational normal

* See § 1.2.

scrolls I use the letter D to denote a directrix curve. To save space I use the symbol $(n)_r$ instead of $\binom{n}{r}$ or ${}_nC_r$.

The fundamental scrolls.

2.1. We first prove a few lemmas.

Any (1, 1) *correspondence between two canonical curves of the same genus lying in independent spaces is projective.*

For suppose that we have two K_{n+1}^{2n}, K and K', in (1, 1) correspondence and lying in independent spaces S_n and S_n'. The lines joining corresponding points of K and K' generate an R_{n+1}^{4n} which lies in the S_{2n+1} containing S_n and S_n' (and is a scroll of speciality 2). A section of K by a prime of S_n is a canonical set and therefore corresponds to a section of K' by a prime of S_n', wherefore the $2n$ generators of R_{n+1}^{4n} through the points of a canonical set of K lie in a S_{2n-1}. Through this S_{2n-1} pass ∞^1 primes of S_{2n+1} which cannot contain any more generators of R_{n+1}^{4n} (for they would then contain K, K', and the whole scroll), and which, therefore, meet R_{n+1}^{4n} in directrix curves of order $2n$. These must all lie in $[n]$, for if one lay in $[n-1]$ the whole scroll would lie in the $[2n]$ determined by this $[n-1]$ and S_n. Thus R_{n+1}^{4n} has a single infinity of directrix K_{n+1}^{2n}, no two of which can meet, and if we take the $[n]$ of one of them as vertex of projection the points of K project into the corresponding points of K', so the lemma is established*. The scroll R_{n+1}^{4n} occurs later†, and we may note that it has no double points, for a double point would imply a common point of S_n and S_n'.

2.2. *Any* (1, 1) *correspondence between the points of a canonical curve* $K(n)$ *in* S_n *is contained in a linear transformation of* S_n.

For take any other canonical curve $K'(n)$ which is in (1, 1) correspondence with $K(n)$ and which lies in a space S_n' not meeting S_n. The (1, 1) correspondence on $K(n)$ gives another (1, 1) correspondence between $K(n)$ and $K'(n)$; both these correspondences between $K(n)$ and $K'(n)$ are projective, so the given correspondence on $K(n)$ is contained in the linear transformation of S_n given by the two projections.

2.3. *The pairs of points of a* γ_2^1 *on a* $K(n)$ *in* S_n *are harmonic conjugates with respect to two independent base spaces* S_λ *and* $S_{n-\lambda-1}$.

* Segre proves this lemma by a different method. S 1, 209

† § 4.2.

For the γ_2^1 gives a (1, 1) correspondence between the points of $K(n)$ which is contained in a linear transformation T of S_n. The square of T contains all the points of $K(n)$ as united points and is therefore the identical transformation. Thus T is an involutory transformation and there are two independent base spaces S_λ and $S_{n-\lambda-1}$ with respect to which pairs of corresponding points are harmonic conjugates*.

2.4. *The number of intersections of two simple curves of orders n and n' lying on a scroll of order N and meeting the generators of the scroll in k and k' points respectively is $kn'+k'n-kk'N$†, and this number cannot be negative.*

This is easily proved by considering the correspondence between primes of a pencil in which two primes correspond when they contain points of the two curves lying on the same generator of the scroll.

2.5. A theorem concerning rational normal curves, scrolls, and $[k]$-scrolls, which will be required later, and which is easily established inductively, is the following.

The number of independent quadrics in $[N+\lambda]$ which contain a rational normal locus $M_{\lambda+1}^N$ generated by a rational ∞^1 of spaces $[\lambda]$ is $(N)_2$, the number of conditions implied being $(\lambda+2)N+(\lambda+2)_2$‡.

2.6. We are now in a position to discuss the fundamental scrolls. Suppose that a canonical curve $K(n)$ of genus $n+1$ in $[n]$ possesses a γ_2^1 of genus $r+1$. The pairs of points of the γ_2^1 are harmonic conjugates with respect to two independent spaces S_λ and $S_{n-\lambda-1}$, and the lines joining pairs of points of the γ_2^1 generate a fundamental scroll $\mathbf{R}(n, r)$ whose order is $n+2r$, by §1.3. The generators of $\mathbf{R}$ meet S_λ and $S_{n-\lambda-1}$ in two directrix curves of genus $r+1$ which are normal since $\mathbf{R}$ is normal. Since these directrix curves lie in independent spaces they have no intersection and the sum of their orders is $n+2r$. If they were both non-special their orders would be $\lambda+r+1$ and $n-\lambda+r$, whose sum is $n+2r+1$, so one of them must be special, say the one in S_λ§. The order of this special directrix curve in S_λ is $\lambda+r$, which cannot be greater than $2r$, wherefore $\lambda \leqslant r$. But by §2.4 the number of intersections of this directrix curve in S_λ and $K(n)$ is $2(\lambda-r)$,

---

* See, for instance, Bertini, *Geometria proiettiva degli iperspazi* (Palermo, 1923), 87.

† See E, 311. The result is due to Segre, *Rom. Acc. Lincei Rend.* (4), 3^2 (1887), 3.

‡ Throughout the paper the symbol $(n)_r$ is used instead of $\binom{n}{r}$.

§ Alternatively we may use the result, $i = i'+i''$, of §1.4.

and since this cannot be negative we have $\lambda \geqslant r$. Thus $\lambda = r$, and the two directrix curves are a C_{r+1}^{2r} in S_r and a C_{r+1}^{n} in S_{n-r-1}. The number of intersections of C_{r+1}^{n} and $K(n)$ is $2(n-2r)$, which cannot be negative, wherefore $n \geqslant 2r$, fixing an upper limit to the genus of a γ_2^1 on $K(n)$. Since the pairs of the γ_2^1 are harmonic conjugates with respect to S_r and S_{n-r-1}, the only intersections of $K(n)$ with these spaces are the $2(n-2r)$ united points of the γ_2^1, which are therefore the intersections of C_{r+1}^{n} and $K(n)$. I call S_r and S_{n-r-1} the *base spaces* of the γ_2^1 on $K(n)$, and we have the following statement*.

The pairs of points of a γ_2^1 of genus $r+1$ on a canonical curve $K(n)$ of genus $n+1$ in $[n]$ are harmonic conjugates with respect to two independent base spaces S_r and S_{n-r-1}. The fundamental scroll $\mathbf{R}(n, r)$ generated by the joins of pairs of the γ_2^1 is of order $n+2r$ and genus $r+1$ and has as directrix curves a canonical curve $K(r)$ in S_r and a normal non-special C_{r+1}^{n} in S_{n-r-1}. The space S_r does not meet $K(n)$, while S_{n-r-1} meets $K(n)$ in $2(n-2r)$ points which are the united points of the γ_2^1 and lie on C_{r+1}^{n}. Also $n \geqslant 2r$.

The canonical curve $K(n)$ projects doubly from S_{n-r-1} into $K(r)$†. It is essential that $K(n)$ shall not be hyperelliptic, but the γ_2^1 on $K(n)$ may be hyperelliptic, in which case $K(r)$ is a rational normal C_0^r counted twice. In particular, $\mathbf{R}(n, 1)$ has a double directrix line, and $\mathbf{R}(n, 0)$ is a cone. Since any special scroll is the projection of a fundamental scroll with a canonical directrix curve, a special scroll has at least one special directrix curve (which may be a multiple curve), whereas a non-special scroll has no special directrix curve‡. Since the intersections of $K(n)$ and C_{r+1}^{n} are united points of the γ_2^1, the tangents to $K(n)$ at these points are generators of $\mathbf{R}$, and if two of the points tend to coincidence along C_{r+1}^{n} the curve $K(n)$ acquires a double point. It is clear moreover that $\mathbf{R}(n, r)$ can have no double point which is not contained in one of the base spaces, for the existence of such a double point would imply a point of intersection of the base spaces.

2.7. Each pair of points of the γ_2^1 are harmonic conjugates with respect to the points in which their join meets the base spaces, which I call the base points of the pair of γ_2^1. Clearly the generators of $\mathbf{R}$ set up a (1, 1) correspondence between $K(r)$ and C_{r+1}^{n}, which I call the base curves of the γ_2^1 on $K(n)$. I denote by Σ the linear system of quadric primals of $[n]$

* This statement, and the proof given above, are due to Segre, S 3, 527.

† *Cp.* H. F. Baker, *Principles of geometry*, 6 (1933), 23, where a more general theorem is given.

‡ *Cp.* S 2, 11.

through $K(n)$. Each corresponding pair of base points is conjugate with respect to every quadric of Σ. Also a quadric of Σ which contains any point of $\mathbf{R}$ not on $K(n)$ contains the generator through this point; in particular a quadric of Σ through a point of $K(r)$ contains the corresponding point of C^n_{r+1}. Quadrics of Σ which contain both base spaces are called *base quadrics*, and ones which have the base spaces as polars are called *harmonic quadrics*. These form two linear systems, since the implied conditions are linear.

Suppose that we have in $[n]$ two independent base spaces S_λ and $S_{n-\lambda-1}$, and let us call the quadrics of $[n]$ which contain these spaces "base quadrics", and those which have the spaces as polars "harmonic quadrics". Suppose that a linear ∞^α system σ of quadrics of $[n]$ contains b independent base quadrics and h independent harmonic quadrics. Then if $b+h=\alpha+1$, I shall say that the system σ is *an harmonic system relative to the base spaces* S_λ *and* $S_{n-\lambda-1}$. Clearly no base quadric can also be an harmonic quadric, since the base spaces are independent, so the base quadrics and harmonic quadrics of an harmonic system σ form two complementary systems in σ. It will be shown in §§ 3 and 6 that the system Σ is an harmonic system relative to the base spaces S_r and S_{n-r-1} of the $\gamma_2{}^1$ of genus $r+1$ on $K(n)$ in $[n]$.

Base quadrics and harmonic quadrics.

3.1. On a canonical curve $K(\nu)$ of genus $\nu+1$ in $[\nu]$ the series cut out by quadrics is a complete non-special $g^{3\nu-1}_{4\nu}$. The number of conditions for a quadric of $[\nu]$ to contain $K(\nu)$ is therefore 3ν, and the number of independent quadrics through $K(\nu)$ is $(\nu-1)_2$*. This breaks down only if $K(\nu)$ is hyperelliptic, consisting of a normal $C_0{}^\nu$ counted twice, when the number of conditions for a quadric to contain the curve is $2\nu+1$, and the number of independent quadrics through the curve is $(\nu)_2$.

Since the base curve C^n_{r+1} in S_{n-r-1} is non-special, the quadric sections of the curve are non-special sets and are therefore contained in a complete linear series of freedom $2n-r-1$. The linear series actually cut out on C^n_{r+1} by quadrics is not necessarily complete, and if it is of freedom $2n-r-1-q$ the curve is said to be quadrispecial when $q>0$, and q is called the quadrispeciality†. The number of conditions on a quadric of S_{n-r-1} to contain C^n_{r+1} is $2n-r-q$, and, since there are $(n-r+1)_2$ independent quadrics in S_{n-r-1}, the number of independent quadrics through C^n_{r+1} is

$$\tfrac{1}{2}\{(n-r-1)^2-n-r-1\}+q.$$

* S 1, 222, foot-note.

† W 1, 383.

This number, of course, is never negative. If the expression in curly brackets is negative the freedom of the complete series containing the quadric sections of C^n_{r+1} is greater than that of the quadrics of S_{n-r-1} and the curve is necessarily quadrispecial. Since $n \geqslant 2r$ the only cases in which this happens are given by the following table.

n	2	2	3	4	4	5	6	7	8
r	0	1	1	1	2	2	3	3	4
q	1	2	2	1	3	2	3	1	2

In other cases q is zero in general, but the possibility of $q > 0$ must be taken into account. For instance, an hyperelliptic C^n_{r+1} for which $n = 2r+4-h$, with $h > 0$, has quadrispeciality $h-1$*. Also a canonical curve $K(\nu)$ in $[\nu]$ which acquires a double point becomes a normal non-special $C^{2\nu}_{\nu}$ of quadrispeciality 1†. Another example of interest is the $C_4{}^7$ which is a (3, 4) curve on a quadric surface in [3] with two double points, and for which $q = 2$, instead of 1, as given by the above table.

3.2. For the present we suppose that the $\gamma_2{}^1$ on $K(n)$ is not elliptic or hyperelliptic, these cases being discussed in §6. We also take $n \geqslant 3$, the two dimensional cases being dealt with in §6.2 and §7.9. We denote by (A) the set of $2(n-2r)$ points in which C^n_{r+1} meets $K(n)$, and these points offer independent conditions to quadrics of $[n]$. For the series of order $4n$ cut out on $K(n)$ by quadrics is complete and non-special, so if the points of (A) offered dependent conditions to quadrics, the series of order $2n+4r$ cut out on $K(n)$ by quadrics through (A) would be special, which is impossible except in the elliptic case, when $r = 0$. The quadrics through (A) therefore meet C^n_{r+1} further in sets (B) of $4r$ points which form a linear series of freedom $3r-1-q$. A quadric of Σ which contains a set (B) contains the generators of $\mathbf{R}$ through the points of (B) and consequently contains the corresponding set (B') on $K(r)$, which is therefore a quadric section of $K(r)$. We notice that, if $q > 0$, the quadrics of Σ do not cut out a complete series on the base curve $K(r)$. It is thus $3r-q$ conditions for a quadric of Σ to contain either $K(r)$ or C^n_{r+1} and consequently to contain $\mathbf{R}$. The numbers of further conditions for a quadric to contain S_r and S_{n-r-1} are

* W 1, 387.

† *Cp.* S 1, 222, where Segre is misleading. For, according to Segre, $q = 0$ whenever the above expression in curly brackets is positive, which is disproved by these examples.

$(r-1)_2$ and $\frac{1}{2}\{(n-r-1)^2-n-r-1\}+q$ respectively, whence we deduce that the number of independent base quadrics in Σ is $r(n-r-2)$ while the number of independent quadrics through the fundamental scroll $\mathbf{R}(n, r)$ is $(n-1)_2-3r+q$.

3.3. To find the number of harmonic quadrics we take an $[n-r-2]$ in S_{n-r-1} meeting C^n_{r+1} in a set (H) of n points. The generators of $\mathbf{R}$ through the points of (H) meet $K(r)$ in a set (H') of n points and meet $K(n)$ in a set (H'') of $2n$ points. Suppose that the number of conditions on a general quadric of $[n]$ to make the n pairs of corresponding points of (H) and (H') conjugate pairs is $n-\lambda$. Then there will be a set of $n-\lambda$ of these pairs such that any quadric which has them as conjugates also has the remaining λ pairs as conjugates. Consider a quadric which passes through the $2(n-\lambda)$ points of (H'') on the generators of $\mathbf{R}$ joining the $n-\lambda$ pairs, and also contains one point of (H'') on each of the λ generators of $\mathbf{R}$ joining the remaining pairs. This quadric has the $n-\lambda$ pairs and consequently the remaining λ pairs as conjugates, and therefore passes through the remaining λ points of (H''). Thus the set (H'') only imposes $2n-\lambda$ conditions on quadrics of $[n]$ to contain it, and the residual series of order $2n$ cut out on $K(n)$ by quadrics through (H'') is of freedom $n-1+\lambda$. But (H'') is the section of $K(n)$ by the prime containing S_r and the $[n-r-2]$ from which we started, so the residual series is the canonical series of freedom n, and $\lambda = 1$.

Now the quadrics of Σ have the corresponding pairs of (H) and (H') as conjugates, so the number of conditions on a quadric of Σ for the spaces S_r and $[n-r-2]$ to be conjugate is $(r+1)(n-r-1)-(n-1)$. But when these conditions are satisfied a general point of $K(r)$ is conjugate to the corresponding point of C^n_{r+1} as well as to $[n-r-2]$, and is therefore conjugate to S_{n-r-1}, so the spaces S_r and S_{n-r-1} are polars. Thus the number of conditions for a quadric of Σ to be an harmonic quadric is $r(n-r-2)$, which is also the number of independent base quadrics, *so* Σ *is an harmonic system relative to the base spaces* S_r *and* S_{n-r-1}.

It is reassuring to note that we cannot have overestimated the number of conditions for base quadrics or for harmonic quadrics, for if the freedom of either of these systems was greater than the values we have obtained they would have a quadric in common, and we know that no base quadric can also be an harmonic quadric.

The freedom of C^n_{r+1} *on* $\mathbf{R}(n, r)$.

4.1. A canonical set of $K(r)$ corresponds to a canonical set of C^n_{r+1} and it is convenient to refer to the $2r$ generators of $\mathbf{R}$ which contain corresponding

pairs of points of these sets as a canonical set of generators. Clearly any directrix C^n_{r+1} on $\mathbf{R}$ is normal and non-special, and does not meet $K(r)$ [except the curve $K(r)$ itself when $n = 2r$]. We can therefore draw a prime of $[n]$ through any C^n_{r+1} and any canonical set of generators, so all the directrix C^n_{r+1} are sections of $\mathbf{R}$ by primes through any canonical set of generators. We first consider what happens in general, leaving particular cases till § 4 . 3. If $n \leqslant 3r$ a canonical set of the base C^n_{r+1} does not lie in an $[n-r-2]$, so a canonical set of generators lies in a prime, not in a secundum of $[n]$, and C^n_{r+1} is unique. But if $n > 3r$ a canonical set of the base C^n_{r+1} will lie in a $[2r-1]$ and a canonical set of generators will accordingly lie in a $[3r-1]$ through which pass ∞^{n-3r} primes of $[n]$ meeting $\mathbf{R}$ further in directrix curves C^n_{r+1}. Thus there are ∞^{n-3r} C^n_{r+1} on $\mathbf{R}$, one of them being singled out as containing the coincidences of the $\gamma_2{}^1$ on the particular $K(n)$ of $\mathbf{R}$ from which we started.

4 . 2. If $n > 3r$, in the $[n-r-1]$ of any C^n_{r+1} the primes through a canonical set of C^n_{r+1} meet the curve further in sets (Z) of $n-2r$ points which form a complete linear series of freedom $n-3r-1$. Since C^n_{r+1} is normal, primes through a set (Z) must cut out the complete canonical series, so each set (Z) lies in an $[n-2r-2]$, which is a secant space of C^n_{r+1}. I call the sets (Z) *secant sets*. Since any C^n_{r+1} is the section of $\mathbf{R}$ by a prime of $[n]$ through a canonical set of generators, any two C^n_{r+1} meet in a secant set (Z)*. Also, since any (Z) lies in a secundum of $[n]$ with any canonical set of generators, there are ∞^1 C^n_{r+1} through any (Z). These C^n_{r+1}, being given by a pencil of primes, meet the generators of $\mathbf{R}$ in related ranges of points, so projecting $\mathbf{R}$ from the $[n-2r-2]$ of a (Z) we obtain in $[2r+1]$ a scroll R^{4r}_{r+1} with ∞^1 canonical directrix curves K^{r+1}_{4r} whose spaces $[r]$ generate the well-known locus M^{r+1}_{r+1} formed by the ∞^1 $[r]$ joining corresponding points of $r+1$ related lines†.

Since there are ∞^{n-3r} C^n_{r+1} each containing ∞^{n-3r-1} sets (Z), each of which lies on ∞^1 C^n_{r+1} there are in all $\infty^{2(n-3r-1)}$ sets (Z) on $\mathbf{R}$. But this is the freedom we should expect for $[n-2r-2]$ meeting a surface of $[n]$ in $n-2r$ points. In fact, any set of $n-2r$ points of $\mathbf{R}$ lying in an $[n-2r-2]$ is a set (Z) provided that the $[n-2r-2]$ does not meet the base space S_r. For, projecting from such an $[n-2r-2]$, we obtain in $[2r+1]$ a normal R^{4r}_{r+1} with one canonical curve K^{2r}_{r+1} as a directrix curve. If we take a set (H) of r independent points of K^{2r}_{r+1} the $[r-1]$ through (H) will meet K^{2r}_{r+1} further

* This is also evident from § 1.4, for since the two C^n_{r+1} are non-special, their $n-2r$ intersections must lie in an $[n-2r-2]$.

† R^{4r}_{r+1} is, of course, the scroll of speciality 2 encountered in § 2.1.

in another set (H') of r independent points*. Denote by (g) and (g') the sets of generators of the scroll through (H) and (H'). The ∞^r primes of $[2r+1]$ through K^{2r}_{r+1} contain sets of $2r$ generators of the scroll which cut out the canonical series on all the directrix curves and therefore on K^{2r}_{r+1}. Thus through (g) and K^{2r}_{r+1} there passes only one $[2r]$ which contains (g'). But through (g) we can draw ∞^1 $[2r]$ which either contain all (g') and give ∞^1 directrix K^{2r}_{r+1}, or meet the scroll in directrix curves of higher order, which are therefore non-special but which meet K^{2r}_{r+1} in sets of independent points. The latter is impossible by § 1 . 4, for the scroll is normal and has speciality 2; so there are ∞^1 directrix canonical curves which must arise from ∞^1 C^n_{r+1} of $\mathbf{R}$ through the $n-2r$ points of $\mathbf{R}$ in the vertex of projection.

4 . 3. Particular cases arise when the dimensions of the space containing a canonical set of C^n_{r+1} are less by k than in the general case. If $n > 3r$ and a canonical set of C^n_{r+1} lies in $[2r-1-k]$, the arguments of § 4 . 1 show that there are ∞^{n-3r+k} C^n_{r+1} on $\mathbf{R}$. The canonical sets of all these C^n_{r+1} lie in $[2r-1-k]$. On each C^n_{r+1} there are $\infty^{n-3r-1+k}$ secant sets (Z) of $n-2r$ points, which are, therefore, special sets of speciality k. The prime sections of C^n_{r+1} are thus contained in the bicanonical series, so we can only have $k > 0$ when $n \leqslant 4r$. Each set (Z) still lies in $[n-2r-2]$ and through each (Z) pass a single infinity of C^n_{r+1}, so in all there are $\infty^{2(n-3r-1+k)}$ sets (Z) on $\mathbf{R}$.

Also, when $n \leqslant 3r$, a canonical set of C^n_{r+1} may lie in an $[n-r-1-k]$, with $k > 0$. If this is so there will be ∞^k C^n_{r+1} each containing ∞^{k-1} secant sets (Z) of $n-2r$ points lying in $[n-2r-2]$, and through each (Z) will pass ∞^1 C^n_{r+1}. This is impossible for $n = 2r$, since the base C^n_{r+1} would be special, and also for $n = 2r+1$. If $n = 2r+2$ and $k = 1$ there will be ∞^1 C^n_{r+1} with a common double point. If $n = 2r+3$ and $k = 1$ there will be ∞^1 C^n_{r+1} with a common trisecant line, and so on.

Thus we may expect the particular cases in which $k > 0$ to arise when $2r+2 \leqslant n \leqslant 4r$.

Freedom of bisecant $K(n)$ on $R(n, r)$.

5 . 1. Starting with a $\gamma_2{}^1$ of genus $r+1$ on a $K(n)$ in $[n]$ we obtained an $\mathbf{R}(n, r)$ with two base curves $K(r)$ and C^n_{r+1} lying in independent spaces S_r and S_{n-r-1}. We have tacitly assumed the existence of such a $K(n)$. However, we may obtain a scroll R^{n+2r}_{r+1} by taking a normal non-special C^n_{r+1}, placing it in (1, 1) correspondence with a $K(r)$ in an independent space, and joining corresponding points. If we can find a proper bisecant curve of

* For if (H') lay in $[r-2]$, (H) would belong to a $g_r{}^1$, would have speciality 2, and would consequently lie in an $[r-2]$.

order $2n$ on this R_{r+1}^{n+2r} it will be a canonical curve, since its genus is given by the formula of § 1 . 3 ; the scroll will then be a fundamental scroll, and we shall have established the existence of a $K(n)$ with a γ_2^1 of genus $r+1$. We first inquire whether the C_{r+1}^n on $\mathbf{R}(n, r)$ are general curves or have particular properties.

We saw in § 3 . 2 that a quadric of S_{n-r-1} through (A) meets the base C_{r+1}^n further in a set (B) of $4r$ points which corresponds to a quadric section of $K(r)$ and is therefore a bicanonical set of C_{r+1}^n. Bicanonical sets of C_{r+1}^n are thus contained in quadric sections, and this turns out to be the only particular property of C_{r+1}^n. Since a quadric through $K(n)$ meets $\mathbf{R}(n, r)$ in a bicanonical set of $4r$ generators, all the C_{r+1}^n on $\mathbf{R}(n, r)$ have this property. Since the freedom of the series cut out by quadrics is in general $2n-r-1$, the C_{r+1}^n on a general $\mathbf{R}(n, r)$ are of general type if $2n \geqslant 5r+1$, but are particular curves if $2n \leqslant 5r$.

We notice also that if $n < 3r$ we can draw a prime of S_{n-r-1} through (A) which will meet C_{r+1}^n further in a set (X) of $4r-n$ points. This prime, together with another prime of S_{n-r-1}, is a degenerate quadric whose intersection with C_{r+1}^n other than (A) is therefore a bicanonical set. The prime through (A) counted twice is a particular quadric of this type, so, denoting corresponding sets on $K(r)$ by dashes, we see that $(A')+2(X')$ is the section of $K(r)$ by a quadric, and that prime sections of C_{r+1}^n correspond to the sections of $K(r)$ by quadrics through (X'). Since the points of (X') are the points of contact of a quadric, (X') is not a general set of points on $K(r)$ if $2(4r-n) > 3r-1$, that is if $2n \leqslant 5r$. Thus we can always obtain a C_{r+1}^n with the required property from a $K(r)$, by making a suitable system of quadrics of $[r]$ correspond to the primes of S_{n-r-1}. (See foot-note, p. 175.)

5 . 2. Consider then the scroll R_{r+1}^{n+2r} in $[n]$ generated by the joins of corresponding points of a $K(r)$ and a normal non-special C_{r+1}^n which are in (1, 1) correspondence and lie in independent spaces S_r and S_{n-r-1}. The only assumptions that we make about C_{r+1}^n are that $n \geqslant 2r$ and that a bicanonical set (B) lies on a quadric of S_{n-r-1} which meets C_{r+1}^n further in a set (A) of $2n-4r$ points. If (B') denotes the corresponding bicanonical set of $K(r)$, the number of conditions for a quadric of $[n]$ to contain the sets (B') and $(A)+(B)$ and to have the spaces S_r and S_{n-r-1} as polars is $(3r-1)+(2n-r-1-q)+(r+1)(n-r)$, where q is the quadrispeciality of C_{r+1}^n, and it can be shown that in every case there are at least ∞^1 quadrics satisfying these conditions. These quadrics all contain the $2r$ generators of R_{r+1}^{n+2r} which join corresponding points of (B) and (B'), and two further conditions are required to make them contain $K(r)$ and C_{r+1}^n and consequently all the generators of R_{r+1}^{n+2r}. Thus the sections of the scroll by these

quadrics are a single infinity of bisecant curves of order $2n$ and, therefore, of genus $n+1$; these are canonical curves, and the scroll R_{r+1}^{n+2r} is therefore a fundamental scroll $\mathbf{R}(n, r)$. Also, these ∞^1 bisecant $K(n)$ all pass through (A) and the pairs of points in which they meet the generators of the scroll are harmonically separated by S_r and S_{n-r-1}. Thus, on a fundamental scroll $\mathbf{R}(n, r)$, through any set (A), which is the residual intersection of a quadric through a bicanonical set of a directrix C_{r+1}^n, pass a single infinity of bisecant $K(n)$ which have this bisecant C_{r+1}^n as base curve of the γ_2^1 cut out by the generators of $\mathbf{R}$.

The residual intersection with C_{r+1}^n of a quadric through any set equivalent to (A) is a bicanonical set, so through any set of the complete series on C_{r+1}^n determined by (A) there pass ∞^1 bisecant $K(n)$ having C_{r+1}^n as base curve. Of course, when C_{r+1}^n is not unique there will be other bisecant $K(n)$ through (A) which do not have the C_{r+1}^n containing (A) as base curve, for evidently any $K(n)$ meets any C_{r+1}^n in a set (A), since the $K(n)$ are sections of $\mathbf{R}$ by quadrics of $[n]$ through a bicanonical set of generators. We note that on any C_{r+1}^n the sets (A) are equivalent to two sets (Z), so no two C_{r+1}^n can have a set (A) in common.

5.3. The sets (A) on a C_{r+1}^n consist of $2n-4r$ points, and cannot be special if $n > 3r$. If $5r < 2n \leqslant 6r$ the sets (A) are not in general special, but may have speciality j. If $4r < 2n \leqslant 5r$ the sets (A) have speciality $5r-2n+1$ in general, but they may have speciality $5r-2n+1+j$. In either case when $j > 0$ the freedom of (A) on each C_{r+1}^n is increased by j, and all C_{r+1}^n of $\mathbf{R}(n, r)$ have the same value of j. When the sets (A) consist of two points and $j = 1$, C_{r+1}^n possesses a g_2^1 and the scroll is hyperelliptic.

Thus for the general $\mathbf{R}(n, r)$ we have the following possibilities:

(a) $n = 2r$. There are no sets (A), C_{r+1}^n is unique, and there are ∞^1 bisecant $K(n)$.

(b) $4r < 2n \leqslant 5r+1$. There is one C_{r+1}^n containing one set (A), through which pass ∞^1 $K(n)$.

(c) $5r+1 \leqslant 2n \leqslant 6r$. There is one C_{r+1}^n containing $\infty^{2n-5r-1}$ sets (A), and the freedom of $K(n)$ on R is $2n-5r$.

(d) $n \geqslant 3r$. There are ∞^{n-3r} C_{r+1}^n each containing $\infty^{2n-5r-1}$ sets (A), and the freedom of $K(n)$ is $3n-8r$.

The only exceptions are when $k > 0$, which can happen only if $2r+2 \leqslant n \leqslant 4r$*, and when $j > 0$, which can happen only if $2r+1 \leqslant n \leqslant 3r$.

* See § 4.3.

In these particular cases the freedoms of C^n_{r+1} and $K(n)$ on $\mathbf{R}(n, r)$ are increased.

5.4. We now show that general fundamental scrolls of the same genus form a system each member of which may be obtained as the projection of one of higher order, and in so doing we see more clearly the nature of the particularity of C^n_{r+1} in the lower cases. We first note that, if a normal scroll $R_p{}^m$ in $[N]$ is the projection of a normal scroll R_p^{m+1} in $[N+1]$ from a point P of itself, the proper bisecant curves $C_\pi{}^\nu$ on $R_p{}^m$ arise from bisecant $C_\pi^{\nu+2}$ of R_p^{m+1} with a double point at P, some of which may split up into the generator through P and a proper bisecant $C_\pi^{\nu+1}$ through P.

A proper bisecant curve of R_p^{m+1} which does not contain P projects into an improper bisecant curve of $R_p{}^m$. Now a $K(n)$ in $[n]$ cannot be the projection of a C^{2n+1}_{n+1} in $[n+1]$ for no such curve exists*, so if an $\mathbf{R}(n, r)$ in $[n]$ is the projection of an $\mathbf{R}(n+1, r)$ in $[n+1]$ from a point P of itself, the bisecant $K(n)$ of $\mathbf{R}(n, r)$ must all be the projections of bisecant $K(n+1)$ of $\mathbf{R}(n+1, r)$ which have acquired a double point at P.

Suppose, first, that $n > 3r$. The freedom of $K(n)$ on $\mathbf{R}(n, r)$ is $3n-8r$. It is three conditions for a $K(n)$ to have a double point at a general point P of the scroll, so projecting from P we obtain $\mathbf{R}(n-1, r)$ in $[n-1]$ with $\infty^{3(n-1)-8r} K(n-1)$ and $\infty^{n-1-3r} C^{n-1}_{r+1}$. This process continues till we obtain $\mathbf{R}(3r, r)$ in $[3r]$ with one C^{3r}_{r+1} and $\infty^r K(3r)$, though when $n = 3r+1$ the $\infty^1 C^n_{r+1}$ all pass through the same secant set (Z) of $r+1$ points, so we must not project from a point of one of the generators of $\mathbf{R}(3r+1, r)$ through this set (Z).

Secondly, if $5r+3 \leqslant 2n \leqslant 6r$, of the $\infty^{2n-5r-1}$ sets (A) on the unique C^n_{r+1} of $\mathbf{R}(n, r)$ $\infty^{2n-5r-3}$ will contain a general point P of C^n_{r+1} counted twice, and the $\infty^1 K(n)$ through each of these sets will have a double point at P†. Projecting from P we obtain $\mathbf{R}(n-1, r)$ in $[n-1]$ with one C^{n-1}_{r+1} and $\infty^{2(n-1)-5r} K(n-1)$. Proceeding thus we reach the case of $2n = 5r+1$ if r is odd, or if r is even we reach the case of $2n = 5r+2$ when there are ∞^1 sets (A) of which a finite number will have a united point; projecting from one of these particular points of the C^n_{r+1} we obtain $R(n, r)$, with $2n = 5r$ possessing $\infty^1 K(n)$.

Finally, if $4r < 2n \leqslant 5r+1$ there is only one C^n_{r+1} containing only one set (A), through which pass $\infty^1 K(n)$. In general, the points of (A) will be distinct, but for particular scrolls two of them will coincide and by projec-

* Such a curve would be special, which is impossible since the order of the curve is greater than that of the canonical series. See § 1.2.

† See the end of § 2.6.

tion from this point we obtain $R(n-1, r)$ with one C_{r+1}^{n-1} and $\infty^1 K(n-1)$. We can proceed thus till we obtain the last fundamental scroll $\mathbf{R}(2r, r)$, a particularity of C_{r+1}^{n} being implied by each projection of $\mathbf{R}(n, r)$ when $2n \leqslant 5r+2$.

Elliptic and hyperelliptic fundamental scrolls.

6.1. The base curves of an elliptic γ_2^1 on a $K(n)$ in $[n]$ are a point O and a normal C_1^n in S_{n-1}. Clearly all the quadrics through $K(n)$ are harmonic, having O and S_{n-1} as pole and polar. The curve C_1^n is not quadrispecial if $n \geqslant 3$*, so $\mathbf{R}(n, 0)$ is an elliptic cone of order n through which pass $(n-1)_2-1$ independent quadrics, all of which are cones. The sets (Z) on C_1^n are the sections by primes of S_{n-1}, and there are $\infty^n C_1^n$, which are just the sections of $\mathbf{R}(n, 0)$ by the primes of $[n]$. The sets (A) on each C_1^n are sections by quadrics of S_{n-1} and are of freedom $2n-1$. Through each (A) of a C_1^n pass $\infty^1 K(n)$ with the C_1^n as base curve, so the freedom of $K(n)$ on $\mathbf{R}(n, 0)$ is $3n$, agreeing with the general result of §5.3.

6.2. If a K_3^4 in a plane π possesses an elliptic γ_2^1 it must be the projection of a K_4^6 on $\mathbf{R}_1^3$ in [3] which has acquired a double point, necessarily on the base C_1^3. Thus the pairs of points of the γ_2^1 on K_3^4 are harmonic conjugates with respect to a point O and a line l, their joins generating a plane pencil with vertex O counted twice, which is $\mathbf{R}_1^2$.

6.3. When the γ_2^1 of genus $r+1$ on $K(n)$ in $[n]$ is hyperelliptic the base curve $K(r)$ in S_r is a rational normal C_0^r counted twice, on which the series cut out by quadrics is of freedom $2r$ and is not the complete bicanonical series if $r > 1$. The arguments of the preceding paragraphs which are affected by this are those used to find the freedoms of base quadrics, of quadrics through $\mathbf{R}(n, r)$, and of $K(n)$ on $R(n, r)$, the other arguments remaining valid. The joins of sets of the g_2^1 on the base curve C_{r+1}^n generate a rational normal scroll R_0^{n-r-2} in S_{n-r-1}, and the planes containing pairs of generators of $\mathbf{R}(n, r)$ generate a rational normal plane-scroll E_0^{n-2}, since they join corresponding points and generators of C_0^r and R_0^{n-r-2}. We know the quadrispeciality of the hyperelliptic C_{r+1}^n, for, provided that $n \geqslant r+4$, if $n \geqslant 2r+4$ the curve is not quadrispecial and the quadrics through it do not all contain R_0^{n-r-2}, while if $n \leqslant 2r+3$ all quadrics through the curve contain R_0^{n-r-2} and the quadrispeciality is $2r+3-n$†. The restriction $n \geqslant r+4$ simply means that C_{r+1}^n lies in space of at least three

* W 1, 387.

† W 1, 387.

dimensions, and the only curves with $r > 1$ which do not satisfy this are C_3^4, C_3^5, and C_4^6, for which q is 3, 2, and 3 respectively, by § 3 . 1. However, in these cases also, $q = 2r+3-n$.

When $q = 0$ the quadrics of S_{n-r-1} through the set (A) cut out the complete bicanonical series of C_{r+1}^n, which corresponds to the complete bicanonical series of $K(r)$ containing the series of freedom $2r$ cut out by the quadrics of S_r. Thus, when $q = 0$, the quadrics of Σ [*i.e.* quadrics through $K(n)$] cut out a series of freedom $2r$ on C_0^r, and $2r+1$ conditions are required to make a quadric of Σ contain C_0^r and consequently $\mathbf{R}(n, r)$. The number of further conditions for a quadric of Σ to contain S_r and S_{n-r-1} is $(r)_2+\frac{1}{2}\{(n-r-1)^2-n-r-1\}$. Thus there are again $r(n-r-2)$ independent base quadrics, but $(n-1)_2-2r-1$ independent quadrics through $\mathbf{R}(n, r)$.

When $q > 0$ all quadrics through $\mathbf{R}(n, r)$ contain R_0^{n-r-2} and therefore contain E_0^{n-2}. But the number of independent quadrics through E_0^{n-2} is $(n-2)_2$*, which is equal to $(n-1)_2-(2r-q+1)$ since $q = 2r+3-n$. Thus the quadrics of Σ cut out a series on C_0^r of freedom $2r-q$. Also the number of conditions for a quadric through E_0^{n-2} to contain S_r and S_{n-r-1} is $(r)_2+(n-r-2)_2$, wherefore the number of independent base quadrics is again $r(n-r-2)$. The argument of § 3 . 3 for the harmonic quadrics is still valid so we have established the following theorem.

When $n > 3$ the quadrics of $[n]$ through a $K(n)$ which possesses a γ_2^1 of genus $r+1$ always form an harmonic system relative to the two base spaces S_r and S_{n-r-1}, which contains $r(n-r-2)$ independent base quadrics and $(n-1)_2-r(n-r-2)$ independent harmonic quadrics. The number of independent quadrics through the $\mathbf{R}(n, r)$ given by the γ_2^1 is $(n-1)_2-3r+q$, where q is the quadrispeciality of the base curve C_{r+1}^n, unless the scroll is elliptic or hyperelliptic when the number is $(n-1)_2-2r-1+q$.

6 . 4. The arguments of §§ 4 and 5 are valid for the hyperelliptic $\mathbf{R}(n, r)$, though a slight change is necessary in § 5 . 2. The number of conditions for a quadric of $[n]$ to contain the sets (B') and $(A)+(B)$ and to have the spaces S_r and S_{n-r-1} as polars is now

$$2r+(2n-r-1-q)+(r+1)(n-r),$$

but the number of further conditions for these quadrics to contain $\mathbf{R}(n, r)$ is still two. Thus there are still ∞^1 bisecant $K(n)$ of $\mathbf{R}$ passing through any set (A) of a C_{r+1}^n and having the C_{r+1}^n as base curve.

---

* § 2.5.

Fundamental scrolls of genus 2.

7.1. A γ_2^1 of genus 2 on a $K(n)$ in $[n]$ gives an R_2^{n+2}. The base spaces are a line d and an S_{n-2} containing a directrix C_2^n which meets $K(n)$ in a set (A) of $2(n-2)$ points. The line d is double on R_2^{n+2} and the scroll is generated by a (1, 2) correspondence between d and C_2^n. The following table shows the results of the general theory up to $n = 8$, and it will be seen that all the numbers increase regularly after $n = 5$. The numbers b, h, and ρ respectively denote the number of independent base quadrics and harmonic quadrics through $K(n)$ and the number of independent quadrics through R_2^{n+2}. The particularity $k > 0$ can only occur when $n = 4$, while $j > 0$ can only occur if $n = 3$*. Also q, the quadrispeciality of C_2^n, can have no values other than those given in the table.

$[n]$	[3]	[4]	[5]	[6]	[7]	[8]
Order of $\mathrm{R}(n, 1)$	5	6	7	8	9	10
Number of points in (Z)	...	2	3	4	5	6
Freedom of (Z) on a C_2^n	...	0	1	2	3	4
Freedom of C_2^n	0	1	2	3	4	5
Number of points in (A)	2	4	6	8	10	12
Freedom of (A) on C_2^n	0	2	4	6	8	10
Freedom of $K(n)$ on $\mathrm{R}(n, 1)$	1	4	7	10	13	16
q	2	1	0	0	0	0
ρ	0	1	3	7	12	18
b	0	1	2	3	4	5
h	1	2	4	7	11	16

7.2. On any C_2^n of R_2^{n+2} the pairs of points corresponding to points of d are sets of the canonical series of C_2^n, and their joins generate a rational

* See § 5.3.

normal scroll R_0^{n-3} in the S_{n-2} containing $C_2{}^n$. The planes containing canonical pairs of generators of $\mathbf{R}_2^{n+2}$ generate a rational normal plane-scroll* E_0^{n-2}, of order $n-2$, since they join the generators of R_0^{n-3} to the corresponding points of d. The freedom of quadrics of $[n]$ through E_0^{n-2} is $\frac{1}{2}(n-1)(n-4)$†, which is less by $n-5$ than the freedom of quadrics through $\mathbf{R}_2^{n+2}$. Now in S_{n-2} the freedom of quadrics through R_0^{n-3} is $\frac{1}{2}(n-2)(n-5)$, and the line-cones projecting them from d contain E_0^{n-2}. Also, the freedom of quadrics of S_{n-2} through $C_2{}^n$ is $\frac{1}{2}n(n-5)$ and the line-cones projecting them from d contain $\mathbf{R}_2^{n+2}$. These freedoms differ by $n-5$, so the system of quadrics through $\mathbf{R}_2^{n+2}$ is determined by $(n-2)_2$ independent quadrics through E_0^{n-2} and $n-5$ independent line-cones with vertex d. Thus, if $n>5$, the fundamental scroll $\mathbf{R}(n, 1)$ is the section of a plane-scroll E_0^{n-2} which possesses a directrix line by a quadric line-cone which has the directrix line as vertex and contains $n-6$ generating planes of E_0^{n-2}.

On E_0^{n-2} there are ∞^{n-3} directrix R_0^{n-3}, each containing a $C_2{}^n$ of $\mathbf{R}_2^{n+2}$. On each R_0^{n-3} there are ∞^{n-4} directrix curves D_0^{n-4} meeting the $C_2{}^n$ on R_0^{n-3} in sets of $n-2$ points which are the secant sets (Z) of § 4 . 2. There are in all $\infty^{2(n-4)} D_0^{n-4}$ on E_0^{n-2}, each meeting $\mathbf{R}_2^{n+2}$ in a set (Z). Through each D_0^{n-4} there pass $\infty^1 R_0^{n-3}$ containing the $\infty^1 C_2{}^n$ through the set (Z) on D_0^{n-4}. Any two $C_2{}^n$ meet in a secant set (Z), and any two R_0^{n-3} meet in a D_0^{n-4}. Any $[n-4]$ which contains $n-2$ points of $\mathbf{R}_2^{n+2}$ and does not meet d contains a D_0^{n-4}. From one of these $[n-4]$ the plane scroll E_0^{n-2} projects into a quadric surface in [3], so each $[n-4]$ containing a D_0^{n-4} is the vertex of a quadric cone containing E_0^{n-2}. The generating $[n-2]$ of one system of the cone meet E_0^{n-2} in generating planes and $\mathbf{R}_2^{n+2}$ in canonical pairs of generators, while generating $[n-2]$ of the other system contain R_0^{n-3} of E_0^{n-2} and $C_2{}^n$ of $\mathbf{R}_2^{n+2}$.

It follows from the above that, of the $\infty^{\frac{1}{2}(n-1)(n-4)}$ quadrics through E_0^{n-2}, $\infty^{2(n-4)}$ are cones with secant $[n-4]$ as vertices. In passing it is worth remarking that the general $E_0{}^3$ in [5] has ∞^2 directrix lines, and that all the quadrics through $E_0{}^3$ are line cones with directrix lines of $E_0{}^3$ as vertices. Such a system of quadric cones is determined by any two quadric cones in [5] with a solid in common, for the residual intersection is an $E_0{}^3$. Any two cones of the system have a solid in common, and the vertices of any pencil in the system generate a regulus in the solid common to the cones of the pencil. This is trivial enough, but the existence of this linear ∞^2

---

* For properties of rational normal plane-scrolls see S 4. A brief summary is given in W 3, 111.

† § 2.5.

system of line cones contradicts Kantor's statement that the locus of $[\mu]$ which are vertices of a linear ∞^k system of quadric $[\mu]$-cones in $[r]$ is a manifold of order $\binom{\frac{1}{2}(r-\mu)}{k}$ and lies in a space of $\frac{1}{2}(r+\mu)$ dimensions common to all the cones*. In fact, systems of quadrics which are all cones need further investigation. Segre only considered singly infinite systems†.

7.3. Suppose that the directrix curve of lowest order of an R_0^{n-3} of E_0^{n-2} is a D_0^{μ} of order μ. Then $2\mu \leqslant n-3$. Also, since R_0^{n-3} contains a $C_2{}^n$, the expression $2\mu+n-2(n-3)$, which gives the number of intersections of $C_2{}^n$ and D_0^{μ}, cannot be negative‡, and therefore $2\mu \geqslant n-6$. Thus, if $n = 2\lambda$, we must have $\mu = \lambda-2$ or $\lambda-3$, and, if $n = 2\lambda+1$, we must have $\mu = \lambda-1$ or $\lambda-2$. In fact for $n > 4$ there are always two distinct types of surface to be considered. In each case we may call the first type general and the second particular.

Supposing first that $n = 2\lambda$, if one R_0^{n-3} possesses a $D_0^{\lambda-3}$, there is a unique minimum directrix $R_0^{\lambda-2}$ on E_0^{n-2} which contains all $D_0^{\lambda-3}$ of E_0^{n-2} and meets every R_0^{n-3} in a $D_0^{\lambda-3}$, so all the R_0^{n-3} are of the same type. On $R_0^{\lambda-2}$ there are $\infty^{\lambda-3} D_0^{\lambda-3}$ through each of which pass $\infty^{\lambda} R_0^{n-3}$, giving all the $\infty^{n-3} R_0^{n-3}$. If, on the other hand, one R_0^{n-3} has a $D_0^{\lambda-2}$ as minimum directrix curve, no R_0^{n-3} can possess a $D_0^{\lambda-3}$ and E_0^{n-2} has a unique minimum directrix $R_0^{\lambda-1}$ which contains all $D_0^{\lambda-2}$ of E_0^{n-2} and meets each R_0^{n-3} in a $D_0^{\lambda-2}$. In this case there are $\infty^{\lambda-2} D_0^{\lambda-2}$ through each of which pass $\infty^{\lambda-1} R_0^{n-3}$, giving all the $\infty^{n-3} R_0^{n-3}$. In the general case the $C_2{}^n$ and the $D_0^{\lambda-2}$ on any R_0^{n-3} meet in two points, so $\mathbf{R}_0^{n+2}$ and $R_0^{\lambda-1}$ have two generators in common, and these two generators contain the intersections of any $C_2{}^n$ with the $D_0^{\lambda-2}$ of its R_0^{n-3}. Projecting from a general point of $\mathbf{R}_0^{n+2}$ we obtain the general scroll in $[n-1]$, while projection from a point of one of the two common generators of $\mathbf{R}_0^{n+2}$ and $R_0^{\lambda-1}$ gives a scroll of particular type in $[n-1]$. In the particular case in $[n]$ a $C_2{}^n$ and a $D_0^{\lambda-3}$ on an R_0^{n-3} have no intersections, so $\mathbf{R}_0^{n+2}$ contains no generators of $R_0^{\lambda-2}$ and projection from a point of $\mathbf{R}_0^{n+2}$ not lying on d gives rise to only one type of scroll in $[n-1]$, the particular type.

Similarly, if $n = 2\lambda+1$, it is easily shown that all R_0^{n-3} are of the same type. If they are of general type any R_0^{n-3} contains $\infty^1 D_0^{\lambda-1}$ each meeting the $C_2{}^n$ on R_0^{n-3} in three points. Each $D_0^{\lambda-1}$ lies on a minimum directrix $R_0{}^{\lambda}$ of E_0^{n-2}, and there are $\infty^1 R_0{}^{\lambda}$ each containing d and meeting R_0^{n+2} in three generators. There are in all $\infty^{\lambda} D_0^{\lambda-1}$ of which $\infty^{\lambda-2}$ pass through

* Kantor, *Sitzungsberichte der Wiener Ak.*, 112, IIa (1903), 815.

† Segre, *Atti Acc. Torino*, 19 (1884), 878.

‡ §2.4.

a general point of E_0^{n-2}, so projection from any point of $\mathbf{R}_2^{n+2}$ not lying on d gives rise to the general type of scroll in $[n-1]$. In the particular case in $[n]$ each R_0^{n-3} has one $D_0^{\lambda-2}$ meeting the $C_2{}^n$ on R_0^{n-3} in one point. E_0^{n-2} has, therefore, a unique minimum directrix $R_0^{\lambda-1}$ containing one generator of R_2^{n+2}. Projection from a point of this generator gives the particular type of scroll in $[n-1]$, while projection from a general point of R_2^{n+2} gives the general type of scroll in $[n-1]$.

7.4. The fundamental scroll $\mathbf{R}_2{}^7$ in [5] is of particular interest. It has a double line d and ∞^2 directrix $C_2{}^5$. In the general case the lines joining pairs of the $g_2{}^1$ on a $C_2{}^5$ generate a regulus whose directrix lines are trisecant lines of $\mathbf{R}_2{}^7$, and meet $C_2{}^5$ in the secant sets (Z). The planes containing canonical pairs of generators of $\mathbf{R}_2{}^7$ generate an $E_0{}^3$ whose ∞^2 directrix reguli are those containing the $\infty^2 C_2{}^5$; the ∞^2 directrix lines of $E_0{}^3$ are trisecants of $\mathbf{R}_2{}^7$; any trisecant line of $\mathbf{R}_2{}^7$ which does not meet d is a directrix line of $E_0{}^3$ and through its intersections with the scroll there pass $\infty^1 C_2{}^5$. The ∞^2 quadrics through $\mathbf{R}_2{}^7$ contain $E_0{}^3$ and are line cones with trisecants of $\mathbf{R}_2{}^7$ as vertices, solids of one system of the cones containing pairs of generators of $\mathbf{R}_2{}^7$ while solids of the other system contain $C_2{}^5$. The scroll $\mathbf{R}_2{}^7$ projects from a trisecant line into a double quadric surface.

In the particular case all the reguli which contain $C_2{}^5$ are cones with a common vertex O, and $E_0{}^3$ is a cone projecting an $R_0{}^3$ of [4] from O. The point O is simple on $C_2{}^5$ and is, therefore, a simple point of $\mathbf{R}_2{}^7$. Of the ∞^7 bisecant K_6^{10} on $\mathbf{R}_2{}^7$ ∞^6 will pass through O. A generating plane of $E_0{}^3$ will meet one of these K_6^{10} in five points, and solids through this plane will cut out a $g_5{}^2$ on the K_6^{10}, which, therefore, lies on a Veronese surface*.

7.5. A $\gamma_2{}^1$ of genus 2 on a $K_5{}^8$ in [4] gives an $\mathbf{R}_2{}^6$ with a double line d and a plane $C_2{}^4$ as base curves. Since $n = 4r$ we should expect a particular case to arise in which $k = 1$ (see §4.3), but in this case $C_2{}^4$ is a double conic and $\mathbf{R}_2{}^6$ is a double cubic scroll containing ∞^2 double directrix conics. Apart from this case there is one secant set (Z) of two coincident points, so $\mathbf{R}_2{}^6$ has ∞^1 directrix $C_2{}^4$ with a common double point O, which is a double point of the scroll. Since $q = 1$ the ∞^2 quadrics through $K_5{}^8$ cut out an involution on d and one of them contains $\mathbf{R}_2{}^6$. This is, of course, the quadric cone with vertex O generated by planes containing canonical pairs of generators of $\mathbf{R}_2{}^6$. The generating planes of the other system of this cone contain the $\infty^1 C_2{}^4$, so the cone is also the one base quadric of each of the $\infty^4 K_5{}^8$ on $\mathbf{R}_2{}^6$†.

* *Cp.* §12.2.

† This scroll is discussed by Edge, E, 253.

The (A) sets on a C_2^4 are the residual intersections of a conic through a bicanonical set, which consists of two canonical sets. The locus of harmonic conjugates of O with respect to canonical sets of C_2^4 is, in general, a cubic with a double point at O. If O is a bi-flecnode the cubic breaks up into the two inflexional tangents at O and a line l which is the polar of O with respect to any conic through two canonical sets of C_2^4. Thus in this particular case all the sets (A) are composed of two canonical sets. Also, any conic with respect to which l is the polar of O meets C_2^4 in four canonical sets. In particular, through a coincidence of the canonical g_2^1 pass an involution of pairs of lines which meet C_2^4 further in three canonical sets. If one C_2^4 of $\mathbf{R}_2^6$ has a bi-flecnode at O, they all have, and all the K_5^8 on $\mathbf{R}_2^6$ project from O into double elliptic quartics. In fact, in this particular case each K_5^8 of $\mathbf{R}_2^6$ has a γ_2^1 of genus 1 as well as one of genus 2.

7.6. A γ_2^1 of genus 2 on a K_4^6 in [3] gives an $\mathbf{R}_2^5$ generated by a (3, 2) correspondence between a double line d and a triple line t, the latter meeting K_4^6 in two points A_1 and A_2*. The unique directrix C_2^3 is t and has quadrispeciality 2. Though much of the general theory is applicable it is better to treat this case separately, particularly as there is only one quadric through the canonical curve. The K_4^6 is the intersection of a quadric Q and a cubic surface. The quadric cannot contain d or t, for it would then contain $\mathbf{R}_2^5$, so Q meets d in two points D_1 and D_2. The intersection of Q and $\mathbf{R}_2^5$ other than K_4^6 consists of four generators, which must be A_1D_1, A_1D_2, A_2D_1, and A_2D_2. The lines d and t are thus polars with respect to Q, which is therefore an harmonic quadric. It should be noted that the lines A_1D_1 and A_1D_2 each contain two points of K_4^6 other than A_1 which are a pair of the γ_2^1, and that the point corresponding to A_1 in the γ_2^1 lies on the third generator of $\mathbf{R}_2^5$ through A_1 and coincides with A_1. This is clear if we regard $\mathbf{R}_2^5$ as the projection of $\mathbf{R}_2^6$ from a general point P. For one C_2^4 of $\mathbf{R}_2^6$ passes through P and on this C_2^4 there is one set (A) consisting of P counted twice and two other points P_1 and P_2. The $\infty^1 K_5^8$ of $\mathbf{R}_2^6$ which have C_2^4 as base curve and pass through this set (A) have double points at P. Also the two lines PP_1 and PP_2 form a conic through the set (A) and their further intersection with C_2^4 consists of two canonical sets. Thus, projecting from P, we obtain our $\mathbf{R}_2^5$ in [3] with a double line and a triple line and ∞^1 bisecant K_4^6 all passing through the same two points A_1 and A_2 of the triple line, the projections of P_1 and P_2. The generator of $\mathbf{R}_2^6$ through P_1 gives a generator of $\mathbf{R}_2^5$ which is the tangent

* *Cp.* A. Wiman, *Bihang till K. Svenska Vet.-Akad. Handlingar*, 21 (1895), 1, 3, 10. Wiman shows that the scroll given by a γ_2^1 on a K_4^6 is an $\mathbf{R}_2^5$ or an elliptic cubic cone.

to all the K_4^6 at A_1, while the generators of $\mathbf{R}_2^6$ through the other two intersections of PP_1 with C_2^4 give the generators A_1D_1 and A_1D_2 of $\mathbf{R}_2^5$.

If we take the particular type of $\mathbf{R}_2^6$ in which the sets (A) are bicanonical sets, and take P to be a united point of the canonical g_2^1 on a C_2^4, then there is an involution of pairs of lines through P which meet C_2^4 in three canonical sets and each of these canonical sets together with P counted twice forms a set (A)*. Thus, projecting from P, we obtain a particular type of $\mathbf{R}_2^5$ in [3] containing ∞^2 bisecant K_4^6 of which three ∞^1 systems pass through each pair of points of a certain involution on the triple line. This is the particular case in which $j=1$, which we should expect from general theory.

7.7. Conversely, suppose that we have a scroll R_2^5 generated by a general (3, 2) correspondence between two lines d and t, given by the equation

$$a_i\lambda^i\mu^2+b_i\lambda^i\mu+c_i\lambda^i=0 \quad \text{(summed for } i=0,\ 1,\ 2,\ 3).$$

The curve in the (x, y, z) plane given by the parametric equations

$$\frac{x}{a_i\lambda^i}=\frac{y}{b_i\lambda^i}=\frac{z}{c_i\lambda^i}$$

is in general a rational cubic with one double point whose parameters give two points D_1 and D_2 on d which correspond to the same pair of points A_1 and A_2 on t. Through the four lines A_1D_1, A_2D_1, A_1D_2, and A_2D_2, which are generators of R_2^5, we can draw ∞^1 quadrics which will meet the scroll further in ∞^1 bisecant K_4^6 all passing through A_1 and A_2. Thus R_2^5 is the fundamental scroll $\mathbf{R}_2^5$, or $\mathbf{R}(3, 1)$.

The particular type of $\mathbf{R}_2^5$ which contains ∞^2 bisecant K_4^6 arises when the plane cubic is a triple line. For there are then ∞^1 sets of three points on d which correspond to the same two points of t. Any other degeneration of the plane cubic would give a degenerate scroll.

7.8. In order to examine the geometrical conditions under which a K_4^6 possesses a γ_2^1 of genus 2 we inquire whether a cubic surface F^3 in [3] can possess an harmonic involution with two lines l and l' as base lines, that is to say an involution contained in the harmonic involution of [3] defined by l and l'. If this happens, any line meeting l and l' contains three points of F^3, two of which must correspond in the involution while the third must be a united point. Thus one of the lines, l, lies on F^3. Now the

* § 7.5.

planes through any line l of a cubic surface meet the surface in a conic with respect to which l has a unique pole, unless the conic is a pair of lines meeting on l, in which case the plane contains three concurrent lines of the surface and is called an *osculatory* tritangent plane*. The points of intersection of the conics with l form an involution with two double points, so the locus of poles of l with respect to the conics is in general a twisted cubic γ meeting l in two points. If γ is degenerate, containing a line l' which does not meet l, the cubic surface has an harmonic involution with l and l' as base lines, and the rest of γ must consist of two lines each meeting l and l', the planes joining these lines to l being osculatory. Thus either of the following conditions is both necessary and sufficient for the surface F^3 to possess an harmonic involution:

(a) The residual points of contact of three of the five tritangent planes of F^3 through l are collinear.

(b) Two of the tritangent planes of F^3 through l are osculatory.

The harmonic involution on F^3 is a particular case of the involution on a general cubic surface given by the residual intersections of lines which meet a line of the surface and an arbitrary line of space. The existence of F^3 is established by its equation, which is of the form

$$ax^3+bx^2y+cxy^2+dy^3+x(fz^2+gzt+ht^2)+y(pz^2+qzt+rt^2)=0,$$

for if (x, y, z, t) lies on this surface so does $(x, y, -z, -t)$. The plane representation of F^3 is simple. Consider in a plane ϖ a line b and a pencil Σ_2 of conics with four base points B_1, B_2, B_3, and B_4. Two conics of Σ_2 will touch b in points B_5 and B_6. The ∞^3 system of cubics through B_1, B_2, ..., B_6 represent the plane sections of a cubic surface F^3 in [3]. The line b gives the line l of F^3 and the osculatory planes of F^3 through l correspond to the degenerate cubics in ϖ consisting of b and the conic of Σ_2 through B_5 or B_6. The points of contact of tangents to conics of Σ_2 from a point H of b generate a cubic curve through B_1, ..., B_6 and the diagonal points of the quadrangle $B_1B_2B_3B_4$, and as H varies on b these cubics generate a pencil Σ_3. The pencils Σ_2 and Σ_3 correspond to the sections of F^3 by planes through l and l', and the harmonic involution on F^3 is given

* This does not imply a node of the surface. Cubic surfaces with osculatory planes were first studied by Eckardt. For references see E. Pascal, *Repertorium der höheren mathematik*, II 2, 828.

by the pairs of free intersections of curves of Σ_2 and Σ_3, or by pairs of points of contact of tangents to conics of Σ_2 from points of b. A particular type of F^3 is the surface with two biplanar nodes having a common tangent plane. Timms gives this surface*, but does not mention that the intersections of the corresponding rays through the two nodes are collinear.

The section of F^3 by a quadric with respect to which l and l' are polars is a $K_4{}^6$ with a $\gamma_2{}^1$ which has l and l' as base spaces and is therefore of genus 2. (It cannot be of genus greater than 2, and if it was of genus 1 the base spaces would be a point and a plane.) Conversely, when a $K_4{}^6$ has a $\gamma_2{}^1$ of genus 2, of the ∞^4 cubic surfaces through the curve ∞^2 will contain the base line t of § 7 . 6. Any one of these surfaces meets d in three points and contains the pairs of generators of $R_2{}^5$ through them. Thus the condition (a) is satisfied and the following theorem is established.

A canonical curve of genus 4 *in* [3] *possesses a* $\gamma_2{}^1$ *of genus* 2 *when and only when it is the intersection of a quadric surface with a cubic surface which possesses an harmonic involution whose base lines are polars with respect to the quadric.*

7 . 9. Since the lowest value of n for which a $K(n)$ can have a $\gamma_2{}^1$ of genus r is $2r$, we should expect to find plane $K_3{}^4$ with involution of genus 2. But this only happens when $K_3{}^4$ is a double conic, for the curve must be the projection of a $K_4{}^6$ in [3] which has a $\gamma_2{}^1$ of genus 2 and has acquired a double point. This only occurs when the points A_1 and A_2 of § 7 . 6 tend to coincidence in A. The $\infty^1 K_4{}^6$ of $R_2{}^5$ which pass through A_1 and A_2 acquire double points at A, but the quadrics containing them become cones with vertex at A. Thus when we project from A we obtain double conics of genus 3 containing involutions of genus 2, and the joins of corresponding points generate a pencil counted twice.

Fundamental scrolls of genus 3.

8 . 1. The following table is deduced from the general theory, and shows the main numerical properties of the general $R(n, 2)$ and the hyperelliptic $R(n, 2)$ for values of n up to 10. The numbers ρ, b, and h have the same meanings as in § 7 . 1. We expect particular types in which $j > 0$ when n is 5 or 6 and types in which $k > 0$ when n is 6, 7, or 8. Also when $R(n, 2)$ is not hyperelliptic there may be types in which q does not have the

* Timms, *Proc. Royal Soc.* (A), 119 (1928), 231 (6, *h*, ix).

value given in the table. The base curves of every $\mathbf{R}(n, 2)$ are a plane K_3^4 and a normal non-special C_3^n. In the hyperelliptic case the K_3^4 is a double conic C^2, the joins of the g_2^1 on C_3^n generate a normal R_0^{n-4}, and the planes containing pairs of generators of $\mathbf{R}(n, 2)$ through points of C^2 generate a normal E_0^{n-2}. The numbers in the table increase regularly after $n = 6$.

$[n]$	[4]	[5]	[6]	[7]	[8]	[9]	[10]
Order of $\mathbf{R}(n, 2)$	8	9	10	11	12	13	14
Number of points in (Z)	...	...	...	3	4	5	6
Freedom of (Z) on a C_3^n	...	...	...	0	1	2	3
Freedom of C_3^n on $\mathbf{R}(n, 2)$	0	0	0	1	2	3	4
Number of points in (A)	0	2	4	6	8	10	12
Freedom of (A) on a C_3^n	...	0	1	3	5	7	9
Freedom of $K(n)$ on $\mathbf{R}(n, 2)$	1	1	2	5	8	11	14
q, for general $\mathbf{R}(n, 2)$	3	2	0	0	0	0	0
q, for hyperelliptic $\mathbf{R}(n, 2)$	3	2	1	0	0	0	0
ρ, for general $\mathbf{R}(n, 2)$	0	2	4	9	15	22	30
ρ, for hyperelliptic $\mathbf{R}(n, 2)$	1	3	6	10	16	23	31
b	0	2	4	6	8	10	12
h	3	4	6	9	13	18	24

8.2. $n = 4$, $r = 2$.

The $\mathbf{R}_3^8$ in [4] is given by a (4, 1) correspondence between a K_3^4 and a line l, which is a fourfold line of the scroll. The ∞^1 bisecant K_5^8 on $\mathbf{R}_3^8$ do not meet K_3^4 or l. The quadrispeciality of the fourfold line of genus 3

is 3, so there are no quadrics through $\mathbf{R}_3^8$, and the ∞^2 quadrics through any bisecant K_5^8, which are all harmonic, cut out a g_8^2 on K_3^4. These results can easily be obtained directly. The sets of points of K_3^4 which correspond to points of l are sets of contacts of 4-tangent conics of K_3^4, so the correspondence between l and K_3^4 is not general. For in a general correspondence the points of l correspond to sets of a g_4^1 on K_3^4 which is cut out by conics through four general points (Y) of K_3^4. But in our particular correspondence the points (Y) are the contacts of a 4-tangent conic*, and consequently the sets of the g_4^1 are also contact sets. We note that the points of l cannot correspond to canonical sets of K_3^4, or the fourfold line l would not be normal.

The hyperelliptic $\mathbf{R}_3^8$ in [4] is given by a (4, 2) correspondence between a conic C^2 and a line l. The one quadric through $\mathbf{R}_3^8$ is, of course, the line cone projecting C^2 from l, which is also the E_0^2 generated by planes containing pairs of generators through points of C^2. The ∞^2 quadrics through a K_5^8 of $\mathbf{R}_3^8$ cut out an involution on l, pairs of points of which correspond to the same four points on C^2, and the correspondence is not general. In fact, if λ and μ are parameters of C^2 and l, a general (4, 2) correspondence is given by an equation of the form

$$\theta_0\lambda^4+\theta_1\lambda^3+\theta_3\lambda^2+\theta_3\lambda+\theta_4=0,$$

where each θ is quadratic in μ. The equations

$$x_0:x_1:x_2:x_3:x_4=\theta_0:\theta_1:\theta_2:\theta_3:\theta_4$$

represent a conic which has no double point in general, so no two points of l correspond to the same four points of C^2. In our particular correspondence the conic is a double line. (If the conic were to degenerate into two distinct lines, the θ's would have a common factor, and we should obtain a degenerate scroll.)

8.3. $n=5,\ r=2$.

The $\mathbf{R}_3^9$ in [5] is given by a (1, 1) correspondence between a K_3^4 and a plane C_3^5. The canonical sets of C_3^5 cannot be collinear, so C_3^5 is always unique. The ∞^1 base quadrics of any K_6^{10} on $\mathbf{R}_3^9$ are the ∞^1 quadrics through the scroll. Since $2n=5r$ the curve C_3^5 is not general, by §5.1. A general C_3^5 has three double points N_1, N_2, N_3 whose joins meet the

* See §5.1.

curve again in points L_1, L_2, L_3. The bicanonical series is cut out by the ∞^5 quartics with double points at N_1, N_2, and N_3, and the remaining points of a bicanonical set which contains any line section of C_3^5 are, therefore, L_1, L_2, and L_3. Now if a bicanonical set of C_3^5 lies on a conic with two further intersections A_1 and A_2, any conic through A_1 and A_2 gives a bicanonical set, and in particular the line A_1A_2 and any other line gives a bicanonical set. Thus the points L_1, L_2, and L_3 lie on a line whose further intersection is the unique set (A), and this is the particular property of the C_3^5 on $\mathbf{R}_3^9$. Denoting corresponding points on K_3^4 by dashes, L_1', L_2', and L_3' are the points of contact of a conic which meets K_3^4 further in A_1' and A_2'. Sections of C_3^5 by lines correspond to sections of K_3^4 by conics through L_1', L_2', and L_3'. The double point N_1 corresponds to the two further intersections of $L_2'L_3'$ with K_3^4, and the sections of C_3^5 by lines through N_1 correspond to sections of K_3^4 by lines through L_1'. (In fact, our particular C_3^5 is given by applying a quadratic transformation to a plane quartic of genus 3, with the points of contact of a 3-tangent conic as base points.)* Projecting $\mathbf{R}_3^9$ from a general point of C_3^5 we obtain an R_3^8 in [4] given by a (4, 1) correspondence between a K_3^4 and a line, but the correspondence is still not general, for the points of the line correspond to a g_4^1 on K_3^4 cut out by conics through four points, of which three are the points of contact of a conic. To obtain the fundamental $\mathbf{R}_3^8$ we must take a particular $\mathbf{R}_3^9$ for which A_1 and A_2 coincide, and project from that point.

If $\mathbf{R}_3^9$ is hyperelliptic, K_3^4 is a double conic and C_3^5 has a triple point T, which is a triple point of the scroll. The C_3^5 is not general, for it can be shown that the three residual intersections of the tangents at T lie on a line whose two further intersections with C_3^5 are the unique set (A). The ∞^2 quadrics through $\mathbf{R}_3^9$ all contain the E_0^3 generated by planes which contain pairs of generators of the scroll.

If $j = 1$ the sets (A) generate a g_2^1, so C_3^5 is hyperelliptic and has a triple point T. The locus of harmonic conjugates of T with respect to the sets (G) of the g_2^1 on an hyperelliptic C_3^5 is in general a C_0^4 with a triple point at T. But since each (A) is a (G) and is also the residual intersection of a conic through four sets (G)†, the polar line l of T with respect to this conic contains five points of C_0^4, and this quartic must split up into l and the three tangents to C_3^5 at T, which must be inflexional tangents. Thus the $\mathbf{R}_3^9$ for which $j = 1$ is hyperelliptic and its directrix C_3^5 has a triflecnode. The scroll contains ∞^2 K_6^{10}.

* See § 5.1.

† Although all bicanonical sets are not composed of four sets (G), of the ∞^2 cut out by conics through (A) ∞^1 will be so composed.

8.4. $n = 6,\ r = 2.$

The general $\mathbf{R}_3^{10}$ in [6] is given by a (1, 1) correspondence between a plane K_3^4 and a general C_3^6 of S_3. The base quadrics all contain R_3^{10}, and the scroll is the further intersection of four quadrics of [6] with a plane and a solid in common. Projecting from two general points of C_3^6 we obtain the R_3^8 in [4] given by a general (4, 1) correspondence between a K_3^4 and a line. To obtain $\mathbf{R}_3^9$ in [5] we must project $\mathbf{R}_3^{10}$ from one of the united points of the g_4^1 on C_3^6 formed by the sets (A). The scroll $\mathbf{R}_3^{10}$ is the incidence scroll generated by lines which meet a plane, a solid, and four [4]*.

Projecting from a general point of $\mathbf{R}_3^{10}$, not on K_3^4 or C_3^6, we obtain the incidence scroll R_3^9 in [5] generated by lines which meet a plane and five solids*.

In the particular case in which $k = 1$ the canonical sets of C_3^6 lie in planes and consequently C_3^6 has a double point O and is a (3, 3) curve on a quadric, and $q = 1$. In this case $\mathbf{R}_3^{10}$ lies on five independent quadrics and contains ∞^1 directrix C_3^6 with a common double point at O, and there are ∞^3 bisecant K_7^{12}. We may also have $j = 1$ when the quadric sections of C_3^6 are tricanonical sets, and in this case the freedom of K_7^{12} is increased by one.

The hyperelliptic $\mathbf{R}_3^{10}$ in [6] is given by a general (1, 2) correspondence between a conic and a C_3^6 which is a (2, 4) curve on a quadric surface Q. All the quadrics through $\mathbf{R}_3^{10}$ contain the E_0^4 generated by planes containing pairs of generators. The case in which $j = 1$ occurs when the eight coincidences of the g_2^1 on C_3^6 lie four by four on two generators of Q. For these generators harmonically separate each pair (G) of the g_2^1, whence it follows that an (A) set is special, being given by a quadric through four sets (G) which will have the two generators as polars. In this case there are ∞^2 (A) on C_3^6 and ∞^3 bisecant K_7^{12} on $\mathbf{R}_3^{10}$. We notice that, projecting from one of the coincidences of the g_2^1 on C_3^6, we obtain a C_3^5 with a triflecnode, as we should expect. The case in which $k = 1$ occurs when Q is a cone and C_3^6 has a double point at the vertex V. There are ∞^1 C_3^6 on this type of hyperelliptic $\mathbf{R}_3^{10}$ all having a double point at V, which is, of course, a node of $\mathbf{R}_3^{10}$. When the tangents to C_3^6 at V are both generators of the cone, we have $j = 1$ as well as $k = 1$, and each of the ∞^1 C_3^6 on $\mathbf{R}_3^{10}$ contains ∞^2 sets (A), so that the scroll contains ∞^4 bisecant K_7^{12}.

8.5. The general $\mathbf{R}_3^{11}$ in [7] has a single infinity of directrix C_3^7 with a common trisecant line, and is the incidence scroll generated by lines which meet a plane, two [4], and three [5]. On the particular $\mathbf{R}_3^{11}$ for which $k = 1$

* See W 2, 241, 242.

there are ∞^2 $C_3{}^7$, each of which has ∞^1 trisecant lines generating an $R_0{}^3$, and also ∞^6 bisecant K_8^{14}.

The hyperelliptic $\mathbf{R}_3^{11}$ in [7] also has a single infinity of directrix $C_3{}^7$ with a common trisecant line which is the directrix line of all the $R_0{}^3$ given by the $g_2{}^1$ on $C_3{}^7$. The quadrics through $\mathbf{R}_3^{11}$ all contain the $E_0{}^5$ generated by planes containing pairs of generators. In the particular case in which $k=1$ the $R_0{}^3$ on which any $C_3{}^7$ is a bisecant curve is a cone, whose vertex V is a simple point of $C_3{}^7$. The sets (Z) are the sections of $C_3{}^7$ by the generators of the cone. On this $\mathbf{R}_3^{11}$ there are ∞^2 $C_3{}^7$ all passing through V, which is a simple point of $\mathbf{R}_3^{11}$.

8.6. The general $\mathbf{R}_3^{12}$ in [8] has ∞^2 $C_3{}^8$ on each of which the secant sets (Z) give ∞^1 quadrisecant planes generating an $E_0{}^3$. The scroll is an incidence scroll generated by the lines which meet a plane, three [5], and two [6]. When $k=1$ there are ∞^3 $C_3{}^8$, each of which is a bicanonical curve whose canonical sets consequently lie in planes.

The general hyperelliptic $\mathbf{R}_3^{12}$ in [8] has ∞^2 directrix $C_3{}^8$, each of which is a bisecant curve on an $R_0{}^4$, the secant sets (Z) being cut out by the ∞^1 directrix conics of $R_0{}^4$. The planes containing pairs of generators of $\mathbf{R}_3^{12}$ generate an $E_0{}^6$ which does not lie on all the ∞^{15} quadrics through $\mathbf{R}_3^{12}$, so the scroll is the section of an $E_0{}^6$ by a quadric which touches it along a directrix conic. There is a particular type of $\mathbf{R}_3^{12}$ in which all the $R_0{}^4$ have a common directrix line which is a directrix line of $E_0{}^6$ and contains two points of $\mathbf{R}_3^{12}$ through which pass all the ∞^2 $C_3{}^8$. When $k=1$ all the $R_0{}^4$ are cones with a common vertex which does not lie on $\mathbf{R}_3^{12}$. $E_0{}^6$ is also a cone in this case and $\mathbf{R}_3^{12}$ has ∞^3 $C_3{}^8$.

When $n>8$ we always have $j=k=0$, but there will be different types of hyperelliptic $\mathbf{R}(n, 2)$ corresponding to the different types of R_0^{n-4} on which the $C_3{}^n$ are bisecant curves*. The general R_3^{13} in [9] is generated by the lines which meet a plane, four [6], and a [7], while the general R_3^{14} in [8] is generated by lines which meet a plane and five [7]†.

Fundamental scrolls of genus 4.

9.1. The base curves of $\mathbf{R}(n, 3)$ in $[n]$ are a canonical curve $K_4{}^6$ in S_3 and a non-special $C_4{}^n$ in S_{n-4}, which is not a general curve when n is 6 or 7. The particular types in which $j>0$ occur when $7\leqslant n\leqslant 9$, and those in which $k>0$ when $8\leqslant n\leqslant 12$. It will be seen from the following table

* *Cp.* §7.3. If R_0^{n-4} has a $D_0{}^m$, then $2m\geqslant n-8$, so there are two types if n is odd and three types if n is even.

† W 2, 242.

that the various numbers in the general and hyperelliptic types increase regularly after $n = 9$.

$[n]$	[6]	[7]	[8]	[9]	[10]	[11]	[12]
Order of $\mathbf{R}(n, 3)$	12	13	14	15	16	17	18
Number of points in (Z)	...	...	...	...	4	5	6
Freedom of (Z) on C_4^n	...	...	...	...	0	1	2
Freedom of C_4^n	0	0	0	0	1	2	3
Number of points in (A)	0	2	4	6	8	10	12
Freedom of (A) on a C_4^n	...	0	0	2	4	6	8
Freedom of $K(n)$ on $\mathbf{R}(n, 3)$	1	1	1	3	6	9	12
q, for general $\mathbf{R}(n, 3)$	3	1	0	0	0	0	0
q, for hyperelliptic $\mathbf{R}(n, 3)$	3	2	1	0	0	0	0
ρ, for general $\mathbf{R}(n, 3)$	4	7	12	19	27	36	46
ρ, for hyperelliptic $\mathbf{R}(n, 3)$	6	10	15	21	29	38	48
b	3	6	9	12	15	18	21
h	7	9	12	16	21	27	34

9.2. The base curves of $\mathbf{R}_4^{12}$ in [6] are a canonical K_4^6 in S_3 and a plane C_4^6, which is not a general curve. For a general C_4^6 has six double points and the bicanonical series is cut out by sextics with the same six points as double points. Since on the C_4^6 of $\mathbf{R}_4^{12}$ a bicanonical set is cut out by a conic, there must be a quartic curve with double points at the double points of C_4^6, which must therefore be the vertices of a quadrilateral. (The double points cannot lie on a conic, for C_4^6 would then be special.) A particular case is when C_4^6 has one triple point and three double points, which must be collinear, and K_4^6 consequently lies on a quadric cone. The C_4^6 cannot have two triple points, for it would then be special. There are four inde-

pendent quadrics through $\mathbf{R}_4^{12}$ of which one is the plane-cone projecting the quadric of S_3 through K_4^6 from the plane π of C_4^6, and the other three contain π and S_3. In fact, $\mathbf{R}_4^{12}$ is the intersection of three quadrics with a plane and a solid in common with a quadric cone having the plane as vertex.

In the hyperelliptic case K_4^6 is a double C_0^3 and C_4^6 has a fourfold point O, but is not the general C_4^6 with a fourfold point. For the locus of harmonic conjugates of O with respect to sets (G) of the g_2^1 on C_4^6 is in general a C_0^5 with a fourfold point at O. The bicanonical series is of freedom 8 and contains the series g_{12}^6 whose sets are composed of six sets (G). Since the sections of the C_4^6 on $\mathbf{R}_4^{12}$ by conics are bicanonical sets, ∞^3 of them will consist of six sets (G), and the polar line l of O with respect to one of these conics contains six points of C_0^5, which therefore splits up into l and the tangents to C_4^6 at O. Thus the four tangents at O to the C_4^6 of $\mathbf{R}_4^{12}$ are inflexional, and the quadrics through a bisecant K_7^{12} which do not contain the plane of C_4^6 meet it in the ∞^3 conics which have O and l as polars. This agrees with general theory, for since $q = 3$ we should expect the quadrics through K_7^{12} to cut out a series of freedom 3 on the double C_0^3 and consequently on C_4^6. We note that the six independent quadrics through $\mathbf{R}_4^{12}$ all contain the E_0^4 generated by planes containing pairs of generators of $\mathbf{R}_4^{12}$.

9.3. The base curves of a general $\mathbf{R}_4^{13}$ in [7] are a K_4^6 in S_3 and a C_4^7 in S_3'. The C_4^7 is not a general curve, for, by § 5.1, if any plane through the unique set (A) meets C_4^7 further in $M_1, \ldots, M_5$, the corresponding points on K_4^6, denoted by dashes, are such that $(A')+2(M')$ is the section of K_4^6 by a quadric*. In fact, if we make the primes of S_3' correspond to the quadrics of S_3 through five points of K_4^6 which are contacts of a quadric Ω, to the K_4^6 in S_3 will correspond a C_4^7 in S_3' of our particular type, the set (A) corresponding to the two residual intersections of Ω and K_4^6. Of course, if the two points A_1 and A_2 coincide we can project the $\mathbf{R}_4^{13}$ into the general $\mathbf{R}_4^{12}$ of [6], the line sections of whose C_4^6 correspond to sections of K_4^6 by quadrics through the contacts of a 6-contact quadric, which also follows from § 5.1.

In [7] we cannot have $j = 1$ except in the hyperelliptic case, but there will doubtless be other particular types. For instance, C_4^7 may be a (3, 4) curve on a quadric, with two double points, in which case $q = 2$. The hyperelliptic R_4^{13} is given by a (1, 2) correspondence between a C_0^3 and a C_4^7 which is a (2, 5) curve on a quadric Q. The particularity of C_4^7 is the fact that two generators of Q contain four pairs (G) of the g_2^1 on C_4^7, the remaining two intersections being the two points A_1 and A_2 of the unique

* While this paper was in the press, R. Frith investigated the particularity of C_4^7 and of the C_{r+1}^n on $\mathbf{R}(n, r)$, *Proc. Camb. Phil. Soc.*, 31 (1935), 327. The notation is somewhat different.

set (A). If these two points coincide in A we can project from A and obtain the $\mathbf{R}_4^{12}$ whose ${C_4}^6$ has a 4-flecnode. The case $j = 1$ arises when the two points A_1 and A_2 are a pair (G), and there is then an involution of pairs of generators of Q which contain five pairs (G). A further particular type arises when ${C_4}^7$ lies on a quadric cone, having a triple point at the vertex, and $j = 1$ when the tangents to ${C_4}^7$ at this triple point are generators of the cone.

It seems unnecessary to consider further examples, and for the scrolls of genus 5 I merely give a table for the general case and the general hyperelliptic case, up to $n = 15$.

Fundamental scrolls of genus 5.

10.

$[n]$	[8]	[9]	[10]	[11]	[12]	[13]	[14]	[15]
Order of $\mathbf{R}(n, 4)$	16	17	18	19	20	21	22	23
Number of points in (Z)	...	...	...	...	...	5	6	7
Freedom of (Z) on a ${C_5}^n$	...	...	...	...	...	0	1	2
Freedom of ${C_5}^n$	0	0	0	0	0	1	2	3
Number of points in (A)	0	2	4	6	8	10	12	14
Freedom of (A) on a ${C_5}^n$	0	0	0	1	3	5	7	9
Freedom of $K(n)$ on $\mathbf{R}(n, 4)$	1	1	1	2	4	7	10	13
q, for general $\mathbf{R}(n, 4)$	2	0	0	0	0	0	0	0
q, for hyperelliptic $\mathbf{R}(n, 4)$	3	2	1	0	0	0	0	0
ρ, for general $\mathbf{R}(n, 4)$	11	16	24	33	43	54	66	79
ρ, for hyperelliptic $\mathbf{R}(n, 4)$	15	21	28	36	46	57	69	82
b	8	12	16	20	24	28	32	36
h	13	16	20	25	31	38	46	55

Special scrolls in space of three dimensions.

11.1. In his classification of scrolls in [3] of order not greater than 6, Edge obtained the fundamental $\mathbf{R}_2^6$ in [4] and investigated the special sextic scrolls in [3] which are its projections*. From the representation of lines of [3] on a quadric in [5] he also obtained two types of R_2^5, four of R_3^6, and two of R_4^6, all of which are normal in [3] and have speciality 1, 2, and 4 respectively†. These scrolls must be the projections of fundamental scrolls and I proceed to show how they are actually obtained.

11.2. One of Edge's types of R_2^5 is the fundamental $\mathbf{R}_2^5$ in [3] with a double line and a triple line, but we saw in § 7.6 that this scroll may be of two distinct types the general one containing a single infinity and the particular one a double infinity of bisecant K_4^6. The $\mathbf{R}_2^5$ is the projection of $\mathbf{R}_2^6$ from a general point. Edge's other type of R_2^5 in [3] is the projection of $\mathbf{R}_2^6$ from a point of one of the two generators through the isolated node, and has a double directrix line and ∞^1 directrix C_2^4 with a common double point on the double line.

11.3. The projection of the general $\mathbf{R}_3^9$ in [5] from a double point of C_3^5 is an R_3^7 in [4] given by a general (3, 1) correspondence between a K_3^4 and a line l, the sets corresponding to points of l being cut out on K_3^4 by lines through a point H of the curve‡. Primes through the plane containing the generators of R_3^7 through a point of l give ∞^1 K_3^4 on R_3^7, all passing through H. Thus the projection of R_3^7 from a general point is the R_3^6 in [3] given by a (3, 3) correspondence between two lines, the correspondence being such as to give a double generator.

The projection of R_3^7 in [4] from a point on the generator through H is the R_3^6 in [3] with only one triple line. This scroll is given by a (3, 1) correspondence between a K_3^4 and a line with one united point H. The points of the line correspond to the sections of K_3^4 by lines through H, and the tangent to K_3^4 at H is a double generator of R_3^6.

The projection of the hyperelliptic $\mathbf{R}_3^8$ in [4] from a point of the double conic is an R_3^6 in [3] given by a (4, 2) correspondence between two lines, which is not a general correspondence, for there is an involution of pairs of points on the fourfold line which correspond to the same set of four points on the double line. However, Edge's R_3^6 in [3], given by a general (4, 2)

* E, 253.
† E, 135, 258.
‡ See § 8.3.

correspondence between two lines, is the projection of the general hyperelliptic $\mathbf{R}_3^9$ in [5] from a point of the double conic and a simple point of C_3^5.

The projection of the hyperelliptic $\mathbf{R}_3^9$ from a point of the double conic and a general point of the scroll is Edge's R_3^6 given by a (1, 2) correspondence with one united point between a line and a C_3^5 with a triple point.

11.4. The projection of the $\mathbf{R}_4^{12}$ in [6], whose C_4^6 has a triple point, from this triple point is an R_4^9 in [5] given by a (1, 3) correspondence between a line and a K_4^6 which lies on a quadric cone. On R_4^9 there are ∞^2 K_4^6 lying on quadric cones with a common vertex V. If we project from a generator of one of these cones, which contains three points of R_4^9, we merely obtain a triple quadric. But if we project from a general point O of R_4^9, any one of the ∞^1 K_4^6 through O will project into a plane C_4^5 with a tacnode T, and we obtain an R_4^8 in [4] given by a (3, 1) correspondence between this C_4^5 and a line l, the sets of points which correspond to points of l being the sections of C_4^5 by lines through T; one point L of l corresponds to T counted twice and the further intersection of the tangent to C_4^5 at T. Projecting again from a point of the line TL, we obtain Edge's R_4^6 in [3] given by a (3, 1) correspondence between a plane C_4^5 with a tacnode and a line through the tacnode; the points of the line correspond to sections of the quintic by lines through the tacnode which is a doubly united point, and the tangent to C_4^5 at the tacnode is a generator of the scroll.

By projection from $\mathbf{R}_4^{13}$ in [7] we can obtain an R_4^{12} in [6] whose directrix curves are a K_4^6 and a plane C_4^6 with a triple point and three non-collinear double points. The two g_3^1 on C_4^6 are distinct, one being given by conics through the four multiple points and the other by lines through the triple point, so the quadric containing K_4^6 is not a cone. Projecting from the triple point of C_4^6 we obtain an R_4^9 in [5] given by a (1, 3) correspondence between a line and a K_4^6 which lies on a non-conical quadric. A general line l in a plane containing three generators of R_4^9 lies in a prime with another such plane, which prime meets R_4^9 in a K_4^6 containing three points of l. Thus the projection of R_4^9 from l is Edge's R_4^6 in [3] given by a general (3, 3) correspondence between two lines.

There are other possible generations of these scrolls which might at first sight appear to lead to distinct types but do not actually do so. For instance, we may take a plane C_4^6 with two triple points T_1 and T_2, and make the points of a line l through T_1 correspond to the sets of points of C_4^6 cut out by lines through T_2, the point T_1 being triply united. This would give an R_4^6 which is not a distinct type, for it may be shown to possess another triple line through T_2.

11.5. It is now possible to mention examples which justify the remark made in § 1.1 that bisecant curves as well as directrix curves should be considered in the classification of special scrolls. For we have seen that the fundamental $\mathbf{R}_2^5$ in [3] may have either a single infinity or a double infinity of bisecant K_4^6, and these two types must surely be regarded as distinct. Also the R_3^6 in [3] given by a (4, 2) correspondence between two lines may possess an improper bisecant curve of genus 5, or the lowest genus of a bisecant curve may be 6*, and these types must be regarded as distinct. Another example is the R_3^8 in [4] given by a (4, 1) correspondence between a plane K_3^4 and a line, which may be of three distinct types according as the lowest genus of a bisecant curve is 5, 6, or 7†.

Harmonic systems of quadrics through canonical curves.

12.1. I have developed the theory of the fundamental scrolls given by a γ_2^1 on a $K(n)$ in $[n]$, and I now propose to discuss conditions under which a $K(n)$ possesses a γ_2^1 of genus $r+1$. I have shown in § 6.3 that necessary conditions are that $n \geqslant 2r$ and that the quadrics of $[n]$ through $K(n)$ form an harmonic system relative to two independent base spaces S_r and S_{n-r-1} which contains $r(n-r-2)$ independent base quadrics. I shall prove that, with a few definite exceptions, these conditions are also sufficient when $n > 3$.

Suppose that these conditions are satisfied and that P is any point of $K(n)$. The line l through P which meets S_r and S_{n-r-1} lies on all the base quadrics and the harmonic conjugate P' of P with respect to the base spaces lies on l and on all the harmonic quadrics. So if $n > 3$ and if all the quadrics through $K(n)$ have no further intersection, the point P' lies on $K(n)$, and we have a γ_2^1 on the curve which has S_r and S_{n-r-1} as base spaces and is, therefore, of genus $r+1$. When $n = 2r$ there is some doubt whether the genus of the γ_2^1 is r or $r-1$, but this is settled by the condition that the number of independent base quadrics is $r(n-r-2)$. For instance, in [4] we may have a line and a plane as base spaces; if there is one base quadric the γ_2^1 is of genus 2, while if there are no base quadrics the genus of the γ_2^1 is 3.

The only cases in which all the quadrics through a $K(n)$ have a further intersection are when a $K(5)$ has a g_5^2 and lies on a Veronese surface, and when a $K(n)$ has a g_3^1 and is a trisecant curve on a rational normal scroll R_0^{n-1}‡. In the latter case the scroll may be of different types, but a directrix

* *I.e.*, it may, or may not be, a projection of $\mathbf{R}_3^8$, as we saw in § 11.3.

† See §§ 8.2, 8.3, and 8.4.

‡ See Enriques-Chisini, *Lezioni sulla teoria geometria delle equazioni*, III, 106.

curve of order m will meet $K(n)$ in $3m+2n-3(n-1)$ points*, so $3m \geqslant n-3$, fixing a lower limit to the order of the minimum directrix curve of R_0^{n-1}†. In both these cases all the quadrics through $K(n)$ contain the whole surface, so the point P' of the above argument need not lie on $K(n)$, and we do not obtain a $\gamma_2{}^1$ on the curve. But there would be an involution on the surface contained in the harmonic involution of $[n]$ which has S_r and S_{n-r-1} as base spaces, and we are led to inquire whether such involutions actually exist, whether the quadrics through the surfaces form harmonic systems relative to S_r and S_{n-r-1}, and whether the number of independent base quadrics is $r(n-r-2)$.

12.2. Suppose that a Veronese surface F^4 in [5] possesses an involution which is contained in an harmonic involution of [5] with two independent base spaces. Any conic of F^4 will correspond to another conic of F^4. Since the surface is represented by all the conics of a plane π, we have a corresponding involution in π which is linear as lines correspond to lines. The involution in π must be an harmonic involution with a base point and a base line. This leads to the following type of involution on F^4, which must be the only possible type. Choose on F^4 a point A and a conic α in a plane ϖ. The tangent planes to F^4 at points of α lie in a prime which meets the tangent plane at A in a line l. The solid σ joining A to ϖ has no intersection with F^4 other than A and α. Any point Q of l lies in the plane of a conic q of F^4 which passes through A and meets α in a point Q'. The polar of Q with respect to q is the line AQ', which lies in σ, so we have an involution on F^4 which is contained in the harmonic involution of [5] with l and σ as base spaces.

The quadrics through F^4 form a system Σ of freedom 5. A quadric of Σ which contains Q also contains the plane of q, so the quadrics of Σ which contain l meet σ in the cone $A\alpha$, and it is only one further condition for these quadrics to contain σ. Thus two independent quadrics of Σ contain l and σ. Also Q and the line AQ' are conjugate with respect to all quadrics of Σ, so two conditions will make Q conjugate to σ. But then any other point of the cone $A\alpha$ is conjugate to l, and therefore σ and l are conjugate. Thus Σ is an harmonic system relative to l and σ, containing two independent base quadrics and four independent harmonic quadrics. The condition of §12.1 is, therefore, not sufficient in the case $n=5$, $r=1$. We really anticipated this result when, in §7.4, we found that a $K(5)$ with a $\gamma_2{}^1$ of genus 2 could lie on a Veronese surface.

---

* §2.4.

† For properties of rational normal scrolls see E, 36, or S 5.

12.3. Certain properties of involutions on rational normal curves are required in the consideration of involutions on R_0^{n-1} in $[n]$. An involution of order 2 on a $C_0^{2\mu+1}$ in $[2\mu+1]$ is necessarily rational, and the joins of pairs of points generate a rational normal scroll $R_0^{2\mu}$. This $R_0^{2\mu}$ is of general type and has ∞^1 directrix D_0^{μ}, for a D^m would meet $C_0^{2\mu+1}$ in $2m-2\mu+1$ points, wherefore $m \geqslant \mu$. The ∞^1 D_0^{μ} are in (1, 1) correspondence with the points of $C_0^{2\mu+1}$, since each D_0^{μ} meets $C_0^{2\mu+1}$ in one point. The D_0^{μ} are cut out by a pencil of primes through μ generators of $R_0^{2\mu}$, and the g_2^1 on $C^{2\mu+1}$ gives an involution among primes of this pencil. It follows that the pairs of points of the g_2^1 on $C_0^{2\mu+1}$ are harmonic conjugates with respect to the spaces S_μ and S_μ' which contain the D_0^{μ} through the two double points of the g_2^1. Notice that one double point of the g_2^1 lies in each of the spaces S_μ and S_μ', which I call base spaces of the g_2^1. I also call the two D_0^{μ} in S_μ and S_μ' the base curves of the g_2^1. It is easily shown that the quadrics through $C_0^{2\mu+1}$ form an harmonic system relative to these base spaces.

Similarly a g_2^1 on a $C_0^{2\mu}$ in $[2\mu]$ gives an $R_0^{2\mu-1}$ of general type with one $D_0^{\mu-1}$ which does not meet $C_0^{2\mu}$ and ∞^2 D_0^{μ} each meeting $C_0^{2\mu}$ in two points. The g_2^1 has two double points I and J, and the ∞^1 D_0^{μ} through I are in (1, 1) correspondence with the points of $C_0^{2\mu}$, since each of them has one further intersection with $C_0^{2\mu}$. The point I corresponds to the degenerate D_0^{μ} consisting of $D_0^{\mu-1}$ and the generator through I. These ∞^1 D_0^{μ} are cut out by a pencil of primes through I and $\mu-1$ generators; the g_2^1 on $C_0^{2\mu}$ gives an involution among these primes, and it follows that the pairs of points of the g_2^1 on $C_0^{2\mu}$ are harmonic conjugates with respect to the space $S_{\mu-1}$ which contains $D_0^{\mu-1}$ and the space S_μ containing the D_0^{μ} through I and J. Notice that in this case both the coincidences of the g_2^1 lie in S_μ. I call $S_{\mu-1}$ and S_μ and the directrix curves which they contain the base spaces and base curves of the g_2^1 on $C_0^{2\mu}$. Again it is easy to show that the quadrics through $C_0^{2\mu}$ form an harmonic system relative to these base spaces.

12.41. Suppose that there is an harmonic involution in $[n]$, with independent base spaces S_r and S_{n-r-1} $(n \geqslant 2r)$, which transforms a rational normal scroll R_0^{n-1} into itself. In this involution generators of R_0^{n-1} must correspond to generators, so either all the generators are self-corresponding or there is an involution among the generators with two double elements which are self-corresponding generators. If all the generators are self-corresponding they must all meet S_r and S_{n-r-1}, which are therefore the spaces of directrix curves D_0^r and D_0^{n-r-1} of R_0^{n-1}. In S_r there are $(r)_2$ independent quadrics containing D_0^r, so of the system Σ of quadrics of $[n]$ through R_0^{n-1}, $(r)_2$ independent ones are cones with vertex S_{n-r-1} and similarly $(n-r-1)_2$ independent ones are cones with vertex S_r. The

spaces S_r and S_{n-r-1} are conjugate with respect to all these cones. Also the number of conditions for a quadric of Σ to contain S_r and S_{n-r-1} is $(r)_2+(n-r-1)_2$, so Σ is an harmonic system relative to S_r and S_{n-r-1}, the number of base quadrics being

$$(n-1)_2-(r)_2-(n-r-1)_2 = r(n-r-1).$$

This number is equal to $r(n-r-2)$ only when $r=0$, but the only canonical curve which can be a trisecant curve on a rational normal cone is the K_4^6 in [3], and we are supposing that $n>3$,

12.42. If only two of the generators of R_0^{n-1} are self-corresponding, each of them must either consist entirely of united points and lie in one of the base spaces or must contain an involution of pairs of corresponding points with two united points, one in each base space. There can be no further united points on R_0^{n-1}, for the generator through a united point is self-corresponding, so the base spaces have no further intersection with R_0^{n-1}. The lines joining pairs of corresponding points of R_0^{n-1}, which I shall call axes, meet S_r and S_{n-r-1} in points of two surfaces F and F'. Clearly the curve in which R_0^{n-1} is met by any prime through S_r is self-corresponding. The pairs of corresponding points on the curve form a g_2^1 (since the curve is rational), and the axes joining them generate a scroll whose directrix curve in S_{n-r-1} is a prime section of F'. Similarly primes through S_{n-r-1} contain scrolls generated by axes whose directrix curves in S_r are prime sections of F. A prime through either base space which contains a generator of R_0^{n-1} also contains the corresponding generator. We have four cases to consider. I shall assume $n \geqslant 2r$ in every case; this merely decides which of two base spaces is to be called S_r.

(*a*) R_0^{n-1} may have two generators lying in the same base space S_{n-r-1}. A prime through S_r meets R_0^{n-1} in a normal D_0^{n-1} with a g_2^1 whose double points are both in S_{n-r-1}. The axes joining pairs of points of the g_2^1 generate a normal R_0^{n-2} whose directrix curves in S_r and S_{n-r-1} divide the pairs of the g_2^1 harmonically. Comparing with §12.3 we see that D_0^{n-1} must be of even order, so $n=2\lambda+1$, and that the directrix curve of R_0^{n-2} which lies in S_{n-r-1} is a normal D_0^{λ}. Since this is a prime section of F', we have $n-r-1=\lambda+1$, and F' is a rational normal scroll of order λ. Similarly the section of R_0^{n-1} by a prime through S_{n-r-1} (or $S_{\lambda+1}$) is a normal D_0^{n-3} whose g_2^1 gives an R_0^{n-4} with a normal $D_0^{\lambda-2}$ in S_r (or $S_{\lambda-1}$). This $D_0^{\lambda-2}$ is a prime section of F, which is therefore a normal $R_0^{\lambda-2}$. The corresponding generators of F and F' arise from the axes which meet two corresponding generators of R_0^{n-1}. Since $n=2\lambda+1$ and $r=\lambda-1$ this case can only occur

when $n = 2r+3$. (We might, of course, have taken the two self-corresponding generators in S_r, which would give $r = \lambda+1$, but this would make $n < 2r$.)

(*b*) R_0^{n-1} may have one generator in S_r and one in S_{n-r-1}. A prime through S_r meets R_0^{n-1} in a self-corresponding D_0^{n-2} with two united points, one of which is in S_r and one in S_{n-r-1}. Comparing with § 12 . 3 we see that again $n = 2\lambda+1$. The g_2^1 on D_0^{n-2} gives an R_0^{n-3} with a $D_0^{\lambda-1}$ in S_{n-r-1} which is a prime section of F'. Thus $n-r-1 = \lambda$, $r = \lambda$, and both F and F' are normal $R_0^{\lambda-1}$. This case only occurs when $n = 2r+1$.

(*c*) R_0^{n-1} may have two generators which meet both spaces. A prime through S_r meets R_0^{n-1} in a self-corresponding D_0^{n-1} whose united points are both in S_r. Again we must have $n = 2\lambda+1$, and the g_2^1 on D^{n-1} gives an R_0^{n-2} with a $D_0^{\lambda-1}$ in S_{n-r-1}. So $r = \lambda$ and again both F and F' are normal $R_0^{\lambda-1}$. This case only occurs when $n = 2r+1$.

(*d*) R_0^{n-1} may have one generator meeting both base spaces and one lying in a base space, say in S_{n-r-1}. A prime through S_r meets R_0^{n-1} in a self-corresponding D_0^{n-1} with one united point in each base space. In this case, therefore, $n-1$ must be odd and $n = 2\lambda$. It follows that $r = \lambda-1$, and $n = 2r+2$, and the surfaces F and F' are rational normal scrolls of orders $\lambda-2$ and $\lambda-1$, lying in $S_{\lambda-1}$ and S_λ respectively. We might equally well have taken S_r as the base space containing a generator of R_0^{n-1}, which would give $r = \lambda$ and $n = 2r$. This case may, therefore, occur only when $n = 2r$ or $n = 2r+2$.

12 . 5. We have found the forms that harmonic involutions on rational normal scrolls must take, and it remains to find whether these involutions actually exist. We again consider the possibilities separately. Take first a normal scroll $R_0^{2\lambda}$ in $[2\lambda+1]$ and choose thereon two generators g_1 and g_2 all of whose points are to be united points. This determines an involution among the generators with g_1 and g_2 as united elements. In the general case, when the scroll has $\infty^1 D_0^\lambda$, these must all be self-corresponding (since each D_0^λ contains two united points and any D_0^λ must correspond to a D_0^λ). An involution is determined on each D_0^λ by the two united points on g_1 and g_2, and an involution on $R_0^{2\lambda}$ is thereby determined. If $R_0^{2\lambda}$ has a unique minimum directrix curve of order m less than λ, this D_0^m must be self-corresponding, and we may choose any pencil of $D_0^{2\lambda-m}$ on $R_0^{2\lambda}$ and say that these shall be self-corresponding. An involution on $R_0^{2\lambda}$ is then determined, each

point P of $R_0^{2\lambda}$ corresponding to the point of intersection of the $D_0^{2\lambda-m}$ of the pencil which passes through P with the generator which corresponds to the generator through P. Pairs of corresponding points on one of these $D_0^{2\lambda-m}$ lie on corresponding generators, and therefore belong to a g_2^1 whose united points are the intersections of $D_0^{2\lambda-m}$ with g_1 and g_2. There are two cases.

12.51. Suppose that $R_0^{2\lambda}$ has a minimum directrix curve of even order 2μ ($\leqslant\lambda$). Then an involution I on $R_0^{2\lambda}$ is determined by g_1, g_2, and a pencil of self-corresponding $D_0^{2(\lambda-\mu)}$. Each $D_0^{2(\lambda-\mu)}$ of the pencil has a g_2^1, contained in I, with normal base curves $C_0^{\lambda-\mu-1}$ and $C_0^{\lambda-\mu}$, the latter containing the united points of the g_2^1 which lie on g_1 and g_2. One $D_0^{2(\lambda-\mu)}$ of the pencil is degenerate (if $2\mu<\lambda$) and contains the minimum directrix curve $D_0^{2\mu}$, which possesses a g_2^1 contained in I whose base curves are a normal $C_0^{\mu-1}$ and a normal $C_0{}^{\mu}$, the latter meeting g_1 and g_2. The $D_0^{2(\lambda-\mu)}$ of the pencil through a point P of $R_0^{2\lambda}$ contains the corresponding point P', and the axis PP' meets the base curves of the g_2^1 on $D_0^{2(\lambda-\mu)}$ in two points which I call the base points of the axis. The axes which meet a pair of corresponding generators l and l' of $R_0^{2\lambda}$ generate a quadric surface, and the curve formed by the base points of these axes, since it does not meet l or l', consists of two generators of the same system as l and l', which I call base lines. One of these base lines meets the base curves $C_0^{\mu-1}$ and $C_0^{\lambda-\mu-1}$ associated with $D_0^{2\mu}$ and any $D_0^{2(\lambda-\mu)}$ of the pencil, while the other base line meets the base curves $C_0{}^{\mu}$ and $C_0^{\lambda-\mu}$. Thus, varying the pair of corresponding generators of $R_0^{2\lambda}$, we obtain two systems of base lines generating two rational normal scrolls $R_0^{\lambda-2}$ and $R_0{}^{\lambda}$, of which the latter contains g_1 and g_2. The spaces $S_{\lambda-1}$ and $S_{\lambda+1}$ of these scrolls are the base spaces of an harmonic involution of $[2\lambda+1]$ which contains I. Thus involutions of type (a) exist when $R_0^{2\lambda}$ has a minimum directrix curve of even order.

The quadrics of $[2\lambda+1]$ through $R_0^{2\lambda}$ form a linear system Σ of freedom $(2\lambda)_2-1$. The number of conditions for a quadric of Σ to contain $R_0^{\lambda-2}$ and consequently $R_0{}^{\lambda}$ is at most $3(\lambda-1)$*, wherefore the number of conditions for a quadric of Σ to contain both base spaces is at most

$$3(\lambda-1)+(\lambda-2)_2+(\lambda)_2=\lambda^2,$$

and the number of independent base quadrics is at least $\lambda(\lambda-1)$. Also the number of independent quadrics of $[2\lambda+1]$ which have the base spaces as polars is $\lambda^2+3\lambda+3$. These quadrics form a system Θ, and a quadric of Θ through any point of $R_0^{2\lambda}$ contains the corresponding point. If we make

* §2.5.

a quadric of Θ contain $2\mu+1$ general points of $D_0^{2\mu}$ and $2(\lambda-\mu)+1$ general points of a self-corresponding $D_0^{2(\lambda-\mu)}$ and $\lambda+1$ general points of $R_0^{2\lambda}$, the quadric will contain $D_0^{2\mu}$, $D_0^{2(\lambda-\mu)}$, and $2\lambda+2$ generators of $R_0^{2\lambda}$, and therefore contains the scroll. It is thus at most $3\lambda+3$ conditions for a quadric of Θ to contain $R_0^{2\lambda}$, and the number of independent harmonic quadrics of Σ is at least λ^2. But no harmonic quadric can be a base quadric, so the numbers of independent base and harmonic quadrics are exactly $\lambda(\lambda-1)$ and λ^2 respectively, since the sum of these numbers is $(2\lambda)_2$. Thus the quadrics of Σ form an harmonic system relative to $S_{\lambda-1}$ and $S_{\lambda+1}$. To compare with previous work we put $n=2\lambda+1$ and $r=\lambda-1$, and the number of independent base quadrics is $r(n-r-2)$. In this case $n=2r+3$.

12.52. Similarly, if $R_0^{2\lambda}$ in $[2\lambda+1]$ has a minimum directrix curve $D_0^{2\mu-1}$, an involution I on $R_0^{2\lambda}$ is determined by two generators g_1 and g_2, all of whose points are united points, and a pencil of self-corresponding $D_0^{2(\lambda-\mu)+1}$, each of which has a g_2^1 contained in I with two base curves of order $\lambda-\mu$, one meeting g_1 and the other meeting g_2. Also $D_0^{2\mu-1}$ has a g_2^1 with two base curves of order $\mu-1$. As before, we obtain two systems of base lines generating two rational normal scrolls of order $\lambda-1$, each containing one of the generators g_1 and g_2. The spaces S_λ and S_λ' of these scrolls are the base spaces of an harmonic involution of $[2\lambda+1]$ which contains I. Thus involutions of type (b) exist when $R_0^{2\lambda}$ has a minimum directrix curve of odd order. It can be shown that the quadrics through $R_0^{2\lambda}$ form an harmonic system relative to S_λ and S_λ' containing $\lambda(\lambda-1)+1$ independent base quadrics. Putting $n=2\lambda+1$ and $r=\lambda$, we find that the number of independent base quadrics is $r(n-r-2)+1$. In this case $n=2r+1$.

12.53. We now consider the possibility of involutions of type (c) with only four united points. On an $R_0^{2\lambda}$ in $[2\lambda+1]$ we can choose two generators g_1 and g_2 as self-corresponding generators and thus determine an involution among the generators. In general $R_0^{2\lambda}$ contains a pencil of D_0^λ and we can determine an involution among these by choosing two self-corresponding D_0^λ. An involution I on $R_0^{2\lambda}$ is then determined, any point P of $R_0^{2\lambda}$ corresponding to the intersection of the generator and the D_0^λ which correspond to those through P. The four united points of I are the intersections of g_1 and g_2 with the self-corresponding D_0^λ. If, however, $R_0^{2\lambda}$ has a unique minimum directrix curve D_0^m, meeting g_1 in G_1 and g_2 in G_2, we may choose a self-corresponding pencil of $D_0^{2\lambda-m}$ passing through fixed points H_1 and H_2

on g_1 and g_2 respectively. An involution I is then determined with G_1, G_2, H_1, and H_2 as its only base points. Proceeding as before, we obtain two systems of base lines which generate two rational normal scrolls F and F', of order $\lambda-1$, each of which contains two of the united points. If the minimum directrix curve is of even order G_1 and G_2 lie on the same base scroll, otherwise G_1 and G_2 lie on different base scrolls. The spaces S_λ and S_λ' are the base spaces of an harmonic involution of $[2\lambda+1]$ which contains I, so involutions of type (c) exist.

It is easily shown that the number of independent quadrics through $R_0^{2\lambda}$ which contain the base spaces S_λ and S_λ' is at least $\lambda(\lambda-1)$. The quadrics which have S_λ and S_λ' as polars form a system Θ. At most $2(\lambda+1)$ conditions are required to make a quadric of Θ contain D_0^m and a self-corresponding $D_0^{2\lambda-m}$, and the quadric then contains g_1 and g_2 (since it contains G_1, G_2, H_1, and H_2), so λ further conditions will make the quadric contain $R_0^{2\lambda}$. Thus at least λ^2 independent quadrics through $R_0^{2\lambda}$ are harmonic. But $\lambda^2+\lambda(\lambda-1)=(2\lambda)_2$. Thus again the quadrics through $R_0^{2\lambda}$ form an harmonic system relative to S_λ and S_λ', containing $\lambda(\lambda-1)$ independent base quadrics. Putting $n=2\lambda+2$ and $r=\lambda$, we find that the number of base quadrics is $r(n-r-2)$. In this case $n=2r+1$.

12.54. Finally, there is the possibility of an involution on $R_0^{2\lambda-1}$ in $[2\lambda]$ with one generator g_1 consisting entirely of united points and another generator g_2 containing only two united points G_2 and H_2. The minimum directrix curve D_0^m is certainly unique, and $m\leqslant\lambda-1$. Choose g_1 and g_2 arbitrarily and take G_2 to be the intersection of g_2 with D_0^m. If H_2 is any other point of g_2 we can choose a pencil of self-corresponding $D_0^{2\lambda-m-1}$ passing through H_2, and an involution I on $R_0^{2\lambda-1}$ is determined. We find that there are two systems of base lines generating scrolls of order $\lambda-2$ and $\lambda-1$, which lie in two base spaces $S_{\lambda-1}$ and S_λ*. The space $S_{\lambda-1}$ contains H_2 if m is even and G_2 if m is odd, and S_λ contains g_1 and the remaining united point on g_2. The harmonic involution with $S_{\lambda-1}$ and S_λ as base spaces contains I, so involutions of type (d) exist. The quadrics through $R_0^{2\lambda-1}$ form an harmonic system relative to $S_{\lambda-1}$ and S_λ, containing $(\lambda-1)^2$ independent base quadrics. In this case the condition $n\geqslant 2r$ gives us two possibilities. If $n=2\lambda$ and $r=\lambda$, the number of independent base quadrics is $r(n-r-2)+1$ and $n=2r$. If $n=2\lambda$ and $r=\lambda-1$, the number of independent base quadrics is $r(n-r-2)$, and $n=2r+2$.

* It is hardly necessary to point out that the two base spaces which occur in all the four cases are independent, for any space containing them both would contain the scroll $R_0^{2\lambda}$ or $R_0^{2\lambda-1}$.

12.6. We have now investigated all possible cases in which the quadrics through a canonical curve form an harmonic system, and the results are summarized in the following theorem, in which it is assumed that $n > 3$ and $n \geqslant 2r$.

The only cases in which the quadrics through a canonical curve $K(n)$ in $[n]$ form an harmonic system relative to two independent base spaces S_r and S_{n-r-1}, containing b base quadrics, are the following.

(1) *When* $b = r(n-r-2)$:

(i) *If* $n = 2r+1$, $2r+2$, *or* $2r+3$ *the curve* $K(n)$ *possesses a* g_3^1, *or a* γ_2^1 *of genus* $r+1$, *or both, except in the case* $n = 5$, $r = 1$, *when* $K(n)$ *may possess a* g_5^2.

(ii) *If* n *and* r *have any other values satisfying* $n > 3$ *and* $n \geqslant 2r$, *the* $K(n)$ *possesses a* γ_2^1 *of genus* $r+1$.

(2) *When* $b = r(n-r-2)+1$ *and* $n = 2r$ *or* $2r+1$ *the* $K(n)$ *possesses a* g_3^1.

(3) *When* $b = r(n-r-1)$ *the* $K(n)$ *possesses a* g_3^1.

In (1)(i), when $K(n)$ has a g_3^1 and lies on an R_0^{n-1}, the involution on the scroll is of type (c), (d), or (a) according as $n = 2r+1$, $2r+2$, or $2r+3$, respectively. In (2) the involution is of type (d) if $n = 2r$ and of type (b) if $n = 2r+1$. In (3) we have the type of §12.41. It will be noticed that (2) and (3) overlap if $r = 1$.

We may add as a corollary that the only involutions of order two that can exist on a $K(n)$ which possesses a g_3^1 are of genus $\frac{1}{2}n$ or $\frac{1}{2}(n \pm 1)$, and that such an involution is itself composed of a g_3^1.

12.7. Finally, we may now state the theorem which was the original object of §12.

When $n > 3$, *with certain exceptions, necessary and sufficient conditions for a canonical curve* $K(n)$ *in* $[n]$ *to possess a* γ_2^1 *of genus* $r+1$ *are that* $n \geqslant 2r$ *and that the quadrics through* $K(n)$ *shall form an harmonic system relative to two independent base spaces* S_r *and* S_{n-r-1}, *containing* $r(n-r-2)$ *independent base quadrics. These conditions are always necessary*, and the only*

* See §6.3.

exceptional cases in which they are not sufficient are (1) $n = 5$, $r = 1$, *when the* $K(n)$ *may lie on a Veronese surface, and* (2) $n = 2r+1$, $2r+2$, *or* $2r+3$, *when the* $K(n)$ *may lie on a rational normal scroll*.*

References.

S 1. Segre, *Math. Annalen*, 30 (1887), 203.
S 2. Segre, *Math. Annalen*, 34 (1889), 1.
S 3. Segre, *Rend. Lombardo* (2), 21 (1888), 523.
S 4. Segre, *Atti Acc. Torino*, 21 (1885), 95.
S 5. Segre, *Atti Acc. Torino*, 19 (1884), 355.
E. Edge, *Theory of ruled surfaces* (Cambridge, 1931).
W 1. Welchman, *Proc. Camb. Phil. Soc.*, 29 (1933), 382.
W 2. Welchman, *Proc. Camb. Phil. Soc.*, 29 (1933), 235.
W 3. Welchman, *Proc. Camb. Phil. Soc.*, 29 (1933), 103.

Sidney Sussex College,
Cambridge.

* When $n = 3$ these conditions are necessary; sufficient conditions for the case $n = 3$, $r = 1$ have been obtained in § 7.8. F. P. White established the particular case of the above theorem in which $n = 4$, $r = 2$ some years ago, but did not publish his result; in this case there are no base quadrics and the condition is simply the existence of a line and a plane which do not meet and which are polars with respect to all quadrics through $K(4)$.

FAILURES IN CONGRUENCES CONNECTED WITH THE FOUR-COLOUR MAP THEOREM

By P. J. HEAWOOD.

[Received 9 August, 1934.—Read 15 November, 1934.]

1. In a former paper on extended congruences connected with normal maps*, a complete analysis was given of all the possibilities for maps of twelve or fourteen divisions (the lowest numbers with which we are concerned); and it was found that, though failures are extremely rare, persistent cases of failure occurred when the numbers associated with two adjacent divisions were 1 and 2 and all the rest zeros. A partial analysis for maps of fifteen and sixteen divisions showed that here again a like assignment of numbers always led to failure. It did not then appear whether there was any special reason for this, or whether the same thing would happen with maps of larger numbers of divisions. The present paper solves the problem.

2. It may be premised that the truth of the four-colour map theorem depends on the possibility of solving, with values of the x's $+1$ or -1 but *not* zero, a system of linear congruences of the form

$$x_r+x_s+x_t+\ldots\equiv 0 \pmod 3,$$

one for each face; where the variables x_1, x_2, ... are associated with the "corner points" of the map, and $x_r, x_s, x_t, \ldots$ are those surrounding the face in question. The *extended* congruences are those of like form, with the right-hand sides 0 or 1 or 2, subject to the condition that their sum is congruent to zero; it being supposed that the map is a *complete* map,

* P. J. Heawood, *Proc. London Math. Soc.* (2), 33 (1932), 253–286.

covering a closed simply-connected surface. *Any* assignment of values $+1$ or -1 to the x's will satisfy some such system of congruences; and, for maps with any considerable number of divisions, the number of possible assignments of such values to the x's so greatly outweighs the number of possible assignments of constants on the right-hand sides of the congruences, that it might seem likely that not only the map-colour system but all such systems could be solved, when the divisions were sufficiently numerous; indeed, the theory of probability was invoked to show that, *merely as a question of averages*, supposing *no essential impediments*, the chance of *any* failures for maps of seventeen divisions would be very small and for a greater number of divisions extremely minute. For fewer divisions there are, as before stated, certain failures; it was, however, left in doubt whether they will persist with larger numbers of divisions, or whether failures will entirely disappear, as probability seemed to suggest.

It will now be shown that, for any such map, if the numbers 1, 2 are assigned to two adjacent divisions, and zeros to all the rest, solution is *necessarily* impossible. The question whether failures entirely disappear, as the number of divisions increases, is therefore answered in the negative.

3. For symmetry we usually suppose the complete map to cover the surface of a sphere. A closed circuit made up of different division-boundaries (Fig. 1) will thus divide it into two parts. We may speak of divisions *outside* or *inside* this circuit (as they appear on the paper), but there is no essential distinction between the two. Let $m_1, m_2, m_3, \ldots$ be the amounts contributed to the successive congruence-sums for divisions adjacent to the circuit on *one* side, by the variables associated with points along its course; $n_1, n_2, n_3, \ldots$ the like for those on *the other*. Thus, if we denote by $x_1, x_2, x_3, \ldots$ the variables assigned to points where boundaries branch off on the "inside", and by y_1, y_2, y_3 those assigned to points where boundaries branch off on the "outside", we have, in the particular case figured:

$$m_1 \equiv x_1+y_1+y_2+x_2, \quad m_2 \equiv x_2+x_3, \quad m_3 \equiv x_3+y_3+y_4+y_5+y_6+x_4, \quad \ldots,$$

$$n_1 \equiv y_1+y_2, \quad n_2 \equiv y_2+x_2+x_3+y_3, \quad n_3 \equiv y_3+y_4, \quad \ldots,$$

and similarly for any other case. There is, therefore, one obvious relation between the m's and the n's; for we have $\Sigma m \equiv \Sigma y - \Sigma x \pmod 3$, since each x will occur twice, and similarly $\Sigma n \equiv \Sigma x - \Sigma y$. Σm and Σn are, therefore, complementary; and, in the case to be more particularly examined, $\Sigma m \equiv 0$, $\Sigma n \equiv 0$.

But there is a more subtle relation between the m's and n's in this case. Suppose that (either for the m's or the n's) we have a sequence of "digits",

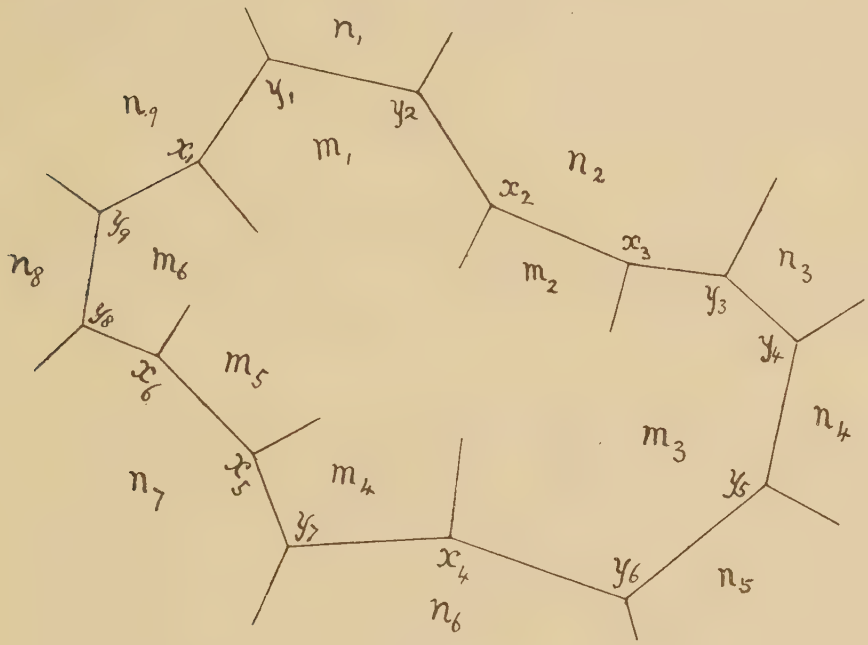


Fig. 1.

such as* 0 1 1 2 1 2 2 2 0 0 1 0, taken cyclically, and that we "reduce" them according to the following rules:

(*a*) a "2" may be cancelled and 1 added to the adjacent digits;

(*b*) a "1" may be cancelled and 2 added to the adjacent digits;

(*c*) a "0" and the two digits adjacent to it may be replaced by their sum.

[3 is, of course, to be replaced by 0 and 4 by 1, in any such result.]

After (*a*) or (*b*) the sequence will include one digit fewer, after (*c*) two digits fewer; but after any of the three, their sum will be congruent to the same value as before, *i.e.*, in the case with which we propose to deal, congruent to zero. After a succession of such steps, then, we necessarily

* -1 and 2 are, of course, equivalent.

arrive finally *either* at two digits 0 0 or 2 1 or 1 2 *or* at a single digit 0. In what follows we regard the essential alternatives for "residuals" as being 0 0 on the one hand, and 2 1 or 1 2 or 0 on the other, the last three being treated as equivalent. At a stage at which three digits are left, the possibilities are 2 2 2 or 1 1 1 or 0 0 0 or 0 1 2 (in some order); and the first two reduce further to 0 0, the last two either to 0 or 1 2 or 2 1, which three we treat as indistinguishable.

4. On this understanding, we are prepared to show (1) that, for any given sequence whose sum is congruent to zero, the residuals will be the same, whatever the order of the steps of "reduction" may be; and (2) that the residuals for the m's and the n's are necessarily the same.

(1) may be proved by mathematical induction. It is easily verified that for all possible arrangements of four digits or of five digits, whose sum is congruent to zero, the order of cancellation is indifferent. Thus the 4-digit sets 1 2 1 2, 0 1 0 2, 0 0 0 0, and the 5-digit sets 2 2 2 0 0, 0 1 2 2 1 have residuals 0 0; while 0 2 2 2, 1 1 2 2, 0 0 2 1 and 2 2 2 2 1, 0 2 0 2 2, 0 2 2 1 1, 0 2 1 2 1, 0 0 0 1 2, 0 1 0 0 2, 0 0 0 0 0 have residuals 0 or 2 1, or 1 2, in whatever order the reduction is made. These, taken cyclically, with their complementaries, include all the essentially distinct cases. Suppose it true that for *any* such sets of r digits, or fewer, the order of reduction is indifferent, and pass to the case of $r+1$ digits. We already know that for these, when one reduction has been made, since there are now r digits (or fewer), the order of subsequent steps is immaterial; the only question is whether the choice of a first step can affect the result—whether the cancelling of (say) a digit p or a digit q, in the first instance, can make any difference in the end. Now *after cancelling* p, we are (by hypothesis) at liberty to proceed in any order and may, *e.g.*, cancel q; or if we first deal with q, we may proceed with p, *supposing only that the two steps do not interfere with each other*, which they cannot do unless p, q are next to each other or next but one. Then in two stages we have reached the same result by the two methods, and it can make no final difference whether we have begun with p or with q. But where p, q are contiguous, the possibility of interference needs special examination. The sequence of digits in the neighbourhood of p, q may be represented by (1) $\alpha\beta pq\gamma\delta$, (2) $\alpha\beta p\gamma q\delta$, and it will be sufficient to consider pairs of values 0, 0; 0, 2; 1, 2; 2, 2 for p and q; other cases merely involving complementary values. In the following table, the first digit shown by a definite number, in the first column, is to be taken as p, the second as q, Greek letters being retained for the adjacent

digits. The columns which follow show the results (*a*) of cancelling p, (*b*) of cancelling q, (*c*) of cancelling first q and then p.

Form	(*a*)	(*b*)	(*c*)
$\alpha\beta00\gamma\delta\ldots$	$\alpha\beta\gamma\delta\ldots$	$\alpha\beta\gamma\delta\ldots$	
$\alpha\beta02\gamma\delta\ldots$	$\alpha(\beta+2)\gamma\delta\ldots$	$\alpha\beta1(\gamma+1)\delta\ldots$	$\alpha(\beta+2)\gamma\delta\ldots$
$\alpha\beta12\gamma\delta\ldots$	$\alpha(\beta+2)1\gamma\delta\ldots$	$\alpha\beta2(\gamma+1)\delta\ldots$	$\alpha(\beta+1)(\gamma+2)\delta\ldots$
$\alpha\beta22\gamma\delta\ldots$	$\alpha(\beta+1)0\gamma\delta\ldots$	$\alpha\beta0(\gamma+1)\delta\ldots$	$\alpha(\beta+\gamma+1)\delta\ldots$
$\alpha\beta0\gamma0\delta\ldots$	$\alpha(\beta+\gamma)0\delta\ldots$	$\alpha\beta0(\gamma+\delta)\ldots$	$\alpha(\beta+\gamma+\delta)\ldots$
$\alpha\beta0\gamma2\delta\ldots$	$\alpha(\beta+\gamma)2\delta\ldots$	$\alpha\beta0(\gamma+1)(\delta+1)\ldots$	$\alpha(\beta+\gamma+1)(\delta+1)\ldots$
$\alpha\beta1\gamma2\delta\ldots$	$\alpha(\beta+2)(\gamma+2)2\delta\ldots$	$\alpha\beta1(\gamma+1)(\delta+1)\ldots$	$\alpha(\beta+2)\gamma(\delta+1)\ldots$
$\alpha\beta2\gamma2\delta\ldots$	$\alpha(\beta+1)(\gamma+1)2\delta\ldots$	$\alpha\beta2(\gamma+1)(\delta+1)\ldots$	$\alpha(\beta+1)(\gamma+2)(\delta+1)\ldots$

In the first line (*a*), (*b*) are the same; the results of dealing with p or with q are indistinguishable, both disappearing together. In the second, (*a*) agrees with (*c*); if we deal with p, q disappears, and the result is the same as if we first cancel q and then p. In all other cases q (or q modified) remains in (*a*), but if we now deal with q the result agrees with (*c*); *i.e.* the result of cancelling first p and then q is the same as the result of cancelling first q and then p. Assuming, then, that for *any* r digits, of zero sum, the order of the steps of reduction is indifferent, the same thing will hold for a like sequence of $r+1$ digits. But it does hold for four or five digits, therefore for 6, 7, ...; *i.e.* we get the same residuals in any given case (*either* 0 0 *or* one of the equivalents 1 2, 2 1, 0) for any order of steps of reduction; the residuals are a definite characteristic of any such cyclic sequence of digits.

5. We have now to show that the "residuals" of the m's are the same as those of the n's (Fig. 1). The values of the m's and n's depend on the values of the x's and y's along the circuit which we are considering; but so far "reduction" has been thought of as an abstract process applied (according to certain arbitrary rules) to the resulting m's or n's. The successive steps of reduction may, however, be brought into relation with *hypothetical* changes in the variables along the circuit, *unrelated to the actual structure of the map*. So long as there are two or more x's between any two consecutive y's, we may suppose reduction of the m's effected, without any alteration of the n's, by appropriate changes in the x's. Suppose, *e.g.*, that $x_2 \equiv x_3$, where x_2, x_3 are adjacent; the substitution of a single variable $x' \equiv x_2 + x_3$ $(\equiv m_2)$ would mean the replacement of three m's, $m_1 m_2 m_3$, by two, congruent to

m_1+x_2, m_3+x_2 respectively [Fig. 2 (*a*), (*b*)], which is in accordance with the

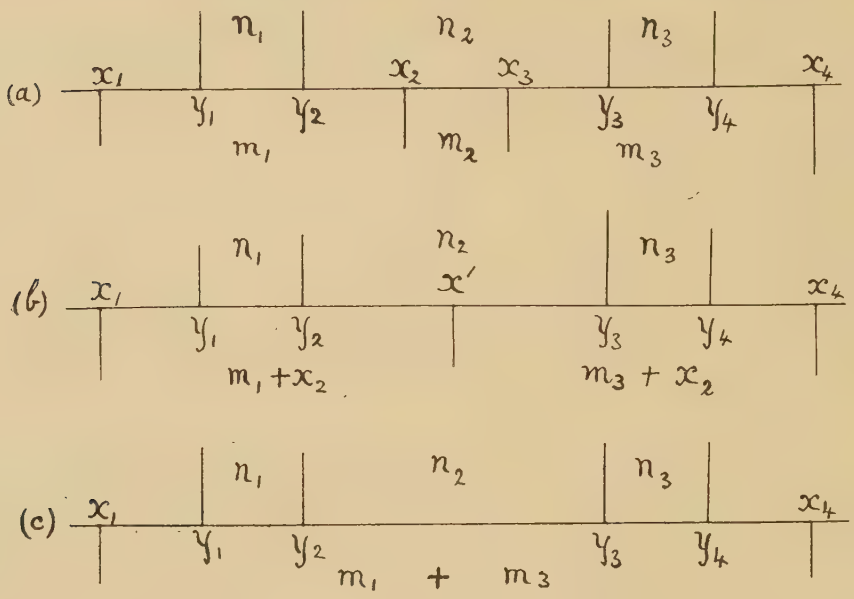


Fig. 2.

rules for reduction. If, however, x_2, x_3 are different, $x_2+x_3 \equiv 0 \equiv m_2$, and the abolition of these two x's would entail the replacement of m_1, m_2, m_3 by a single $m \equiv m_1+m_3$ [Fig. 2 (*a*), (*c*)], which is again in accordance with rule. In neither case will the values of the n's be affected by these changes. Similarly, so long as there are two y's between any consecutive x's, an appropriate alteration of the y's would give a reduction of the n's without affecting the m's. In this way we can proceed with successive reductions of the m's and n's (in a related scheme), until a stage is reached at which the x's and y's are strictly alternate, as in Fig. 3.

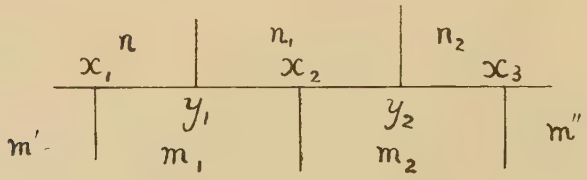


Fig. 3.

Suppose that, at such a stage, there are at least three x's and three y's, and that five successive values are $x_1 y_1 x_2 y_2 x_3$. The cancelling of *one* or

two of these x's, with appropriate changes of others of these x's and y's (but so that each y is still congruent to either 1 or 2), may or may not give a valid reduction of the m's, but it will leave the *number* of n's unaltered; and the *values* of the n's will be unaltered, if x_1+y_1, $y_1+x_2+y_2$, y_2+x_3 remain the same (treating a cancelled x as zero). The following table shows the changed values of $x_1 y_1 x_2 y_2 x_3$ which will fulfil the last condition, and also give valid reductions of the m's, in all the cases which it is necessary to examine separately. The first column shows the original values, the second the substituted values, "—" being put where an x has been cancelled. The third column shows the original sequence of m's from that before x_1 (denoted by m') to that after x_3 (denoted by m''), the two intermediate values being $\equiv x_1+y_1+x_2$, $x_2+y_2+x_3$. The fourth column shows, in terms of m', m'', the reduced m's corresponding to the changes made.

Original. $x_1\ y_1\ x_2\ y_2\ x_3$	Modified values.	Original m's	Reduced m's
1 1 1 1 1	– 2 – 1 1	m', 0, 0, m''	m', m''
,,	1 1 – 2 –	,,	m', m''
,,	– 2 2 2 –	,,	m', m''
1 1 1 1 2	– 2 – 1 2	m', 0, 1, m''	$m'+1$, m''
,,	– 2 2 2 1	,,	m', 2, $m''+2$
,,	1 1 – 2 1	,,	m', 2, $m''+2$
1 1 1 2 1	– 2 – 2 1	m', 0, 1, m''	$m'+1$, m''
1 1 1 2 2	– 2 – 2 2	m', 0, 2, m''	$m'+2$, m''
,,	– 2 1 1 –	,,	$m'+2$, m''
,,	1 1 2 1 –	,,	m', 1, $m''+1$
1 1 2 1 1	– 2 1 1 1	m', 1, 1, m''	$m'+2$, 0, m''
,,	1 1 1 2 –	,,	m', 0, $m''+2$
1 1 2 1 2	– 2 1 1 2	m', 1, 2, m''	$m'+2$, 1, m''
1 1 2 2 1	– 2 1 2 1	m', 1, 2, m''	$m'+2$, 1, m''
,,	– 2 2 1 2	,,	m', 2, $m''+1$
,,	1 1 – 1 2	,,	m', 2, $m''+1$
1 2 1 1 2	1 2 – 2 1	m', 1, 1, m''	m', 0, $m''+2$
1 2 2 2 1	2 1 – 2 1	m', 2, 2, m''	$m'+1$, 0, m''
,,	1 2 – 1 2	,,	m', 0, $m''+1$
1 2 1 2 1	*No valid change*	m', 1, 1, m''	—

It will be seen that changes that give a valid reduction can be carried out in every case but the last, often in more than one way. We can therefore proceed as above with reduction from the stage where x's and y's alternate, save only when they have the values 1, 2, 1, 2, ... alternately all round the circuit; *i.e.* the x's all 1's and the y's all 2's, or *vice-versa*.

But in that case the m's and n's will now consist, the one entirely of 1's, the other entirely of 2's, in equal numbers, and corresponding reductions

will obviously give equivalent residuals. In all other cases we can proceed as indicated in the table; then we shall have consecutive y's without intermediate x's, and the original method will apply for reduction of the n's without affecting the m's; and so on continually, by one or the other method, applied to the reduction of the n's or the m's. Needless to say the x's and y's in these schemes are hypothetical; they have no place in the map, and only serve to form a link between valid reductions of the m's and n's along a given circuit.

Suppose that, by a continual succession of such linked reductions, we reach a stage at which there are only two m's and two n's, defined by two x's and two y's; these will be arranged either as in (a) or (b), Fig. 4.

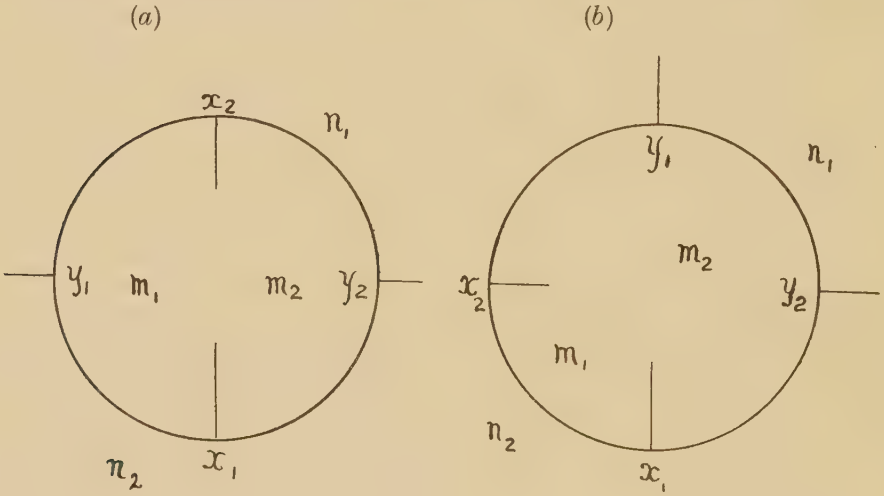


Fig. 4.

Since, throughout, $\Sigma m \equiv 0$, $\Sigma n \equiv 0$, we have $x_1 + x_2 \equiv y_1 + y_2$; and the following are the only possibilities that need be considered:

Values of $x_1\ x_2\ y_1\ y_2$	(a) $m_1\ m_2,\ n_1\ n_2$	(b) $m_1\ m_2,\ n_1\ n_2$
1 1 1 1	0 0, 0 0	2 1, 2 1
2 2 2 2	0 0, 0 0	1 2, 1 2
1 2 1 2	1 2, 2 1	0 0, 0 0
1 2 2 1	2 1, 2 1	0 0, 0 0

Thus the residuals for the m's and n's are either both 0 0 or both 1 2. If we examine what may happen at stages where there are three m's and

three n's or three m's and two n's (supposing a difficulty in reducing to just 2 and 2 as above), it will be found (though the cases to be examined are more numerous) that the same thing holds. If, *e.g.*, there are three m's and three n's, and the m's are 1 1 1 or 2 2 2, the n's will likewise be 1 1 1 or 2 2 2; if the m's are 0 1 2, so will be the n's. The residuals for both are 0 0 in the one case and 1 2 or single 0 in the other.

6. Now suppose that a circuit such as that we have been considering, *e.g.* that in Fig. 1, is surrounded by a ring of zero divisions, as shown diagrammatically in Fig. 5 (where the original circuit is represented by

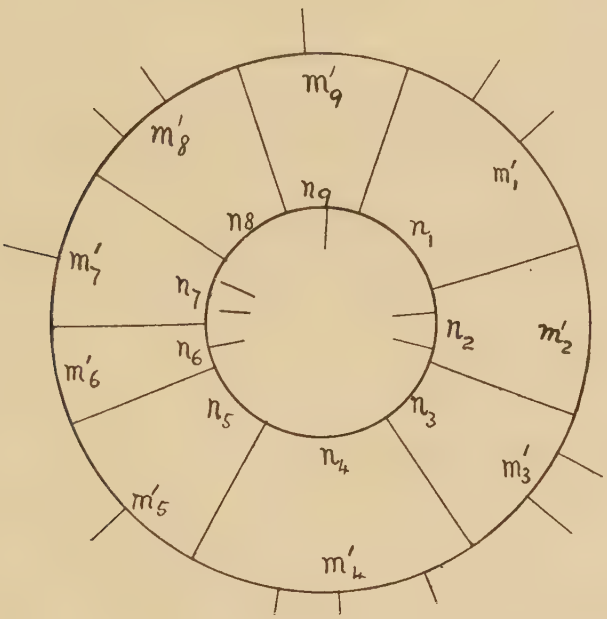


Fig. 5.

the inner circle); we shall have for the new circuit, embracing all these divisions, a sequence of m's, $m_1' m_2' \ldots$, equal in number to the n's of the original circuit, with $m_1' + n_1 \equiv 0$, $m_2' + n_2 \equiv 0$, ...; *i.e.* each m' is the complement of the corresponding n. Therefore, not only will $\Sigma m' \equiv \Sigma m$ (*i.e.* $\equiv 0$ in the case we are considering), but the "residuals" will be the same for the new m's as for the original n's, and therefore for the original m's; and so again if further rings of zero divisions are added. As we cannot contemplate in all cases the continual addition of complete zero rings, as we build up the map, it is important to show that the same thing holds for the new outer boundary, if a single zero division is

added. Suppose that the added division stretches along the old boundary from u to u' (Fig. 6), where u and u' denote the variables associated with

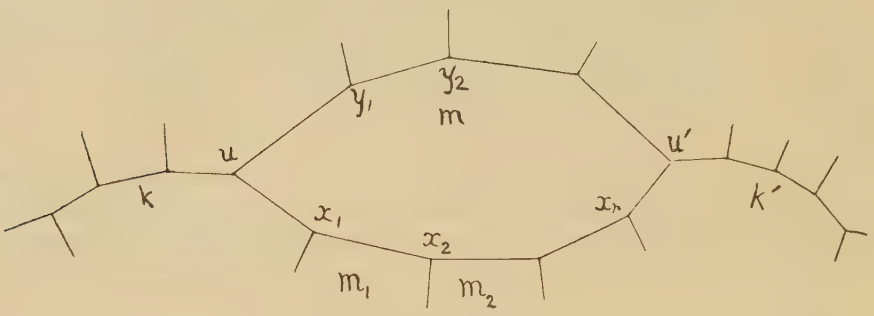


Fig. 6.

these points (which for the old circuit will be points where branches go outwards, for the new where branches go inwards). Between u and u' the new outer boundary will have branches outwards only (at $y_1, y_2, \ldots$, say), while the old has branches inwards only (at $x_1, x_2, \ldots, x_r$, say). We shall have inside the *new* boundary three m's, k, m, k', where

$$m \equiv u+y_1+y_2+\ldots+u',$$

in place of the old values

$$k+x_1, \quad x_1+x_2, \quad x_2+x_3, \quad \ldots, \quad x_r+k'.$$

But, since the added division is a zero division,

$$u+y_1+y_2+\ldots+u' \equiv -(x_1+x_2+\ldots+x_r)$$
$$\equiv 2s, \text{ if } s \equiv x_1+x_2+\ldots+x_r.$$

The three *new* m's, k, $2s$, k', will reduce to $k+k'$ *or* to $k+s$, $k'+s$, according as s is or is not congruent to zero. But, as in § 5, the reduction of the *old* m's,

$$k+x_1, \quad x_1+x_2, \quad x_2+x_3, \quad \ldots, \quad x_r+k',$$

may be connected with successive alterations of the x's, whereby *all* or *all but one* will disappear. (We cancel two adjacent *unlike* x's, their sum being congruent to zero; and we substitute for two adjacent *like* x's a single x congruent to their sum.) If, finally, *all* the x's disappear, which will happen when $s \equiv 0$, we are left with one $m \equiv k+k'$ (since the sum of the m's will be at every stage the same); if *one* x remains, it must be congruent to s, to give the proper sum, and we are left with two m's, congruent to $k+s$, $k'+s$. This is exactly the same as the result for the *new* m's above, in the respective cases. The complete sequences of m's (old and new) will, there-

fore, have the same residuals. If between u, u' there are no x's or no y's, it will not invalidate the argument. A special case requiring notice, however, is where the added division lies along the whole of the old boundary except a single "edge", so that u and u' belong to the same m, and k, k' are identical. Then the new m's are simply $u+u'$ and $2s$, where $2s \equiv 2(x_1+x_2+\ldots+x_r)$, as in the normal case. The old m's will be $u+u'+x_1+x_r$, x_1+x_2, x_2+x_3, ..., $x_{r-1}+x_r$, reducible by the same method as before to (say) $u+u'+x+x'$, $x+x'$, where $x+x' \equiv s$. Since $\Sigma m \equiv 0$, $u+u'+2s \equiv 0$, *i.e.* $u+u' \equiv s$; and we are left with residuals s and $2s$ both for the old m's and the new. If the reduction is to $u+u'+x$ and x (with one x), where $x \equiv s$, the result is the same. (We do not contemplate a circuit bounding a single division.)

It is because the map here treated is on a simply-connected surface, such as a sphere, that we can suppose divisions added, as above, one by one, and arrange to finish at the end with two outstanding divisions, which have a connected piece of boundary in common with each other.

7. We are now in a position to draw final conclusions. Suppose, in any such map, that we have the variables so arranged that they give a "nucleus" of two divisions whose constants are 1, 2, as at A (Fig. 7),

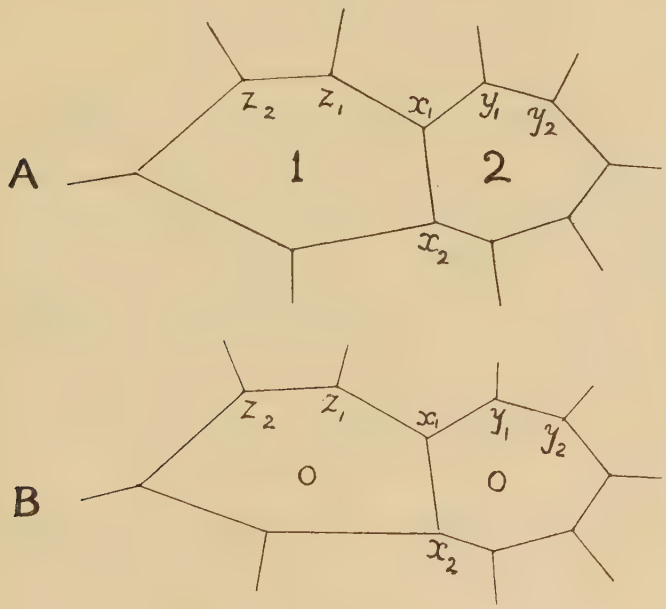


Fig. 7.

surrounded by zero divisions. The circuit immediately enclosing these

two divisions will have residuals 1, 2; therefore, all the boundary circuits, as we proceed outwards, embracing more and more zero divisions, will have residuals 1, 2. It will, therefore, be impossible for the map, completed on the sphere, to close in finally with zero divisions. For, if the two divisions last to be included were zero divisions, it would imply a circuit surrounding them with residuals 0, 0. Similarly, if we begin, as at B, with two adjacent divisions 0, 0, we shall be precluded from closing in with a pair of adjacent divisions 1, 2, and all the rest zero. It is, therefore, impossible, however many the divisions of a map may be, to have two adjacent divisions associated with constants 1, 2, and all the rest with zeros, if the corresponding congruences are to be solved.

This is the case of failure that proved so persistent in the examples analysed in the former paper; and the main question there raised, whether, when the number of divisions is sufficiently increased, all "failures" will disappear, is definitely answered in the negative.

8. On the other hand, suppose that we omit, not a single congruence corresponding to a single division of the *complete* map on the sphere, but those corresponding to *two adjacent divisions**; *i.e.* suppose that there are $n+2$ divisions in the *complete* map, we take n congruences involving $2n$ variables. It would appear that there is nothing to prevent these being duly satisfied for *any* values of the constants.

Consider first the maps of twelve and fourteen divisions, for which the "failures" were fully analysed in the preceding paper. The results *there* given (Figs. 5 and 7) show that, if two adjacent divisions are left out of account, all constants will be possible for the rest. For in *any* of the cases there pictured, we shall find that we can obviate failure by replacing the constants of *any* two adjacent divisions by another pair, whose sum is congruent to the same value, *e.g.* 2, 2 by 0, 1. For in this way we are able to get an arrangement *not* equivalent to any of those figured; and as these include all possible "failures" under the conditions given, the modified system of congruences for the complete map is *not* a case of failure. It follows that the system got by *omitting* congruences for the two divisions in question, without altering the constants of the rest, is one that can be solved in a valid manner.

But, further, the proposition that, if we omit two adjacent divisions, the congruences corresponding to the rest can be solved for any values of the

* The omission of a single congruence is immaterial, since (if used) its constant is determined by the necessity that the sum of all the constants, for the complete map, must be congruent to zero.

constants, holds even in the rudimentary cases, which were before left out of account. For the "tetrahedron", *e.g.*, we have two congruences,

$$(1)\ x_1+x_2+x_3\equiv a,\quad (2)\ x_1+x_2+x_4\equiv b\ \text{(say)};$$

and $x_3-x_4\equiv a-b$ can always be satisfied in at least one way, with values 1 or 2 for the x's, and then $x_1+x_2\equiv a-x_3$ in at least one way; the total number of solutions ranging from one to three, according to the values of a and b (0, 1, or 2).

So for the "triangular prism" we have (say)

$$(1)\ x_1+x_2+x_3\equiv a,\quad (2)\ x_1+x_4+x_5+x_2\equiv b,$$

and *either*

$$(3)\ x_4+x_5+x_6\equiv c,\quad \textit{or}\ (3)\ x_2+x_5+x_6+x_3\equiv c,$$

according as the two divisions omitted are two quadrilaterals or a quadrilateral and a triangle; in either case solutions are possible for all values of a, b, c.

Similarly, for the "cube" we have

$$(1)\ x_1+x_2+x_3+x_4\equiv a,$$

$$(2)\ x_5+x_6+x_7+x_8\equiv b,$$

$$(3)\ x_1+x_5+x_2+x_6\equiv c,$$

$$(4)\ x_2+x_6+x_3+x_7\equiv d;$$

which again can be shown to be possible for all values of the constants.

If now we take the more irregular map of six divisions, consisting of two triangles, two quadrilaterals, and two pentagons, and omit two adjacent divisions, we have five cases, according as we omit two quadrilaterals, two pentagons, quadrilateral and pentagon, triangle and quadrilateral, or triangle and pentagon. The corresponding sets of four congruences will be such as will be got from the following:

$$(1)\quad x_1+x_2+x_3\equiv a,$$

$$(2)\quad x_2+x_3+x_4+x_5\equiv b,$$

$$(3)\quad x_4+x_5+x_6+x_7\equiv c,$$

$$(4)\quad x_1+x_2+x_5+x_6+x_8\equiv d,$$

$$(5)\quad x_1+x_3+x_4+x_7+x_8\equiv e,$$

$$(6)\quad x_6+x_7+x_8\equiv f,$$

by omitting A—(2), (3); B—(4), (5); C—(3), (5); D—(3), (6); E—(5), (6).

So again for the map consisting of four triangles and two hexagons, the only other case of six divisions that we have to examine, the sets of congruences are (say)

$$(1)\ x_1+x_2+x_3\equiv a, \quad (2)\ x_1+x_3+x_4\equiv b,$$

$$(3)\ x_2+x_3+x_4+x_5+x_6+x_7\equiv c,$$

with

$$(4)\ x_6+x_7+x_8\equiv d \quad or \quad (4)\ x_1+x_2+x_4+x_5+x_7+x_8\equiv d,$$

according as a triangle and a hexagon or two adjacent triangles are omitted.

In each of these essentially distinct cases, it can be verified that solution is possible for all values of the constants. Such special verification does not amount to proof of the general result; but it must be remembered that the chance of valid solution becomes greater, not less, as the number of divisions is increased.

9. Granting, however, the general proposition that, if we omit two adjacent divisions from any complete map, the congruences corresponding to the rest can be duly solved, with any values of the constants on the right-hand sides, we can, by the doctrine of residuals developed in §§ 3–7, complete the proof that the special system associated with the map-colour theorem can always be solved. For, if we satisfy the system of congruences corresponding to *all but two adjacent divisions* of the completed map, with zeros on the right-hand sides, the proper constants for the two omitted are bound to be zeros. The alternative of 1 and 2 is excluded; therefore, in closing in the map, no final difficulty can possibly arise.

High Close,
Durham.

ON THE SYNTHESIS AND ANALYSIS OF SIMPLY-STIFF FRAMEWORKS*

By H. ROXBEE COX.

[Received 31 August, 1934.—Read 15 November, 1934.]

Contents.

1. Introduction and nomenclature.
2. Theorems.
3. Procedure for detecting simply-stiff frameworks.
4. The symmetrical simply-stiff frameworks.
5. Example.

1. *Introduction and nomenclature.*

1.1. If a simply-stiff three-dimensional framework has j joints, it has necessarily $3j-6$ members disposed among its joints. This fact is well known and easily proved. It cannot, however, be inferred that, if a framework with j joints has $3j-6$ members, it is simply-stiff: the absurdity of such an inference is easily demonstrable.

It is frequently desirable to know in the course of design or calculation whether or not a given framework is simply-stiff. The first and obvious test is to count the numbers of joints and members. If, having j joints, it is found *not* to have $3j-6$ members, it cannot be simply-stiff. If it *has* $3j-6$ members, then it may or may not be simply-stiff, and it is necessary to devise further tests before its character can be determined.

In this paper four simple theorems are given which may be applied to test whether or not a framework is simply-stiff or used for building frame-

* Air Ministry Official Report. Crown copyright reserved.]

works which are bound to be simply-stiff. Of these theorems, the first two are merely restatements of the conditions of equilibrium of a particle and of a rigid body. The third is an extremely useful adjunct to the first and second, and may be new to many who are concerned in framework calculations. The fourth is a theorem dependent on the third with a more restricted scope than the others, and is perhaps only one of several which could be devised each with the analysis of a particular type of framework in view. From the writer's experience, however, it seems unlikely that theorems additional to those given will be required in practical problems.

1.2. The enunciation and application of these theorems is simplified by the adoption of a special nomenclature. It is convenient to refer to

2	members meeting at a point at which no other members occur as a	*pair,*
3		*trio,*
4		*quartet,*
5		*quintet,*
.		.....
.		.....
.		.....
n		n-*tet,*
.		.....
.		.....
.		.....
$j-1$		$(j-1)$-*tet,*

the joint common to the members of an n-tet as its *vertex,*

the ends of the members of an n-tet remote from its vertex as its *feet,*

a joint at which an odd number of members meet as an *odd joint.*

1.3. In connection with these definitions it is worth noticing that

1.31. any framework must have an even number of odd joints.

1.32. in a simply-stiff framework, the average number of members at a joint is $6-12/j$, which, since j is a positive integer, is necessarily less than 6; so that, however great j is, a simply-stiff framework *must* contain quintets, quartets, or trios.

2. *Theorems.*

In the following theorems no provisos relating to special geometry or external loading are made. It is well known that special geometry may change a stable distribution of members among joints into a mechanism, and that a mechanism may be stable for special orientations of the items of a load system.

2.1. The first theorem states merely that

If to a given simply-stiff space framework an additional joint is attached by a trio to joints of the framework, the result is a simply-stiff framework.

Conversely, if a simply-stiff framework is the result of attaching an additional joint by means of a trio to another framework, this other framework must be itself simply-stiff.

This theorem, proof of which would be superfluous here, permits the synthesis of a large number of simply-stiff frameworks of a special type. Its converse can be applied as an analytical process. For a given framework with j joints and $3j-6$ members is simply-stiff if, on removal of all its trios, the remaining framework is simply-stiff; equally, this reduced framework is simply-stiff if on removal of all *its* trios, the further reduced framework is simply-stiff; and so on. If this successive reduction can be performed until a framework recognisable as simply-stiff, such as the tetrahedron, is reached, the original framework must have been itself simply-stiff. If the reduction produces what is clearly a mechanism, the original framework could not have been simply-stiff.

This reduction procedure can be followed to a definite conclusion in a considerable number of cases, since many frameworks are built on the basis of the theorem. The procedure must fail, however, if at any stage of the reduction the framework contains no trios and it is not recognisable as either simply-stiff or a mechanism.

For the less general case of plane frameworks, the first theorem becomes

If to a given simply-stiff plane framework an additional joint is attached by a pair to joints of the framework, the result is a simply-stiff framework.

This theorem and its converse have applications in two-dimensional problems corresponding to the more general procedures described above, and with similar limitations.

2.2. The second theorem states that

A simply-stiff space framework will be formed if two simply-stiff frameworks A and B are connected by

(*a*) *coalescing a member of A with a member of B and introducing one member between a joint of A and a joint of B neither of which is at an end of the common member*;

or (*b*) *coalescing a joint of A with a joint of B and introducing three members so that on each framework three ends are distributed between two or three joints other than the common joint*;

or (*c*) *coalescing no members or joints and introducing six members so that on each framework six ends are distributed between more than two joints with not more than three at a joint.*

This theorem is merely a restatement of the laws of equilibrium of a rigid body and needs no proof here. Its converse is equally true. The theorem can be used for building up simply-stiff frameworks which could not be achieved by application of the first theorem, and the converse can frequently be used in testing whether a given framework is simply-stiff. Thus, when the reduction process of § 2.1 has been applied to the utmost and a framework of doubtful character remains, sometimes it can be seen that, by simultaneously removing six members whose ends are distributed as stated in the theorem, two simpler frameworks are left. If one or neither of these is recognisable as simply-stiff, the converses of the first and second theorems are applied further until a successful conclusion is reached, or until frameworks reducible by neither converse are achieved.

The version of this second theorem applicable to plane frameworks is

A simply-stiff plane framework will be formed if two simply-stiff frameworks A and B are connected by

(*a*) *coalescing a member of A with a member of B*;

or (*b*) *coalescing a joint of A with a joint of B and introducing a member between a joint of A and a joint of B neither of which is the common joint*;

or (*c*) *coalescing no members or joints and introducing three members so that on each framework three ends are distributed between more than one joint.*

2.3. The third theorem states that

If in a simply-stiff space framework the member between any two joints is replaced by a quartet with feet at these two joints and at two other joints of the framework, the resulting framework is simply-stiff.

For suppose that it is intended to replace a member joining two joints B and C by a quartet $A.BCDE$ with feet at B and C and at two joints D and E.

Suppose the member BC removed. Then relative motion between the joints B and C will be possible and each of the remaining joints of the framework will have or have not gained one degree of freedom. If by any means relative motion between B and C is prevented, the framework will become stiff again.

In the most general case, after removal of BC, it will be possible for the points B, C, D, and E to move relative to the original configuration with interdependent motions along line paths. Then, if a trio is introduced with feet at B, C, and D, its vertex A traces a line path. In emphasis of this statement it is remarked that, for a *given* position of this vertex, the relative positions of B, C, and D are fixed.

Since in general the distance between the points A and E when moving along their line paths must vary, introduction of a member AE must prevent their relative motion, and consequently the relative motion of B, C, D, and E.

Thus the replacement of BC by the quartet $A.BCDE$ has produced a stiff framework. Since the original framework was simply-stiff and since the change has not disturbed the relation of j joints and $3j-6$ members, the new framework is also simply-stiff.

The converse of this theorem is not in general true.

The third theorem can be used for the building of frameworks which cannot be erected by application of the first and second theorems. Since this third theorem has no converse however, there is no general reduction process corresponding to it as there was corresponding to the first and second theorems.

If it is applied to the tetrahedron $ABCD$ in Fig. 1 (a), the side AD can be replaced by the quartet $E.ABCD$, thus producing the framework of Fig. 1 (b). Similarly, replacing CE by the quartet $F.ACDE$ produces the framework of Fig. 1 (c), which cannot be produced by using the more familiar first theorem.

If the side EF is replaced by the quartet $G.AEDF$, the framework of Fig. 1 (d) can be formed and by repetition of the process a series of frame-

works, of which Fig. 1 (*e*) gives a typical symmetrical example, can be erected. In the figures of this class there are four members at all joints except two, at each of which there are $j-2$.

It should be noted that any space framework produced by the third theorem must contain at least one quartet.

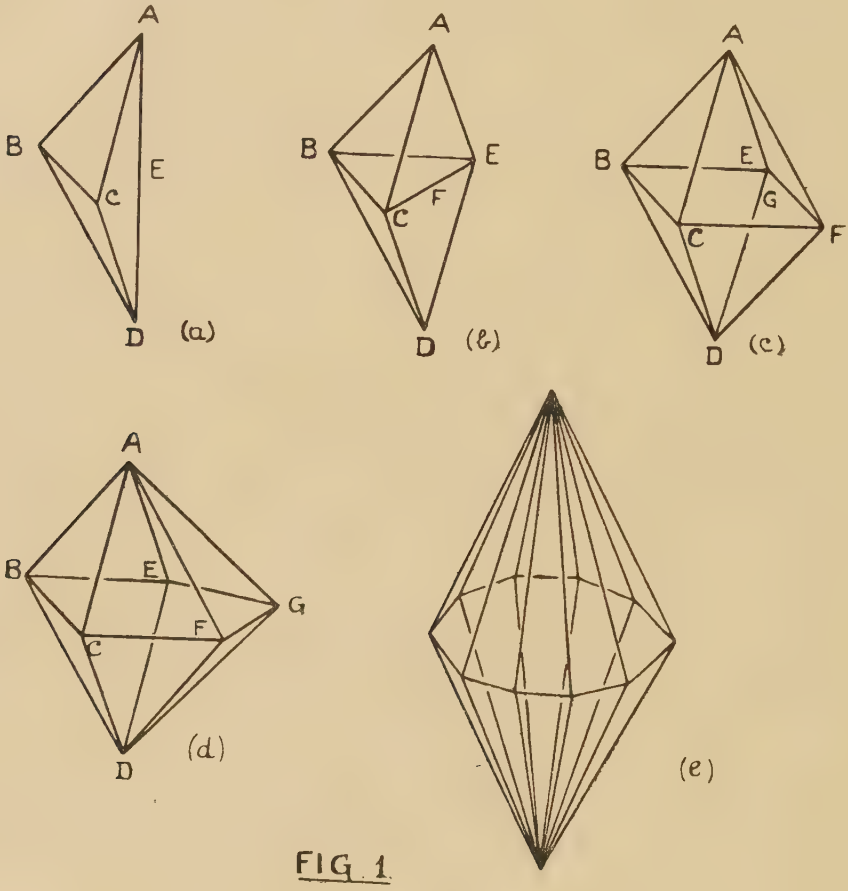


FIG. 1.

This third theorem becomes for the case of two-dimensional frameworks

If in a simply-stiff plane framework the member between any two joints is replaced by a trio with feet at these two joints and at some other joint of the framework, the resulting framework is simply-stiff.

The converse of this degenerate version of the third theorem appears to be true.

2.4. The fourth theorem states that

If in a simply-stiff space framework there occurs among n particular joints an arrangement of $2n-3$ members which, if projected on to a plane, would represent a simply-stiff plane framework, then a simply-stiff space framework results if the $n-3$ of these members which do not form part of the polygon embracing the n joints are removed and replaced by an n-tet with feet at the n-joints.

To prove this theorem we must recognise this lemma. *If the motion of each of a number of points is restricted to a surface, relative motion between the points will be prevented if they are connected by an arrangement of bars which, if projected on to a plane, delineates a simply-stiff plane framework.*

Referring to Fig. 2 (*a*), consider an *n*-tet with vertex *A* and feet *B*, *C*, *D*, *E*, ... If the vertex *A* is held, *each* of the feet *B*, *C*, *D*, *E*, ...

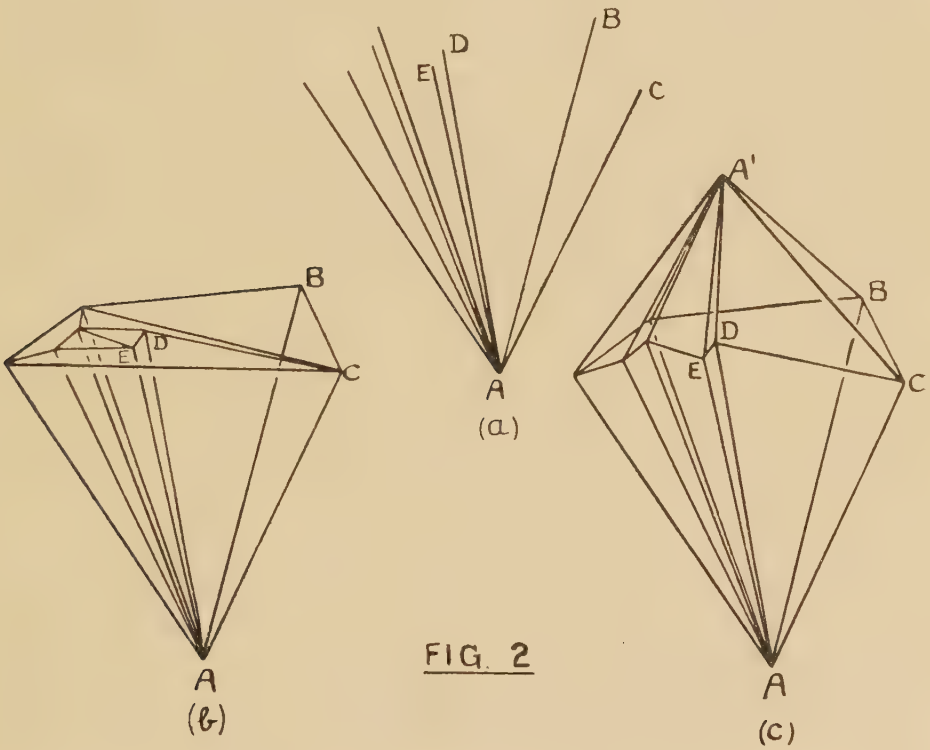


FIG. 2

has two degrees of freedom. In other words, if the members AB, AC, AD, AE, ... are displaced, the movement of each of the points B, C, D, E, ... is confined to a surface. It follows then that if a two-dimensional simply-

stiff arrangement of bars devised by the plane versions of the foregoing theorems is introduced among the joints B, C, D, E, ..., these joints will with A form a simply-stiff space framework such as that of Fig. 2 (b). But by the third theorem the joints B, C, D, E, ... can be braced by introducing the n-sided polygon (or "n-gon") $BCDE...B$ among the joints and adding the n-tet $A'.BCDE...B$ as shown in Fig. 2 (c). It follows that, in so far as the simply-stiff stabilisation of $ABCDE...$ is concerned when A is joined to each of B, C, D, E, ..., the members among the points B, C, D, E, ... which do not form part of the n-gon $BCDE...B$ are equivalent to the n-tet $A'.BCDE...B$.

So far we have proved that the replacement of the bracing of an n-gon by an n-tet is justified for the case of an $(n+1)$-jointed framework of a special kind.

Now, referring to Fig. 2 (b), suppose an n-gon $BCDE...B$ with $n-3$ bracing members occurs in a simply-stiff framework P. The stability of the n-gon is unaffected if that part of the framework other than the n-gon and its bracings is replaced by an n-tet $A.BCDE...B$ with feet at the joints of the n-gon. That is, in so far as the stability of the joints of the n-gon is concerned, the framework P and the simply-stiff framework of Fig. 2 (b) are equivalent. It follows that the framework P and the simply-stiff framework of Fig. 2 (c) are equivalent. Consequently, in the simply-stiff framework P, the $n-3$ bracings of the n-gon $BCDE...B$ may be replaced by the n-tet $A'.BCDE...B$, and the theorem is proved.

This theorem does not appear to have a counterpart in two-dimensions, and its converse is not in general true.

3. *Procedure for detecting simply-stiff frameworks.*

The procedure which suggests itself for detecting whether or not a given framework is simply-stiff falls naturally into three parts:

3.1. A test whether the necessary condition of j joints and $3j-6$ members is fulfilled. If it is not, there is no more to be done.

3.2. Continued reduction of the framework by the converses of the first and second theorems until

3.21. an obvious mechanism is reached.

3.22. an obviously simply-stiff framework is reached.

3.23. a framework N having no trios and indivisible into smaller units is reached.

In the event of §3.21 or §3.22 arising, there is no more to be done.

3.3. Attempted synthesis of the framework N of § 3.23 from a known simply-stiff framework by means of the theorems of § 2. Illustrations of the use of the second, third, and fourth theorems occur in § 4 and a special example on the use of the third theorem, and involving the first theorem, is given in § 5.

4. *The symmetrical simply-stiff frameworks.*

4.1. When performing § 3.2 of the detective procedure, an ability to recognise simply-stiff frameworks is advantageous. The tetrahedron [Fig. 1 (*a*)] and the simple structures formable upon it by the first theorem [*cf.* Fig. 1 (*b*)] are easily recognisable. Less familiar are the octahedral framework of Fig. 1 (*c*) and the series represented by Fig. 1 (*e*).

4.2. In a quest for easily recognisable units, "symmetrical" frameworks, such as have the same number of members at each joint, suggest themselves as interesting quarries. For a symmetrical space framework

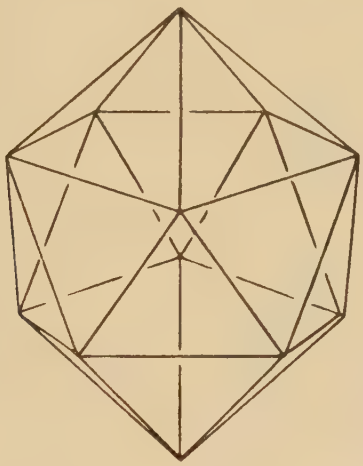


FIG. 3.

to be simply-stiff it is clearly necessary (though not of course sufficient) that the number of members n meeting at each joint must be $6-12/j$. It follows that the only possible finite simply-stiff symmetrical space frameworks are those with $n = 2$, 3, 4, and 5, for which $j = 12/(6-n)$ must be respectively 3, 4, 6, and 12. These are the triangle, the tetrahedron, the octahedron, and the icosahedron. We are satisfied of the stiffness of the triangle and tetrahedron [Fig. 1 (*a*)] and the regular octahedron of Fig. 1 (*c*).

Although, by distorting the geometry, the appearances of octahedral frameworks can be made very different, they appear always to consist of two quartets with feet on a common (distorted) quadrilateral, so that structurally they are essentially of the same type as regular octahedra and consequently are all simply-stiff.

4.3. The regular icosahedron is shown in Fig. 3. We can apply the third and fourth theorems to test frameworks of this type. Starting with the framework of Fig. 1(*d*), replacing each of the members AF, AG, AE, AB by an appropriate quartet, a framework such as that of Fig. 4 is easily achieved. Using the fourth theorem, we replace the bracings AC, $F'C$ of the pentagon $AF'FCB'$ by the quintet $C'.AF'FCB'$ [Fig. 4 (*b*)] and so produce a distorted version of the icosahedron of Fig. 3. So the regular icosahedron and its isomorphs are, like the tetrahedron and the octahedron, simply-stiff.

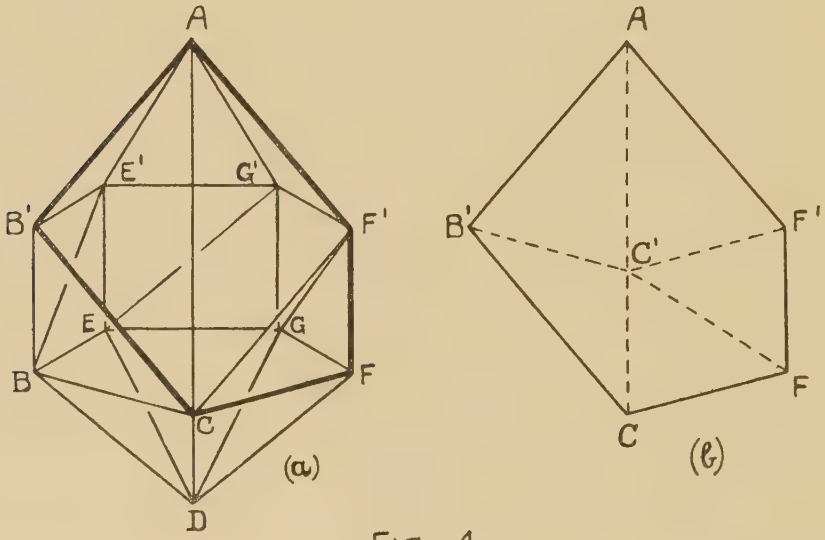


FIG. 4.

By applying part (*c*) of the second theorem to join together six-jointed simply-stiff frameworks, "irregular" icosahedra* which are simply-stiff

* The essential characteristic of these twelve-jointed frameworks is that each has twelve quintet vertices. It is difficult to decide upon the number of "faces" each has. The term "icosahedron", meaning as it does "twenty bases", does not express the essential characteristic and may, therefore, be thought inappropriate.

can be produced. Examples of these are given in Figs. 5, 6, 7, and 8. In these diagrams each of the four possible simply-stiff six-jointed frameworks is shown connected to its reflection. Further irregular icosahedra are formable by connecting pairs of *different* six-jointed frameworks.

In Fig. 5, two octahedra are shown joined together to give a framework in which each joint is the vertex of a quintet.

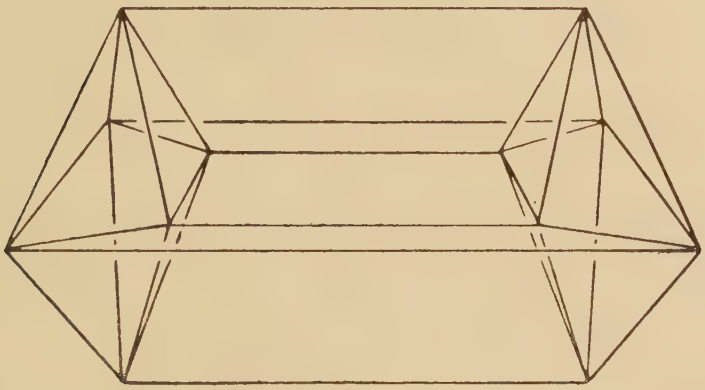


FIG. 5.

In Fig. 6, two six-jointed frameworks, each having three quintet-vertices and three trio-vertices, are shown joined by bars between the six trio-vertices.

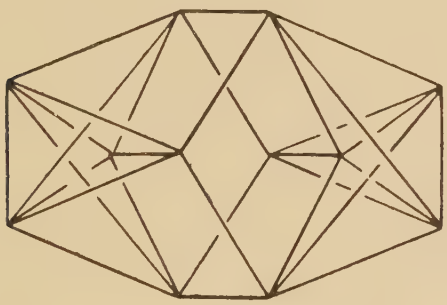


FIG. 6.

In Fig. 7, two six-jointed frameworks, each having two trio-vertices, two quartet-vertices, and two quintet-vertices, are shown joined by six bars between the trio and quartet-vertices.

In Fig. 8, two six-jointed frameworks, each having one quintet-vertex, four quartet-vertices, and one trio-vertex, are shown joined by six bars between all but the quintet-vertices.

The six-jointed frameworks shown in Figs. 5 and 6 could not be joined in ways essentially different from those shown. In this sense, the icosahedra of Figs. 5 and 6 are unique arrangements.

There are, however, alternative methods of joining the six-jointed frameworks of Figs. 7 and 8. These alternative methods are shown schematically in Figs. 7 and 8.

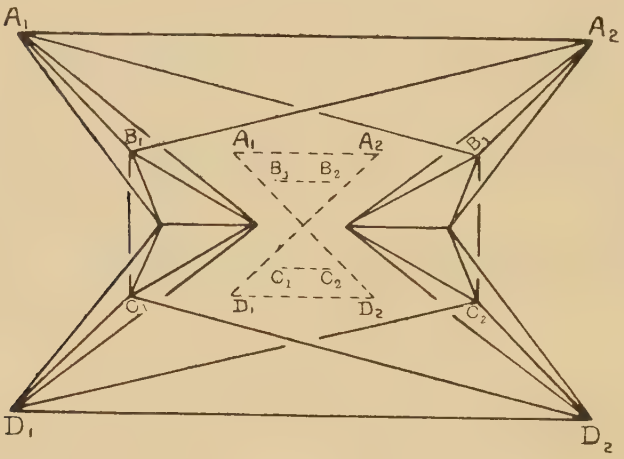


FIG. 7.

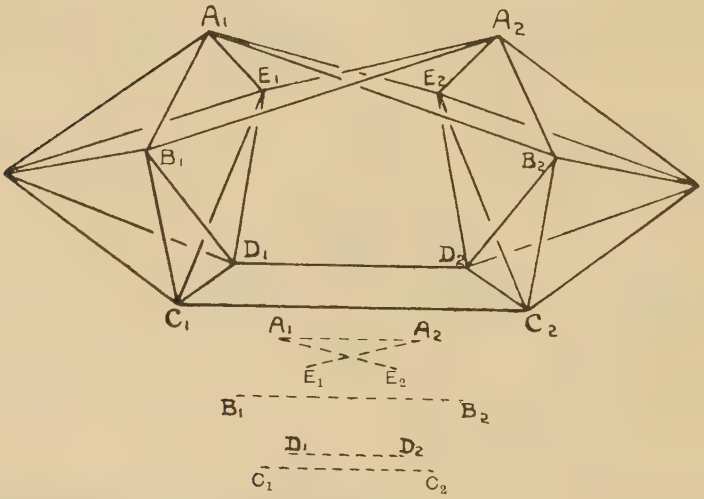


FIG. 8.

4.4. It is of interest to notice that, of the five regular polyhedra, the cube and the dodecahedron cannot form simply-stiff frameworks; nor can any of the Archimedean regular polyhedra.

4.5. There are only two "symmetrical" simply-stiff plane frameworks. One is the triangle and the other is the trio-verticed six-jointed framework

of Fig. 9, which can easily be developed from the basic triangle by the plane version of the third theorem. An "irregular" six-jointed framework with trio-vertices is shown in Fig. 10. This can be produced by using the plane version of the second or of the third theorem.

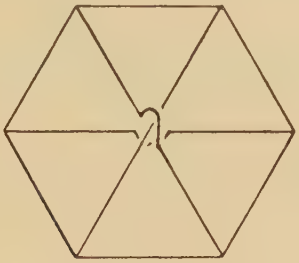


FIG. 9.

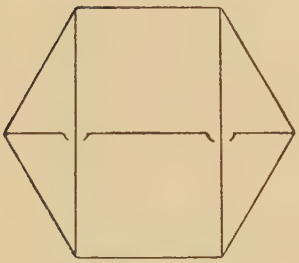


FIG. 10.

5. *Example.*

Let us suppose that to a framework having j joints and $3j-6$ members the reduction process of § 3 . 2 has been applied to the utmost, and that the residual framework is that represented by the diagram of Fig. 11.

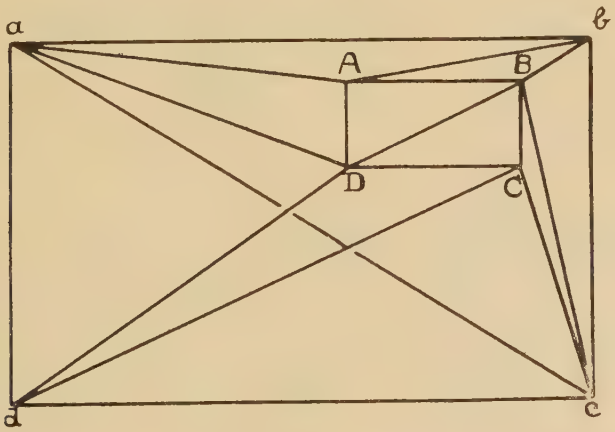


FIG. 11.

We wish to discover if this framework can be erected by application of any of the theorems of § 2. Referring to Fig. 12 (*a*) we first draw the simply-

stiff framework *abABD*. Replacing *Db* by the quartet *c*.*abDB* and building the trio *C*.*DBc*, we get Fig. 12 (*b*). Replacing *Dc* by the quartet *d*.*aDCc*, the framework of Fig. 11 is obtained. It could, of course, have been produced by other series of steps.

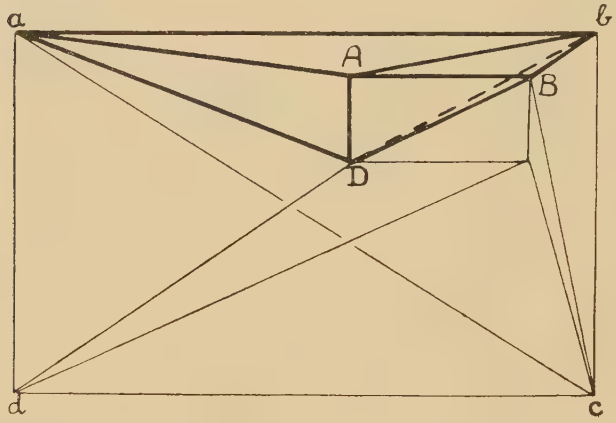


FIG. 12(a).

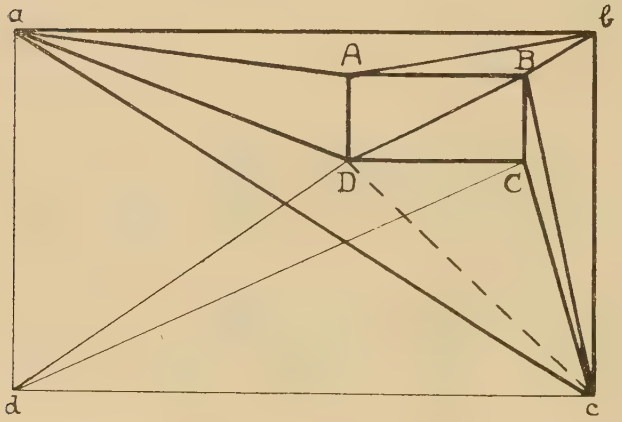


FIG. 12(b).

Royal Aircraft Establishment,
South Farnborough, Hants.

ADJOINT PRIMALS IN FOUR DIMENSIONS

By L. ROTH.

[Received 29 September, 1934.—Read 15 November, 1934.]

The object of the present paper is to provide the enumerative groundwork necessary for the discussion of the invariants and covariants of threefolds, in analogy with results which have long been known for surfaces. The preliminary problem in such a discussion is naturally the following: given a primal V_3^n of S_4, having ordinary singularities of the most general type, to obtain the virtual characters of the surface $F^{(l)}$ residual to the double surface of V_3^n, with respect to an adjoint primal of any order l. In §§ 3–4 we solve this problem and thence determine the curvilinear genus and grade of the system $|F^{(l)}|$; in particular, for $l = n-5$ we derive the virtual characters of the canonical system of V_3^n, and thus establish by direct calculation the classical relations (1), (44), and (45) connecting the characters of the system. From these formulae we obtain expressions for the invariants as linear functions of the additive characters of V_3^n, analogous to the well-known formulae for the invariants of surfaces. We then give formulae for the characters of $|F^{(l)}|$, and for the i-canonical system, in terms of the additive characters of V_3^n. Finally we consider the values assumed by the invariants when V_3^n is generated by ∞^1 planes or ∞^2 lines.

The relation

$$2P_a = \Omega_0 - \Omega_1 + \Omega_2 + 4 \qquad (1)$$

between the invariants of a variety V_3 without singularities was established by Severi* in an indirect manner involving a good deal of labour: as Severi has remarked, the relation, which is essentially an enumerative one, could no doubt be obtained more rapidly by enumerative methods—as, in fact, is accomplished in § 7. B. Segre† has calculated the invariants of a V_3^n of S_4, in the very special case where its double surface is the complete inter-

* Severi, 1.

† B. Segre, 1.

section of two non-singular primals; and Todd* has recently obtained a formula for P_a in the general case, from equations given by the author†. With these exceptions it is believed that the results obtained in the sequel are now given for the first time‡.

1. *Characters of* V_3. The projective characters of a variety V_3 without singularities, which we may suppose situated in S_5, are those characters which specify the nature of the aggregate of its tangent spaces. Three of these characters are the order, rank, and class of a prime section, and a fourth is the class of V_3, that is, the number of tangent primes in an arbitrary pencil. Further, the ∞^1 tangent spaces passing through an arbitrary point have a curve of contact whose order, rank, and class of immersion in V_3 furnish three more characters; and it has been shown§ that all the other enumerative characters of V_3 can be expressed in terms of these seven. If V_3 is projected on to S_4 from an arbitrary point, the characters of the projection V and its multiple loci are defined as follows.

The primal V is of order n, rank a, first class m, and class w. It contains a double surface B of order μ_0, rank μ_1, class μ_2, and *ceto* ν_2. (It will be observed that B can have no improper nodes.) V contains a curve C_t, triple on B and on V, of order t, rank r_t, and classes t_1, t_0 of immersion in V and B respectively; on C_t there is a number τ of points at which two of the three tangent primes to V coincide, and these points are actual pinch-points of B; there is also a number ξ of quadruple points on V, which are quadruple on C_t and sextuple on B. Finally, there is a curve C_j of pinch-points, which is simple on B and meets C_t in the τ stationary points; C_j has order j, rank r_j, and classes j_1, j_0 of immersion in V and B respectively.

A primal having multiple loci of these types is said to possess "ordinary singularities". Such are the primals considered in the present work, though they are not, of course, necessarily the projections of varieties in S_5.

2. *The adjoint primals.* Suppose that V^l is a primal of order l, adjoint to V; that is, V^l contains B simply, has C_t for double curve, and has triple points at the ξ quadruple points of V. Suppose that its residual intersection with V is a surface $F^{(l)}$ of order α_0, rank α_1, class α_2, and *ceto* β_2, which meets B in a curve C of order ϵ_0 and rank ϵ_1, say, having classes z, z' of immersion

* Todd, 1.
† Roth, 1.
‡ A summary of the results of this paper has been given in Roth, 8.
§ Roth, 1.

in B and $F^{(l)}$ respectively. Evidently V and V^l touch along C, and their common tangent primes, which envelop a developable of class Z, suppose, contain the pairs of tangent planes to B and $F^{(l)}$ at points of C. It is clear that V^l can have isolated nodes only at the improper nodes of $F^{(l)}$; for if V^l has a node at a simple point P of B, its osculating cone contains the tangent plane to B at P, and meets each of the tangent primes to V in a second plane which must be tangent to $F^{(l)}$; and, further, such a point P must be a node of C. It is clear also that the ι points in which C meets C_t are double on C and simple on $F^{(l)}$. Thus the singularities of V^l consist of a double curve C_t on which there are ξ triple points, and d isolated nodes, where

$$\beta_2+2d=\alpha_0(\alpha_0-1)-\alpha_1. \tag{2}$$

3. *The equations of intersection.* First, to obtain α_1 and ϵ_0 we apply a method due to Severi* which for convenience we reproduce here. An arbitrary prime section of V is a surface v of order n, containing a double curve b of order μ_0 and rank μ_1; a surface v^l adjoint to v meets it in a residual curve f of order

$$\alpha_0=ln-2\mu_0, \tag{3}$$

and f meets b in ϵ_0 points at which v and v^l touch.

Consider the correspondence set up on an arbitrary line g in the section by the tangent planes to v and v^l having the same points of contact on b. Since the classes of immersion of b in v and v^l are

$$\mu_0(n-2)-3t \quad \text{and} \quad \mu_0(l-1)-3t$$

respectively, the indices of the correspondence are $\mu_0(n-2)-3t$ and $2\{\mu_0(l-1)-3t\}$. The united points arise from the ϵ_0 intersections of b and f, and the μ_1 points of contact of tangents to b which meet g, these last counted twice. Hence

$$\epsilon_0+2\mu_1=\mu_0(n-2)-3t+2\{\mu_0(l-1)-3t\}. \tag{4}$$

To obtain α_1 we use the principle† that adjoint surfaces of order $l+n-4$ cut out canonical sets on f, residual to the ϵ_0 intersections with b. Thus

$$\alpha_1+2\epsilon_0=(l+n-2)\,\alpha_0. \tag{5}$$

3.1. Next, the correspondence set up, on an arbitrary plane, by the tangent planes to B and $F^{(l)}$ at points of C gives

$$z+z'=Z+\epsilon_1. \tag{6}$$

* Severi, 2, § 19.

† Severi, 3.

3.2. Consider the intersection of C with the first polar of a point with respect to V^l. Since C has nodes at the d nodes of V^l and at the ι intersections of C and C_t, we have

$$\epsilon_0(l-1) = Z+2d+2\iota. \tag{7}$$

3.3. To find ι we consider the correspondence set up on a line g by the tangent primes to V and V^l at points of C_t; the indices of the correspondence are $(2t_1, 3t_1')$, where t_1' is the class of immersion of C_t in V^l, given by

$$t_1' = t(l-2)-4\xi.$$

The coincidences arise from the t_0 tangent planes to B which meet g, and the ι intersections of C with C_t; so that

$$2t_1+3\{t(l-2)-4\xi\} = 2t_0+\iota. \tag{8}$$

3.4. The intersections of $F^{(l)}$ with the first polars of a point P with respect to V and V^l consist of the Z points of contact (counted twice) of common tangent primes to V and V^l which pass through P; the β_2 points of contact of tangent planes to $F^{(l)}$ which pass through P; and the $d+\iota$ nodes of C, each counted twice. We thus obtain the equation

$$\alpha_0(l-1)(n-1) = 2Z+\beta_2+2d+2\iota. \tag{9}$$

3.5. Consider the curve of contact Γ' of tangent planes to $F^{(l)}$ which meet a given line g; the tangent primes to V and V^l at points of Γ' set up, on a second line g', a correspondence with indices $\alpha_1(l-1)$, $\alpha_1(n-1)$, since Γ' is of order α_1. The united points arise from the tangent planes which meet g and g', in number $\alpha_2+\beta_2$, and the z' tangent planes meeting g which have their contacts on C, counted twice. Thus

$$2z'+\alpha_2+\beta_2 = \alpha_1(l+n-2). \tag{10}$$

3.6. Similarly, consider the curve of contact Γ of tangent planes to B which meet g; this is of order μ_1, and passes through the t_0 points of C_t the tangent planes at which meet g, and through the τ stationary points. Hence the tangent primes to V and V^l at points of Γ set up on g' a correspondence with indices

$$\mu_1(n-2)-t_0-\tau, \quad 2\{\mu_1(l-1)-t_0-\tau\}.$$

The united points are due to the $\mu_2+\nu_2$ tangent planes which meet g and g', and the τ improper tangent planes, both counted twice, and the z tangent planes meeting g and touching B in points of C. Thus

$$z+2(\mu_2+\nu_2+\tau) = \mu_1(n+2l-4)-3t_0-3\tau. \tag{11}$$

3.7. Equations (5) and (10) admit of an interesting verification. For the adjoint primals of order $l+n-5$ cut on $F^{(l)}$ the improper canonical system*, residual to the curve $2C$. Hence, if $K^{(l)}$ denotes a virtual canonical curve of $F^{(l)}$, and $C^{(l)}$ a prime section, we have, symbolically,

$$|K^{(l)}+2C| = |(l+n-5)C^{(l)}|.$$

Now the canonical numbers† of $K^{(l)}$, C, and $C^{(l)}$ are $\omega'-1$, $z'-3\epsilon_0$, and $\alpha_1-3\alpha_0$ respectively, where ω' is the arithmetical linear genus of $F^{(l)}$. Therefore

$$\omega'-1+2(z'-3\epsilon_0) = (l+n-5)(\alpha_1-3\alpha_0). \tag{12}$$

Writing for ω' its value $\alpha_2+\beta_2-6\alpha_1+9\alpha_0+1$ and eliminating z' between (10) and (12), we obtain (5).

3.8. The number λ' of lines passing through a point P and meeting C and $F^{(l)}$ again is given by

$$\lambda'+z'+2d = \epsilon_0(\alpha_0-1). \tag{13}$$

The corresponding character λ of B is given by

$$\lambda+z+4\iota = \epsilon_0(\mu_0-1). \tag{14}$$

The cone projecting C from P meets V^l in a residual curve C' of order $\epsilon_0(l-1)$, and the intersections of C' and V, other than those lying on C, give rise to lines through P which meet C and B or $F^{(l)}$ again. Now C' meets C in the d nodes of V^l, which are double on C', and in the ι intersections of C with C_ι, which are triple on C'; and also in the Z points of contact of the tangent primes to V^l, in points of C, which pass through P; each of these points counts three times since C' has three-point contact there with V. We thus obtain the equation

$$2\lambda+\lambda'+4d+6\iota+3Z = \epsilon_0(l-1)n. \tag{15}$$

By means of (7) we have

$$2\lambda+\lambda'+2Z+2d+4\iota = \epsilon_0(l-1)(n-1). \tag{16}$$

Hence, from (6), (13) and (14), we derive

$$\epsilon_1 = \epsilon_0(l+n-4)-4\iota+Z-z. \tag{17}$$

* Severi, 1.

† Or "immersion characters", as they are more often called.

4. From (4) and (5) we obtain

$$\alpha_1 = ln(l+n-2)-2\mu_0(3l+2n-6)+18t+4\mu_1, \tag{18}$$

$$\epsilon_0 = \mu_0(2l+n-4)-9t-2\mu_1. \tag{19}$$

To solve the remaining equations we first determine the value of ι. We have*

$$2t_1 = 2r_t+\tau \quad \text{and} \quad 2t_0 = 4r_t+\tau,$$

where $$\tfrac{1}{2}\tau = t(n-3)-4\xi-r_t.$$

Hence, from (8), $$\iota = 3\{t(l-2)-4\xi\}-2r_t. \tag{20}$$

Subtracting (2) from (9) we have, by using (5),

$$Z = \alpha_0\mu_0-\epsilon_0-\iota. \tag{21}$$

Thus

$$Z = \mu_0(ln-2\mu_0)-\mu_0(2l+n-4)+2\mu_1-3\{t(l-5)-4\xi\}+2r_t. \tag{22}$$

Then, from (7), we obtain

$$d = \mu_0\{l(l-2)+\mu_0\}-\mu_1 l-3t(2l-1)+6\xi+r_t. \tag{23}$$

This formula gives *the number of isolated nodes which an adjoint primal V^l must have at simple points of B.* If there is no triple curve C_t, it reduces to the ordinary formula† for the case of a general surface without improper nodes.

Substituting in (11) for t_0 and τ, we have

$$z = \mu_1(2l+n-4)-2(\mu_2+\nu_2)-13t(n-3)+52\xi+7r_t. \tag{24}$$

From (6) and (17),

$$z' = \mu_0\{(l+n-4)(2l+3n-8)-2(n-1)(n-4)\}-4\mu_0{}^2+4(\mu_2+\nu_2)$$
$$-2\mu_1(3l+2n-10)-t(27l-17n-12)-32\xi-2r_t. \tag{25}$$

Using (21), (24), and (25) we obtain, from (6),

$$\epsilon_1 = \mu_0\{2(l+n-3)(l+n-4)-(n-1)(n-4)\}-2\mu_0{}^2+2(\mu_2+\nu_2)$$
$$-\mu_1(4l+3n-14)-4t(6l-n-9)+3r_t+8\xi. \tag{26}$$

Substituting for d and α_1 in (2), we have

$$\beta_2 = ln(l-1)(n-1)+2\mu_0{}^2-2\mu_0\{(l+n-2)(l+n-3)-(n^2-3n+1)\}$$
$$+2\mu_1(l-2)+12t(l-2)-12\xi-2r_t. \tag{27}$$

---

* Roth, 1 (36)–(38).

† Severi, 2, § 7.

Hence, eliminating z' between (10) and (25), we have finally

$$a_2 = ln\{(l+n-2)^2-(l-1)(n-1)\}-2\mu_0\{4(l+n-3)^2+l-(n-1)(n-5)\}$$
$$+6\mu_0{}^2-8(\mu_2+\nu_2)+2\mu_1(7l+6n-22)+4t(15l-4n-9)+76\xi+6r_t. \quad (28)$$

Thus, in conclusion:

If V is a primal of S_4 having for singularities a double surface B containing a triple curve C_t on which there are ξ quadruple points, the residual intersection with V of an adjoint primal V^l is a surface $F^{(l)}$ of general type, whose characters are given by equations (3), (18), (27), and (28). The surfaces B and $F^{(l)}$ intersect in a curve C, along which V and V^l touch, whose order and rank are given by (19) and (26). The isolated nodes of V^l are at the improper nodes of $F^{(l)}$, which are nodes of C, as also are the intersections of C and C_t.

5. *Examples.* As a verification of the equations given in § 3 we propose to consider some cases in which the nature of the residual surface $F^{(l)}$ is known *a priori*. Suppose, in the first place, that V is generated by ∞^1 planes, forming a primal of rank a. The characters of V are all known functions of n and a, and have been obtained in a previous paper*; for present purposes we recall that each generating plane meets the double surface B in a curve c_{n-2} of order $n-2$, and that each curve c_{n-2} meets the triple curve C_t in

$$\tfrac{1}{2}(n-2)(n-3)-\tfrac{1}{2}a+2$$

points, which are nodes of c_{n-2}; also, the class of immersion of c_{n-2} in B is† $a+n-7$.

Consider now the intersection of V with an adjoint primal V^{n-2}; evidently the residual surface $F^{(n-2)}$ must consist of generating planes, in number

$$n(n-2)-2\mu_0 = a-n.$$

Hence $\alpha_0 = a-n$, $\alpha_1 = \alpha_2 = \beta_2 = 0$, and $d = \binom{a-n}{2}$. Further, $z' = 0$ and $z = (a-n)(a+n-7)$, since the curve C consists of $a-n$ curves c_{n-2}. Thus

---

* Roth, 3. Thus, $\mu_1 = an-6a+4n$, $\mu_2 = n^2+\binom{a}{2}-12a+11n$, $\nu_2 = \frac{1}{2}a^2-8(a-n)$, $\tau = 2(a-n)n-12a+14n$, $t = \binom{n}{3}-\frac{1}{2}an+2a-n$, $r_t = a\binom{n}{2}-6an+4n^2-\frac{1}{2}a^2+29a-28n$, $\xi = \binom{n}{4}-\frac{1}{2}a\binom{n}{2}+\frac{1}{8}a^2+2an-n^2-\frac{29}{4}a+6n$.

† Roth, 2.

the intersections of C and C_t are in number $\iota = \frac{1}{2}(a-n)\{(n-2)(n-3)-a+4\}$, while $\epsilon_0 = (a-n)(n-2)$ and $\epsilon_1 = (a-n)(a-4)$. If the known characters of B are substituted in the equations (4)–(17), those stated above for $F^{(n-2)}$ and C will be obtained.

5.1. If V is ruled, *i.e.* generated by ∞^2 lines, its generators are $(n-3)$-secant to the double surface*. Hence, the adjoint V^{n-3} must meet V in a *ruled surface* $F^{(n-3)}$. Consider, for example, the Castelnuovo primal V (whose plane sections are of genus 2); the residual intersection is of order $\alpha_0 = n(n-3)-(n^2-3n-2) = 2$, *i.e.* $F^{(n-3)}$ is a quadric. In fact, the characters of B are†

$$\mu_0 = \tfrac{1}{2}(n^2-3n-2), \quad \mu_1 = 2(n-1)(n-4), \quad \mu_2 = 2(n-4)(n+1),$$

$$\nu_2 = 2(n-3)(n-4), \quad \tau = 4(n-4)(n-5), \quad t = \tfrac{1}{6}n(n-4)(n-5),$$

$$r_t = (n-4)(n-5)^2, \quad \xi = \binom{n-3}{4} \qquad (4 \leqslant n \leqslant 8),$$

whence it follows at once that $\alpha_1 = 2$, $\epsilon_0 = 2(n-3)$. Evidently C meets all the generators of the quadric in $n-3$ points; hence, by calculation of $\iota = (n-4)(n-5)$, we obtain $\epsilon_1 = 2(n-3)^2-2(n-4)(n-5) = 2(3n-11)$. And $z' = 2(n-3)$. With these numbers we verify the preceding equations, which give $\alpha_2 = 2$, $\beta_2 = 0$, $d = 0$, and $z = 8(n-4)$, $Z = 4(n-4)$.

5.2. If V is a Bordiga primal† (with plane sections of genus 3) its characters are

$$\mu_0 = \tfrac{1}{2}(n^2-3n-4), \quad \mu_1 = 2(n^2-4n-3), \quad \mu_2 = 2n(n+1)-57,$$

$$\nu_2 = 2(n^2-6n+6), \quad \tau = (n-5)(4n-18), \quad t = \tfrac{1}{6}(n-5)(n^2-4n-6),$$

$$r_t = (n-2)(n-5)(n-6), \quad \xi = \frac{n}{4}\binom{n-5}{3} \qquad (5 \leqslant n \leqslant 11).$$

We obtain at once $\alpha_0 = 4$, $\alpha_1 = 6$, so that $F^{(n-3)}$ is a rational quartic scroll, having $\alpha_2 = 4$, $\beta_2 = 4$, $d = 1$. Also, from (19), $\epsilon_0 = 4n-13$, and, from (20), $\iota = (n-5)(2n-9)$. The generators of the scroll meet C in $n-3$ points, and C has a node at the improper node of $F^{(n-3)}$: by the well-known formulae it follows that $\epsilon_1 = 4(n-3)^2-2(n-5)(2n-9) = 2(7n-27)$, $z' = 6n-19$. Hence $z = 17n-75$ and $Z = 9n-40$.

---

* Roth, **4**.

† Roth, **4, 5**.

5.3. Suppose that V is a Del Pezzo primal; its double surface is therefore ruled*, having characters

$$\mu_0 = \tfrac{1}{2}n(n-3), \quad \mu_1 = 2(n-3)^2, \quad \mu_2 = \tfrac{1}{2}n(n-3), \quad \nu_2 = (n-3)(n-4),$$

$$\tau = 2(n-3)(n-4), \quad t = \binom{n-2}{3}, \quad r_t = (n-3)(n-4)(n-5), \quad \xi = \binom{n-3}{4}$$

$$(3 \leqslant n \leqslant 8).$$

V has no residual intersection with an adjoint V^{n-3}; but, taking $l = n-2$, we obtain $\alpha_0 = n$, $\alpha_1 = 2n$. Hence $F^{(n-2)}$ is a Del Pezzo surface (for it cannot be an elliptic scroll) and therefore has the characters $\alpha_2 = 12$, $\beta_2 = 4(n-3)$, $d = \binom{n-3}{2}$. We find that $\iota = \frac{1}{2}(n-3)(n-2)(n-4)$, and verify the equations, which give

$$z' = 2n(n-3), \quad z = 4(n-3)^2, \quad Z = 2(n-2)(n-3), \quad \epsilon_1 = 4(n-2)(n-3).$$

6. *Characters of the system* $|F^{(l)}|$. Another adjoint U^l of the same order as V^l meets $F^{(l)}$ in C and in a residual curve Γ of order $\eta_0 = l\alpha_0 - \epsilon_0$. Suppose that Γ has rank η_1 and that it meets C in δ points. From the correspondence between tangent primes to U^l and tangent planes to $F^{(l)}$ which have the same point of contact on C, we obtain the relation

$$\epsilon_1 + \delta = z' + \zeta, \tag{29}$$

where $\zeta = \epsilon_0(l-1) - 2\iota$ is the class of immersion of C in U^l. Hence

$$\delta = z' - \epsilon_1 + \epsilon_0(l-1) - 2\iota. \tag{30}$$

The rank η_1 of Γ is given by the equation

$$l\alpha_1 + l(l-1)\alpha_0 = \epsilon_1 + \eta_1 + 2(\delta + \iota),$$

whence

$$\eta_1 = l\alpha_1 + l(l-1)\alpha_0 + \epsilon_1 - 2z' - 2\epsilon_0(l-1) + 2\iota. \tag{31}$$

From this follows the genus of the curve Γ of intersection of any two surfaces $F^{(l)}$. Finally, the grade $n^{(l)}$ of the system $|F^{(l)}|$ is given by

$$n^{(l)} = \eta_0 l - \delta = (l\alpha_0 - \epsilon_0)l - \delta. \tag{32}$$

* Roth, 4, 5.

Substituting for the characters of $F^{(l)}$ their values obtained in § 4, we have

$$\eta_1 = l^2 n(2l+n-3)+6(\mu_0{}^2-\mu_2-\nu_2)-\mu_0(14l^2+12ln+n^2-44l-13n+36)$$
$$+\mu_1(16l+5n-30)+6t(12l-5n-3)+48\xi+3r_l, \quad (33)$$

$$n^{(l)} = l^3 n+2(\mu_0{}^2-\mu_2-\nu_2)-\mu_0(6l^2+3ln-12l-2n+8)$$
$$+\mu_1(6l+n-8)+t(27l-13n+3)+16\xi+r_l. \quad (34)$$

6.1. For example, in the case of the Bordiga primal, the surfaces $F^{(n-3)}$, which are rational quartic scrolls, meet in pairs in curves of order $\eta_0 = 4(n-3)-(4n-13) = 1$, *i.e.* in generators; hence $\eta_1 = n^{(n-3)} = 0$, as is verified by substituting in the previous equations.

If V is plane-ruled, $F^{(n-2)}$ consists of $a-n$ planes, where a is the rank of V (§ 5), so that Γ is of order zero, and thus $\eta_1 = n^{(n-2)} = 0$. Thus, if the characters of V are substituted in (33) and (34), the equations are found to vanish identically.

7. *The canonical system.* The virtual canonical system $|K|$ is cut out by adjoint primals of order $n-5$. From the equations given in § 4 we obtain, for the characters of a canonical surface K, the expressions

$$\alpha_0 = n(n-5)-2\mu_0. \quad (35)$$

$$\alpha_1 = n(n-5)(2n-7)-2\mu_0(5n-21)+18t+4\mu_1. \quad (36)$$

$$\beta_2 = n(n-1)(n-5)(n-6)-2\mu_0\{2(2n-7)(n-4)-(n-1)(n-2)\}$$
$$+2\mu_0(\mu_0-1)+2\mu_1(n-7)+12t(n-7)-12\xi-2r_l. \quad (37)$$

$$\alpha_2 = n(n-5)\{(2n-7)^2-(n-1)(n-6)\}-2\mu_0\{16(n-4)^2-(n-2)(n-5)\}$$
$$+6\mu_0{}^2-8(\mu_2+\nu_2)+2\mu_1(13n-57)+4t(11n-84)+76\xi+6r_l. \quad (38)$$

Also

$$\varepsilon_0 = \mu_0(3n-14)-9t-2\mu_1. \quad (39)$$

The arithmetic genus, curvilinear genus, and grade of $|K|$ are denoted by Ω_2, Ω_1, and Ω_0 respectively. Thus

$$12(\Omega_2+1) = 12\alpha_0-8\alpha_1+2\alpha_2+\beta_2$$
$$= n(n-5)(7n^2-65n+160)-2\mu_0(33n^2-309n+727)+14\mu_0{}^2$$
$$-16(\mu_2+\nu_2)+2\mu_1(27n-137)+100t(n-9)+140\xi+10r_l, \quad (40)$$

and from (34),

$$\Omega_0 = n(n-5)^3 - \mu_0(9n^2-89n+218) + 2(\mu_0{}^2 - \mu_2 - \nu_2) + \mu_1(7n-38) + 2t(7n-66) + 16\xi + r_t. \quad (41)$$

From (33) we have

$$2(\Omega_1 - 1) = \tau_1 - 2\tau_0,$$

where

$$\tau_1 = n(n-5)^2(3n-13) - \mu_0\{14(n-5)^2 + 12n(n-5) + n^2 - 44(n-5) - 13n + 36\} + 6(\mu_0{}^2 - \mu_2 - \nu_2) + \mu_1(21n-110) + 6t(7n-63) + 48\xi + 3r_t, \quad (42)$$

and
$$\tau_0 = (n-5)\alpha_0 - \varepsilon_0$$
$$= n(n-5)^2 - \mu_0(5n-24) + 9t + 2\mu_1 \quad \text{from (35) and (39).} \quad (43)$$

We thus establish the classical relation* between the invariants,

$$2\Omega_1 - 2 = 3\Omega_0. \quad (44)$$

The arithmetic linear genus of K is defined by the expression

$$\omega = \alpha_2 + \beta_2 - 6\alpha_1 + 9\alpha_0 + 1.$$

On substitution we at once obtain the result*

$$\omega = 4\Omega_0 + 1. \quad (45)$$

7.1. More convenient forms of the above results may be found by expressing them in terms of the additive characters of V, namely, n, a, j, m, w, j_1, and j_0. We have†

$$\mu_0 = \binom{n}{2} - \tfrac{1}{2}a, \quad \mu_1 = a(n-1) - \tfrac{5}{2}j - m,$$

$$\mu_2 = \tfrac{1}{2}a^2 + mn + 2j - \tfrac{1}{2}(3m + 3w + 33j_1 - 11j_0),$$

$$\nu_2 = \tfrac{1}{2}(a^2 - m - w - 13j_1) + 2j_0,$$

$$t = \binom{n}{3} - \tfrac{1}{2}an + \tfrac{1}{3}(2a + 2j + m),$$

$$r_t = a\binom{n}{2} - n(a + \tfrac{5}{2}j + m) - \tfrac{1}{2}a^2 + a + 4j + 2m + w + 14j_1 - 4j_0,$$

$$\xi = \binom{n}{4} - \tfrac{1}{2}a\binom{n}{2} + \tfrac{1}{8}a^2 + \tfrac{1}{3}n(2a + 2j + m) - \tfrac{1}{4}(3a + 5j + 3m + w + 13j_1) + j_0.$$

* Pannelli, 1, B. Segre, 1.

† Roth, 1, (1), (9), (10), (12), (14), (38), (43)–(45).

Substituting these expressions in the previous formulae, we obtain

$$\alpha_0 = a-4n. \tag{46}$$

$$\alpha_1 = -13a+2m+20n+2j. \tag{47}$$

$$\alpha_2 = 85a-27m-100n+3w-26j+21j_1-8j_0. \tag{48}$$

$$\beta_2 = 20a-9m+w-14j+11j_1-4j_0. \tag{49}$$

$$\varepsilon_0 = (a-4n)n+3a-j-m+4n. \tag{50}$$

$$\eta_0 = -8a+16n+m+j. \tag{51}$$

For the invariants Ω_0 and Ω_2 we derive the formulae

$$\Omega_0 = 48a-12m-64n+w-13j+8j_1-3j_0, \tag{52}$$

$$12(\Omega_2+1) = 306a-79m-408n+7w-82j+53j_1-20j_0. \tag{53}$$

Now Todd* has shown that the arithmetic genus of V is given by

$$24P_a = 24(1-n)+18a-7m+w-4j-2j_0+5j_1. \tag{54}$$

Hence, from (44), (52), (53), and (54), follows the fundamental relation (1),

$$2P_a = \Omega_0-\Omega_1+\Omega_2+4,$$

established by Noether† and Severi‡.

8. *Applications.* We consider first some simple examples in which the invariants may be found from first principles, thereby verifying the above formulae.

(i) If V is a sextic having a double plane, K is evidently a general quartic surface, and the geometrical genus $P_g = 2$. Using the formulae in § 7, we write $\mu_0 = 1$, $\mu_1 = \mu_2 = \nu_2 = 0$, $t = \xi = r_l = 0$, and obtain $\alpha_0 = 4$, $\alpha_1 = 12$, $\alpha_2 = 36$, $\beta_2 = 0$; and $\Omega_2 = 1$, $\Omega_1 = 1$, $\Omega_0 = 0$, whence $P_a = 2$. Alternatively, we find from § 7.1 that the additive characters of V are $a = 28$, $m = 120$, $w = 456$, $j = 8$, $j_1 = 16$, $j_0 = 0$, and thence use equations (46)–(53).

(ii) Suppose that V is a septimic with a double plane; then the adjoint primals of order $n-5$ are quadric cones $\phi^2(P_g = 9)$ which meet in pairs in

* Todd, 1.
† Noether, 1.
‡ Severi, 1.

cubic scrolls R^3, residual to the double plane ϖ. From the plane representation of R^3 we find at once that V meets R^3 in a conic lying in ϖ, and a residual curve represented on the plane by a curve (12, 7) of order 12 having a 7-ple point. Hence $\Omega_1 = 55-21 = 34$. Now ϕ^2 may be represented on a space S_3 by means of quadrics having a base-line and two base-points*, or symbolically by (2; 1, 12). Hence K corresponds to the symbol

$$7(2;\ 1,\ 12)-2(1;\ 0,\ 1^2) = (12;\ 7,\ 5^2),$$

i.e. is represented by a surface of order 12 having a 7-ple line and two 5-ple points; and we may find the invariants ω and Ω_2 of this surface by formulae due to Noether†. We thus obtain $\omega = 89$, $\Omega_2 = 26$, and hence $P_a = 9$.

Alternatively we may use the equations given in § 7 . 1, writing

$$a = 40,\quad m = 215,\quad w = 1060,\quad j = 10,\quad j_1 = 25,\quad j_0 = 0.$$

(iii) If V is a sextic with a double quadric, then K is a quadric and $P_g = 1$. We have $a = 26$, $m = 98$, $w = 314$, $j = 12$, $j_1 = 24$, $j_0 = 12$, giving for the invariants the values $\Omega_0 = 2$, $\Omega_1 = 4$, $\Omega_2 = 0$, $P_a = 1$. From (51) we observe that the virtual characteristic curve of the system $|K|$ is a sextic.

(iv) If V is a septimic having a double quadric, $P_g = 6$, and the surfaces K are decimics whose characters, by § 7, are

$$\alpha_0 = 10,\quad \alpha_1 = 50,\quad \alpha_2 = 210,\quad \beta_2 = 40,\quad \gamma_0 = 10,$$

so that $\omega = 41$ and $\Omega_2 = 14$. Hence $\Omega_0 = 10$, $\Omega_1 = 16$, $P_a = 6$. The same results are obtained from § 7 . 1 by substituting

$$a = 38,\quad m = 186,\quad w = 802,\quad j = 16,\quad j_1 = 40,\quad j_0 = 16,$$

and they may be verified as follows. The surfaces K are cut out by adjoint quadrics ϕ^2, which may be represented on S_3 by quadrics (2, 1) having a base-line; in this representation K corresponds to a surface (10, 5) whose invariants, by Noether's formulae‡, are $\omega = 41$, $\Omega_2 = 14$.

(v) Let V be a septimic with a double cubic scroll R^3, so that $P_g = 3$. We have, for the characters of V,

$$a = 36,\quad m = 157,\quad w = 540,\quad j = 22,\quad j_1 = 55,\quad j_0 = 30,$$

---

* Roth, **7**, 273.
† Noether, **2**.
‡ Noether, **2**.

leading to

$$\alpha_0 = 8, \quad \alpha_1 = 30, \quad \alpha_2 = 84, \quad \beta_2 = 24, \quad \gamma_0 = 3.$$

Hence $\omega = 1, \quad \Omega_2 = 3, \quad$ and $\quad \Omega_1 = 1, \quad \Omega_0 = 0, \quad P_a = 3.$

If an adjoint quadric cone, which cuts out K, is represented on S_3 by the quadrics $(2;\ 1,\ 1^2)$, R^3 corresponds to the surface $(3;\ 2,\ 0^2)$, and K to the surface $(8;\ 3,\ 5^2)$, whence $\omega = 1,\ \Omega_2 = 3$, as above.

(vi) Finally, let V be a septimic having a double Segre quartic surface. The characters of V are

$$a = 34, \quad m = 136, \quad w = 424, \quad j = 24, \quad j_1 = 60, \quad j_0 = 48, \quad P_g = 2.$$

Thus $\alpha_0 = 6, \quad \alpha_1 = 18, \quad \alpha_2 = 42, \quad \beta_2 = 12, \quad \gamma_0 = 0,$

so that

$$\omega = 1, \quad \Omega_2 = 1, \quad \text{and} \quad \Omega_0 = 0, \quad \Omega_1 = 1, \quad P_a = 2.$$

These results are verified by representing an adjoint quadric on S_3 by surfaces (2, 1), in which case the double surface corresponds to a surface (4, 2) and K to a surface (6, 3), having invariants $\omega = 1,\ \Omega_2 = 1$.

9. *Plane-forms and ruled primals.* Suppose that V is plane-ruled and of genus π; the characters of V, expressed in terms of n and a, have been given in § 5. From these, by substituting in the equations of § 7, we obtain for the invariants the values

$$P_a = \pi, \quad \Omega_0 = 54(\pi - 1), \quad \Omega_1 = 1 + 81(\pi - 1), \quad \Omega_2 = 29(\pi - 1) - 1.$$

These may be compared with the values

$$p_a = -\pi, \quad \omega = 1 - 8(\pi - 1),$$

assumed by the arithmetic and linear genera of a prime section.

9.1. Suppose instead that V is simply ruled, *i.e.* that it is generated by ∞^2 lines, one of which passes through an arbitrary point of V. In this case V is completely defined by the five characters n, a, m, w, j, the remaining pair being given by the formulae*

$$j_1 = -18a + 3m + 16n + w + 8j,$$

$$j_0 = -44a + 6m + 40n + 3w + 19j.$$

* Roth, **4**, 399.

The invariants of V may now be expressed in terms of the characters of a prime section. Writing, as usual,

$$\omega = j+m-6a+9n+1, \quad 12(p_a+1) = 12n-8a+2m+j, \tag{55}$$

we obtain*, from § 7.1,

$$P_a = -p_a, \quad \Omega_0 = -6(\omega-1)-2n, \quad \Omega_1 = 1-9(\omega-1)-3n,$$
$$-\Omega_2 = 3\omega+2p_a+n.$$

10. *Systems adjoint to the prime sections.* The system $|F^{(l)}|$ considered in §§ 3–6 is the h-th adjoint† to the system $|F|$ of prime sections of V, *i.e.*

$$|F^{(l)}| = |hF+K| \quad (l = h+n-5).$$

We may obtain its characters in terms of the additive characters of V by writing $l = h+n-5$ in the formulae of § 4 and § 6 and using the results of § 7; but it is perhaps simpler to use the expressions which have been given‡ for the projective characters of a composite surface. Now the $h+1$ components of the surface $hF+K$ meet in pairs in curves whose total order is

$$\gamma_0 = h\alpha_0+\binom{h}{2}n, \tag{56}$$

and whose total rank is

$$\gamma_1 = h\alpha_1+\binom{h}{2}a. \tag{57}$$

They meet in threes in I points, given by

$$I = \binom{h}{2}\alpha_0+\binom{h}{3}n. \tag{58}$$

The spaces containing pairs of tangent planes to the surfaces along their curves of intersection envelop a developable of class

$$X = \binom{h}{2}a+hx,$$

where x is the class of immersion in V of a prime section of K; thus

$$x = \alpha_0(n-1)-\varepsilon_0 = j+m-4a,$$

by (46) and (50), whence

$$X = \binom{h}{2}a+h(j+m-4a). \tag{59}$$

---

* *Cf.* Severi, **1**.

† This is not to be confused with the h-adjoint system of $|F|$.

‡ Roth, **6**; Semple, **1**.

We then have*

$$\alpha_0 = \alpha_0 + hn,$$
$$\alpha_1 = \alpha_1 + la + 2\gamma_0,$$
$$\beta_2 = \beta_2 + hj + 2X,$$
$$\sigma_2 = \alpha_2 + hm + 3\gamma_1 + 2\gamma_0 - X + 6I.$$

Substituting in these formulae from (56)–(59) and simplifying, we obtain

$$\alpha_0 = a + (h-4)n. \tag{60}$$
$$\alpha_1 = \alpha_1 + 3h(a-3n) + h^2 n. \tag{61}$$
$$\beta_2 = \beta_2 + h(3j + 2m - 9a) + h^2 a. \tag{62}$$
$$\alpha_2 = \alpha_2 + h(5j + 6m + 65n - 37a) + 2h^2(2a - 7n) + h^3 n. \tag{63}$$

Also

$$\eta_0 = \gamma_0 + 2h\alpha_0 + h^2 n, \tag{64}$$
$$\eta_1 = \gamma_1 + 4h(2j + 2m + 28n - 15a) + h^2(7a - 26n) + 2h^3 n, \tag{65}$$
$$n^{(l)} = \Omega_0 + 3h\gamma_0 + 3h^2\alpha_0 + h^3 n,$$

from (31)–(34).

10.1. Consider, for example, the first adjoint system $|F'|$. From the previous equations we obtain for the order, grade, and curvilinear genus of its characteristic system the expressions

$$\eta_0' = \gamma_0 + 2a - 7n = \omega - 1, \tag{67}$$
$$n' = \Omega_0 + n + 3(j + m - 7a + 12n), \tag{68}$$
$$\pi' = \Omega_1 - \tfrac{57}{2}a + 4(j + m) + 51n. \tag{69}$$

The arithmetic genus of F' is

$$p_a' = \Omega_2 + p_a + \tfrac{1}{2}\alpha_1 - \alpha_0 + 1 = \Omega_2 + p_a - \tfrac{15}{2}a + j + m + 14n + 1. \tag{70}$$

From (68)–(70) we have

$$p_a' + n' - \pi' = p_a + \Omega_0 - \Omega_1 + \Omega_2 + 1,$$

or, by (1),

$$p_a' + n' - \pi' = p_a + 2P_a - 3, \tag{71}$$

an equation given by Severi†.

* Roth, 6; Semple, 1.
† Severi, 1.

Eliminating Ω_1 and Ω_0 from (68), (69), and (44), we obtain the relation

$$3n'-2(\pi'-1)=\omega-1-3a. \tag{72}$$

10.2. As remarked in § 5.1, if V is a ruled primal, the second adjoint system $|F''|$ consists of ruled surfaces, while the first adjoint and canonical systems are non-existent. Moreover, the characteristic system of $|F''|$ is composed of generators. These facts are confirmed by equations (60)–(66); for, inserting the values of Ω_0, Ω_1, Ω_2, j_1, and j_0 given in § 9.1, we obtain for $|F''|$ the characters

$$\alpha_0''=a-2n, \quad \alpha_1''=2(\omega-1)+5a-4n,$$

$$\beta_2''=2(\alpha_1''-\alpha_0''), \quad \alpha_2''=\alpha_0''.$$

Hence F'' is a scroll. Also,

$$\eta_0''=j+m+4(n-a), \quad \eta_1''=0, \quad n''=0,$$

whence it follows that two surfaces F'' intersect in $j+m+4(n-a)$ generators.

11. *Characters of the system* $|iF^{(l)}|$. By the method of the preceding section we can determine the characters of the i-th multiple of the system $|F^{(l)}|$, which includes as a particular case the i-canonical system. Using a similar notation, we have, for the composite surface $iF^{(l)}$,

$$\gamma_0=\binom{i}{2}\eta_0, \quad \gamma_1=\binom{i}{2}\eta_1, \quad I=\binom{i}{3}n^{(l)},$$

$$X=\binom{i}{2}\{\eta_0(n-1)-\delta\},$$

where, from (32),

$$\delta=\eta_0 l-n^{(l)}.$$

Hence, as in § 10, the characters of $iF^{(l)}$ are

$$\alpha_0^{(i)}=i\alpha_0, \quad \alpha_1^{(i)}=i\alpha_1+i(i-1)\eta_0, \tag{73}$$

$$\beta_2^{(i)}=i\beta_2+i(i-1)\{\eta_0(n-l-1)+n^{(l)}\}, \tag{74}$$

$$\alpha_2^{(i)}=i\alpha_2+3\binom{i}{2}\eta_1+i(i-1)\eta_0-\binom{i}{2}\{\eta_0(n-l-1)+n^{(l)}\}+6\binom{i}{3}n^{(l)}. \tag{75}$$

The system $|iF^{(l)}|$ has order, rank, and grade given by

$$\eta_0^{(i)}=i^2\eta_0, \quad \eta_1^{(i)}=i^2\eta_1+4n^{(l)}\{i(i-1)+\tfrac{1}{3}i(i-1)(i-2)\}, \quad n^{(il)}=i^3n^{(l)}. \tag{76}$$

References,

Noether, 1, *Math. Annalen*, 8 (1875), 495.
————, 2, *Annali di Mat.* (2), 5 (1871), 164.
Pannelli, 1, *Rend. Lincei* (5), 15 (1906), 483.
Roth, 1, *Proc. London Math. Soc.* (2), 35 (1933), 540.
————, 2, *ibid.* (2), 39 (1935), 334.
————, 3, *ibid.*, 33 (1931), 115.
————, 4, *ibid.*, 35 (1933), 380.
————, 5, *Proc. Camb. Phil. Soc.*, 28 (1932), 300.
————, 6, *ibid.*, 29 (1933), 88.
————, 7, *ibid.*, 30 (1934), 271.
————, 8, *Rend. Lincei* (6), 21 (1935), 314.
B. Segre, 1, *Mem. Acc. Italia*, 5 (1934), 479.
Semple, 1, *Proc. Royal Irish Acad.*, 41 (1933), 70.
Severi, 1, *Rend. di Palermo*, 28 (1909), 33.
————, 2, *Mem. Acc. Torino* (2), 52 (1903), 61.
————, 3, *Rend. di Palermo*, 17 (1903), 73.
Todd, 1, *Journal London Math. Soc.*, 9 (1934), 206.

Imperial College of Science,
South Kensington, London, S.W.7.

ON THE CESÀRO DERIVATES OF A FUNCTION

By W. L. C. SARGENT.

(Received 13 November, 1934.—Read 15 November, 1934.]

1. *Introduction.* This paper is divided into three parts. In the first part we consider the behaviour of a function which is C-continuous at every point of a closed interval, and finally prove the mean value theorem for the case when the C-derivative exists at every interior point of the interval.

The second part contains results concerning the disposition of the C-derivates and their relation to the approximate derivative.

In the third part functions are constructed which illustrate the theorems already proved and show that the disposition of the C-derivates may be quite different from that of either the ordinary derivates or the approximate derivates.

2. *Definition of the derivates.* We suppose throughout that $f(x)$ is finite and integrable in the Denjoy-Perron sense.

The Cesàro-derivates or C-derivates of $f(x)$ have been defined by Burkill†. The upper right C-derivate $CD^+f(x)$ is given by the equation

$$CD^+f(x) = \overline{\lim_{h\to+0}} \frac{\frac{1}{h}\int_x^{x+h} f(t)\,dt - f(x)}{\frac{1}{2}h}$$

$$= \overline{\lim_{h\to+0}} \frac{2}{h^2}\int_x^{x+h} [f(t)-f(x)]\,dt,$$

and there are corresponding definitions for the other derivates $CD_+f(x)$, $CD^-f(x)$ and $CD_-f(x)$.

The upper and lower (two-sided) derivates $CD^*f(x)$, $CD_*f(x)$ are defined to be $\max\{CD^+f(x),\ CD^-f(x)\}$ and $\min\{CD_+f(x),\ CD_-f(x)\}$ respectively. If they are equal, $f(x)$ is said to have a C-derivative $CDf(x)$.

† J. C. Burkill, " The Cesàro-Perron integral ", *Proc. London Math. Soc.* (2), 34 (1932), 314–322 (315).

The function $f(x)$ is said to be Cesàro-continuous or C-continuous at x if

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x);$$

that is, if it coincides with the derivative of its indefinite integral at the point x.

Properties of C-continuous functions.

3. We first prove that the C-derivates obey a generalized form of the mean value theorem.

THEOREM I. *If $f(x)$ is C-continuous in the closed interval (a, b), then the sets of points ξ, η of (a, b) for which*

$$(3.1)\qquad CD_*f(\xi) \geqslant \frac{f(b)-f(a)}{b-a} \geqslant CD^*f(\eta)$$

are each of cardinal c and contain perfect sub-sets.

If we write

$$(3.2)\qquad g(x) = f(x) - \frac{f(b)-f(a)}{b-a}\,(x-a),$$

then $g(a) = g(b)$, $g(x)$ is C-continuous in the closed interval (a, b), and (3.1) reduces to

$$(3.3)\qquad CD_*\,g(\xi) \geqslant 0 \geqslant CD^*g(\eta).$$

In order to find a point η for which (3.3) is satisfied, let

$$\phi(a) = g(a),$$

$$\phi(x) = \frac{1}{x-a}\int_a^x g(t)\,dt \qquad (a < x \leqslant b).$$

Then $\phi(x)$ is a continuous function of x in the closed interval (a, b).

If $\phi(x)$ is constant throughout (a, b), it is easily seen that

$$\frac{1}{h}\int_x^{x+h} g(t)\,dt = g(a),$$

whenever $a \leqslant x < x+h \leqslant b$, so that the C-continuous function $g(x)$ is equal to $g(a)$ at every point of (a, b) and the theorem is trivial.

We may therefore suppose that $\phi(x)$ attains an upper bound M, greater than $g(a)$, at some point of (a, b); otherwise we work with the lower bound and the procedure is similar.

Let η be the lower bound of the points x of (a, b) for which $\phi(x) = M$. The point η clearly cannot coincide with a, and

$$\frac{1}{\eta - a}\int_a^\eta g(t)\,dt = M. \tag{3.4}$$

Since

$$\frac{1}{x-a}\int_a^x g(t)\,dt < M,$$

whenever $a < x < \eta$, it follows that

$$\frac{1}{\eta - x}\int_x^\eta g(t)\,dt > M \qquad (a < x < \eta), \tag{3.5}$$

since otherwise (3.4) would not be satisfied.

By (3.5), on letting x tend to η,

$$g(n) \geqslant M, \tag{3.6}$$

so that η cannot coincide with b.

Since

$$\frac{1}{x-a}\int_a^x g(t)\,dt \leqslant M,$$

whenever $\eta < x \leqslant b$, it follows that

$$\frac{1}{x-\eta}\int_\eta^x g(t)\,dt \leqslant M \qquad (\eta < x \leqslant b), \tag{3.7}$$

since otherwise (3.4) would not be satisfied.

By (3.7),

$$g(\eta) \leqslant M,$$

and hence, in view of (3.6),

$$g(\eta) = M. \tag{3.8}$$

It then follows from (3.5) and (3.7) that

$$g(\eta) - \frac{1}{h}\int_{\eta-h}^{\eta} g(t)\,dt < 0,$$

$$\frac{1}{h}\int_{\eta}^{\eta+h} g(t)\,dt - g(\eta) \leqslant 0,$$

whenever $0 < h < \min(b-\eta, \eta-a)$, so that

$$CD^* g(\eta) \leqslant 0.$$

In order to find a point ξ for which (3.3) is satisfied, let

$$\psi(x) = \frac{1}{\eta - x}\int_x^\eta g(t)\,dt \qquad (a \leqslant x < \eta),$$

$$\psi(\eta) = g(\eta).$$

It follows from (3.4), (3.5), and (3.8) that the continuous function $\psi(x)$ attains an upper bound N, greater than $g(\eta)$, at some point interior to (a, η). Let ξ be the upper bound of the points x of (a, η) for which $\psi(x) = N$. Then $a < \xi < \eta$ and

$$\frac{1}{\eta - \xi} \int_{\xi}^{\eta} g(t)\, dt = N.$$

Proceeding as before, we can show that

$$\frac{1}{x - \xi} \int_{\xi}^{x} g(t)\, dt > N \qquad (\xi < x < \eta),$$

$$\frac{1}{\xi - x} \int_{x}^{\xi} g(t)\, dt \leqslant N \qquad (a \leqslant x < \xi).$$

It follows that $g(\xi) = N$, and hence that

$$CD_{*}\, g(\xi) \geqslant 0,$$

so that (3.3) is satisfied.

Now let E denote the set of points x of (a, b) for which $CD^{*} g(x) \leqslant 0$. It remains to show that E is of cardinal c and contains a perfect sub-set.

Since $g(x)$ is C-continuous in the closed interval (a, b), it coincides with a derivative at every point of the interval. It therefore takes every value between $g(a)$ and $g(\eta)$ at least once in the interval (a, η) and once in the interval (η, b). Hence, if

$$g(a) < y' < g(\eta),$$

there are points α, β of (a, η), (η, b) respectively, such that

$$g(\alpha) = y' = g(\beta).$$

Proceeding as before, we can find a point η' of (α, β) for which

$$CD^{*} g(\eta') \leqslant 0.$$

Corresponding to two different numbers y_1, y_2 between $g(a)$ and $g(\eta)$, we always obtain different points η_1, η_2.

For suppose, if possible, that η_2 coincides with η_1. Then

$$\frac{1}{\eta_1 - \alpha_1} \int_{\alpha_1}^{\eta_1} g(t)\, dt = g(\eta_1),$$

$$\frac{1}{\eta_1 - \alpha_2} \int_{\alpha_2}^{\eta_1} g(t)\, dt = g(\eta_1),$$

and hence

$$\frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(t)\,dt = g(\eta_1), \tag{3.9}$$

where we may suppose that $a_1 < a_2$. It follows from (3.9) that $a_2 \geqslant \eta_1$, so that $\eta_2 > \eta_1$. Hence η_2 cannot coincide with η_1.

As the numbers y' describe the interval $\{g(a), g(\eta)\}$, the points η' therefore describe a set of cardinal c. The set E is thus of cardinal c.

Since $f(x)$ is the derivative of a continuous function, it is easily seen that the C-derivates are measurable B. Hence the set E is measurable B and, being of cardinal c, contains a perfect sub-set.

By a similar method, it can be shown that the set of points x of (a, b) for which $CD_{*}\,g(x) \geqslant 0$ is of cardinal c and contains a perfect sub-set.

4. The following theorem can be deduced from Theorem I in the usual way:

THEOREM II. *If $f(x)$ is C-continuous in the closed interval (a, b), and the set of points x of (a, b) for which $CD^{*}f(x) < 0$ contains no perfect sub-set, then $f(x)$ increases in (a, b).*

5. THEOREM III. *If $f(x)$ is C-continuous in the closed interval (a, b) and $CDf(x)$ exists (finite or infinite) for $a < x < b$, then there is a point ξ interior to (a, b) such that*

$$\frac{f(b)-f(a)}{b-a} = CDf(\xi).$$

Defining $g(x)$ *by* (3.2), it is sufficient to find a point ξ for which

$$CDg(\xi) = 0.$$

We first obtain a decreasing sequence of sub-intervals (a_n, b_n) of (a, b) which are such that, for every positive integer n, $a_{n-1} < a_n < b_n < b_{n-1}$ and

$$g(a_{n-1}) < g(b_n) < g(a_n), \tag{5.1}$$

$$\frac{1}{b_n-a_{n-1}}\int_{a_{n-1}}^{b_n} g(t)\,dt = g(b_n), \tag{5.2}$$

$$\frac{1}{x-a_{n-1}}\int_{a_{n-1}}^{x} g(t)\,dt \leqslant g(b_n) \qquad (a_{n-1} < x \leqslant b_{n-1}), \tag{5.3}$$

$$\frac{1}{b_n-a_n}\int_{a_n}^{b_n} g(t)\,dt = g(a_n), \tag{5.4}$$

$$\frac{1}{b_n-x}\int_{x}^{b_n} g(t)\,dt \leqslant g(a_n) \qquad (a_{n-1} \leqslant x < b_n). \tag{5.5}$$

If we take the intervals (a, b), (ξ, η) of Theorem I for the intervals (a_0, b_0), (a_1, b_1) respectively, these conditions are satisfied when $n = 1$, and, proceeding as in Theorem I, the sequence of intervals can easily be defined by induction.

Let ξ, η be the limits of the increasing and decreasing sequences of points $\{a_n\}$, $\{b_n\}$ respectively.

First suppose that ξ and η do not coincide. Then, by (5.2) and (5.3), letting n tend to infinity, we get

$$\frac{1}{x-\xi}\int_{\xi}^{x} g(t)\,dt \leqslant \frac{1}{\eta-\xi}\int_{\xi}^{\eta} g(t)\,dt \qquad (\xi < x < \eta).$$

In the same way, it follows from (5.4) and (5.5) that

$$\frac{1}{\eta-x}\int_{x}^{\eta} g(t)\,dt \leqslant \frac{1}{\eta-\xi}\int_{\xi}^{\eta} g(t)\,dt \qquad (\xi < x < \eta).$$

These two relations can only be true if the sign of equality holds in each case, when the C-continuous function $g(x)$ is constant in (ξ, η) and $CDg(x)$ is zero at every point of (ξ, η).

It remains to consider the case when η coincides with ξ. It then follows from (5.1), (5.2), and (5.4) that $g(a_n)$ and $g(b_n)$ each increases to the limit $g(\xi)$. Hence, letting x coincide with ξ in (5.3) and (5.5), we have

$$g(\xi) - \frac{1}{\xi - a_{n-1}}\int_{a_{n-1}}^{\xi} g(t)\,dt \geqslant 0,$$

$$\frac{1}{b_n - \xi}\int_{\xi}^{b_n} g(t)\,dt - g(\xi) \leqslant 0.$$

Letting n tend to infinity in these two inequalities, we see that

$$CDg(\xi) = 0.$$

Relations between C-derivates and the approximate derivative.

6. We first prove two lemmas.

LEMMA I. *If $CD^+f(x) < \infty$ in a measurable set E in which $ADf(x)$ is zero, then*

$$CD^+f(x) \leqslant 0 \leqslant CD^-f(x)$$

almost everywhere in E.

Let E_0 be the sub-set of E in which $CD^+f(x) > 0$, and suppose that E_0 is of positive measure.

Then we can find positive numbers M, η_1, and a sub-set E_1 of E_0, of measure greater than zero, such that, if x is a point of E_1,

$$(6.1) \qquad \frac{2}{h^2}\int_x^{x+h}[f(t)-f(x)]\,dt < M \qquad (0 < h \leqslant \eta_1).$$

Since $ADf(x)$ is zero at points of E_1, it follows, from a lemma due to Khintchine†, that there is a perfect sub-set P of E_1 such that $mP > 0$ and, if x is a point of P,

$$\frac{f(t)-f(x)}{t-x} \to 0$$

as $t \to x$, t being a point of P.

By a lemma due to G. C. Young‡, we can assume that every portion of P is of positive measure.

Since $f(t)$ is integrable in the Denjoy-Perron sense, there is a portion π of P with complementary intervals

$$(c_1, d_1),\ (c_2, d_2),\ \ldots,\ (c_n, d_n),\ \ldots,$$

such that

$$\int_\pi |f(t)|\,dt < \infty,$$

and

$$\Sigma \left\| \int_{\gamma_n}^{\delta_n} f(t)\,dt \right\| < \infty,$$

where $\left\| \int_{\gamma_n}^{\delta_n} f(t)\,dt \right\|$ denotes the upper bound of $\left| \int_{\gamma_n}^{\delta_n} f(t)\,dt \right|$ for

$$c_n \leqslant \gamma_n < \delta_n \leqslant d_n.$$

Let x_0 be a point of density of π, and denote by π_h the portion of π contained in the interval (x_0, x_0+h).

Given ϵ, we can find η_2 such that

$$(6.2) \qquad \frac{m\pi_h}{h} > 1 - \frac{\epsilon}{M+2\epsilon} \qquad (0 < h \leqslant \eta_2),$$

and

$$(6.3) \qquad \left| \frac{f(t)-f(x_0)}{t-x_0} \right| < \epsilon,$$

whenever t is a point of π such that $0 < t-x_0 \leqslant \eta_2$.

---

† A. Khintchine, "Structure des fonctions mesurables", *Fundamenta Math.*, 9 (1927), 212–279 (228).

‡ G. C. Young, "On the derivates of a function", *Proc. London Math. Soc.* (2), 15 (1916), 360–384 (365).

Let $\eta = \min(\eta_1, \eta_2)$, and choose h such that $0 < h \leqslant \eta$.

Denote by

$$(a_1, b_1),\ (a_2, b_2),\ \ldots,\ (a_n, b_n),\ \ldots$$

the complementary intervals of π_h. Let c be the upper bound of points of π_h on the left of x_0+h.

It follows from the definition of π that

$$(6.4)\quad \int_{x_0}^{x_0+h} [f(t)-f(x_0)]\,dt = \int_{\pi_h} [f(t)-f(x_0)]\,dt + \Sigma\int_{a_n}^{b_n} [f(t)-f(x_0)]\,dt + \int_c^{x_0+h} [f(t)-f(x_0)]\,dt.$$

Now

$$\int_{a_n}^{b_n} [f(t)-f(x_0)]\,dt = \int_{a_n}^{b_n} [f(t)-f(a_n)]\,dt + \int_{a_n}^{b_n} [f(a_n)-f(x_0)]\,dt$$

$$< \tfrac{1}{2}M(b_n-a_n)^2 + \epsilon(a_n-x_0)(b_n-a_n),$$

by (6.1) and (6.3), so that

$$(6.5)\quad \int_{a_n}^{b_n} [f(t)-f(x_0)]\,dt < \tfrac{1}{2}h(M+2\epsilon)(b_n-a_n).$$

In the same way,

$$(6.6)\quad \int_c^{x_0+h} [f(t)-f(x_0)]\,dt < \tfrac{1}{2}h(M+2\epsilon)(x_0+h-c).$$

Also, by (6.3),

$$\int_{\pi_h} [f(t)-f(x_0)]\,dt < \epsilon\int_{x_0}^{x_0+h} (t-x_0)\,dt$$

$$= \tfrac{1}{2}\epsilon h^2.$$

Together with (6.4), (6.5), and (6.6), this yields

$$\int_{x_0}^{x_0+h} [f(t)-f(x_0)]\,dt < \tfrac{1}{2}\epsilon h^2 + \tfrac{1}{2}h(M+2\epsilon)\{\Sigma(b_n-a_n)+x_0+h-c\}$$

$$< \tfrac{1}{2}\epsilon h^2 + \tfrac{1}{2}\epsilon h^2,$$

by (6.2), so that

$$\frac{2}{h^2}\int_{x_0}^{x_0+h} [f(t)-f(x_0)]\,dt < 2\epsilon.$$

Since ϵ is arbitrary,

$$CD^+f(x_0) \leqslant 0,$$

and this is impossible if x_0 is a point of E_0.

It follows that $CD^+f(x) \leqslant 0$ almost everywhere in E.

Now let α be any positive number, and let F_n be the set of points x of E for which

$$\frac{2}{h^2}\int_x^{x+h} [f(t)-f(x)]\,dt < \tfrac{1}{2}\alpha, \tag{6.7}$$

whenever $0 < h \leqslant 1/n$.

Let x_0 be a point of density of F_n.

Since $ADf(x_0) = 0$, given any positive number δ, we can find a point ξ of F_n such that

$$0 < x_0 - \xi < \min(\delta, 1/n), \tag{6.8}$$

$$\frac{f(\xi)-f(x_0)}{\xi-x_0} > -\tfrac{1}{4}\alpha. \tag{6.9}$$

Now

$$\frac{2}{(\xi-x_0)^2}\int_{x_0}^{\xi} [f(t)-f(x_0)]\,dt = 2\,\frac{f(\xi)-f(x_0)}{\xi-x_0} - \frac{2}{(x_0-\xi)^2}\int_{\xi}^{x_0} [f(t)-f(\xi)]\,dt$$

$$> -\alpha,$$

by (6.7), (6.8), and (6.9), so that

$$CD^-f(x_0) \geqslant -\alpha.$$

Hence $CD^-f(x) \geqslant -\alpha$ almost everywhere in F_n.

Since $CD^+f(x) \leqslant 0$ in E, F_n increases to E. It therefore follows that $CD^-f(x) \geqslant -\alpha$ almost everywhere in E.

Since α is arbitrary, $CD^-f(x) \geqslant 0$ at almost all points of E.

LEMMA II. *If $CD^+f(x) < \infty$ in a measurable set E in which $f(x)$ is approximately differentiable, then*

$$CD^+f(x) \leqslant ADf(x) \leqslant CD^-f(x)$$

almost everywhere in E.

Let $f_1(x)$ be the function which coincides with $ADf(x)$ at points of E and is zero outside E.

Since $f_1(x)$ is measurable and finite, there is a continuous function $F(x)$ such that

$$F'(x) = f_1(x)$$

almost everywhere. Let E_0 be the sub-set of E at which this relation is satisfied.

If we write

$$g(x) = f(x) - F(x),$$

then, if x is a point of E_0,

$$CD^+ g(x) = CD^+ f(x) - ADf(x),$$

$$CD^- g(x) = CD^- f(x) - ADf(x),$$

and $$ADg(x) = 0.$$

Hence $CD^+ g(x) < \infty$ in E_0, while $ADg(x)$ is zero at points of E_0. By Lemma I,

$$CD^+ g(x) \leqslant 0 \leqslant CD^- g(x)$$

almost everywhere in E_0.

It follows that

$$CD^+ f(x) \leqslant ADf(x) \leqslant CD^- f(x)$$

almost everywhere in E_0, hence almost everywhere in E.

7. THEOREM IV. *If $CD^* f(x) < \infty$ in a measurable set E, then $f(x)$ is approximately differentiable almost everywhere in E, and*

$$CD^+ f(x) = ADf(x) = CD^- f(x) \tag{7.1}$$

at almost all points of E.

Given ϵ, we can find positive numbers M, η, and a subset E_1 of E, such that

$$mE_1 > mE - \epsilon,$$

and, if x is a point of E_1,

$$\frac{2}{h^2} \int_x^{x+h} [f(t) - f(x)]\, dt < M$$

whenever $0 < |h| \leqslant \eta$.

If we write

$$g(x) = f(x) - Mx, \tag{7.2}$$

then

$$\frac{2}{h^2} \int_x^{x+h} [g(t) - g(x)]\, dt < 0 \tag{7.3}$$

whenever x is a point of E_1 and $0 < |h| \leqslant \eta$.

Let x_1 and x_2 be any two points of E_1 such that $0 < x_2 - x_1 \leqslant \eta$. Then, by (7.3),

$$g(x_1) > \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} g(t)\, dt > g(x_2).$$

It follows that $g(x)$ is approximately differentiable almost everywhere in E_1, hence almost everywhere in E.

In view of (7.2), the function $f(x)$ is approximately differentiable almost everywhere in E.

By Lemma II, since $CD^+f(x) < \infty$ in E,

$$CD^+f(x) \leqslant ADf(x) \leqslant CD^-f(x)$$

almost everywhere in E.

Also, since $CD^-f(x) < \infty$ in E, it follows, from a result similar to Lemma II, that

$$CD^-f(x) \leqslant ADf(x) \leqslant CD^+f(x)$$

almost everywhere in E, so that (7.1) is satisfied.

The following corollaries can be deduced at once from Theorem IV.

COROLLARY I. *If $CD^+f(x) = -\infty$ in E, then $CD^-f(x) = +\infty$ almost everywhere in E.*

COROLLARY II. *A function can have an infinite C-derivative only in a set of measure zero.*

8. THEOREM V. *If $CD^+f(x)$ and $CD_+f(x)$ are finite at points of a measurable set E, then $ADf(x)$ and $CDf(x)$ exist and are equal almost everywhere in E.*

Given ϵ, we can find M, η_1, and a closed sub-set Q of E, such that

$$mQ > mE - \epsilon \tag{8.1}$$

and

$$\left| \frac{2}{h^2} \int_x^{x+h} [f(t) - f(x)]\, dt \right| \leqslant M, \tag{8.2}$$

whenever $0 < h \leqslant 2\eta_1$ and x is a point of Q.

Let x, y be any two points of Q such that $0 < y - x \leqslant \eta_1$. We shall show that

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leqslant 3M. \tag{8.3}$$

Let $z = y + (y - x)$. Write $y - x = h$, so that $z - y = h$, $z - x = 2h$, where $0 < 2h \leqslant 2\eta_1$.

It follows from (8.2) that

$$\left|\int_x^z f(t)\,dt - 2hf(x)\right| \leqslant 2Mh^2,$$

$$\left|\int_x^y f(t)\,dt - hf(x)\right| \leqslant \tfrac{1}{2}Mh^2,$$

$$\left|\int_y^z f(t)\,dt - hf(y)\right| \leqslant \tfrac{1}{2}Mh^2,$$

and hence that

$$|hf(y) - hf(x)| \leqslant 3Mh^2,$$

so that (8.3) is satisfied.

Let x_0 be any point of density of Q.

Then we can find η such that $\eta \leqslant \eta_1$, $x_0 - \eta$ is a point of Q, and

$$\frac{m\{Q.(x, x_0)\}}{x_0 - x} \geqslant \tfrac{1}{2}, \tag{8.4}$$

whenever $0 < x_0 - x \leqslant \eta$.

Let x be any point of the interval $(x_0 - \eta, x_0)$.

If x_1 is the upper bound of points of Q to the left of x, then x_1 is a point of Q and $x_0 - x_1 \leqslant \eta$. Since the interval (x_1, x) contains no points of Q, it follows from (8.4) that

$$\frac{x_0 - x_1}{x_0 - x} \leqslant 2 \tag{8.5}$$

and

$$\frac{x - x_1}{x_0 - x} \leqslant 1. \tag{8.6}$$

Now

$$\frac{2}{(x_0 - x)^2}\int_{x_0}^{x} [f(t) - f(x_0)]\,dt$$

$$= 2\,\frac{f(x_0) - f(x_1)}{x_0 - x} - \frac{2}{(x_0 - x)^2}\int_{x_1}^{x_0} [f(t) - f(x_1)]\,dt + \frac{2}{(x_0 - x)^2}\int_{x_1}^{x} [f(t) - f(x_1)]\,dt,$$

so that, by (8.2) and (8.3),

$$\left|\frac{2}{(x_0 - x)^2}\int_{x_0}^{x} [f(t) - f(x_0)]\,dt\right| \leqslant 6M\,\frac{x_0 - x_1}{x_0 - x} + M\left(\frac{x_0 - x_1}{x_0 - x}\right)^2 + M\left(\frac{x - x_1}{x_0 - x}\right)^2$$

$$\leqslant 17M,$$

by (8.5) and (8.6).

It follows that $CD^-f(x_0)$ and $CD_-f(x_0)$ are finite.

Thus $CD^-f(x)$ and $CD_-f(x)$ are finite almost everywhere in Q, and hence, since ϵ is arbitrary, almost everywhere in E.

By Theorem IV, $f(x)$ is approximately differentiable almost everywhere in E and $CD^+f(x)$ and $CD^-f(x)$ are equal to $ADf(x)$ at almost all points of E. Applying Theorem IV to the function $\{-f(x)\}$, we see that $CD_+f(x)$ and $CD_-f(x)$ are also equal to $ADf(x)$ almost everywhere in E.

Construction of functions.

9. We proceed to construct functions which exhibit certain properties at points of a perfect set P, and show that the finiteness of either one or two C-derivates does not imply the existence of the C-derivative, unless the two finite derivates are on the same side.

We first obtain the perfect set P by subtraction of complementary intervals from the interval (0, 1).

With the middle point of (0, 1) as centre, describe an interval of length $1/(2^2 2!)$. This will be called the complementary interval of the first set.

We now show how to find the complementary intervals of the n-th set, assuming that those of the r-th set have been found for $r < n$. Let (c, d) be any interval that remains after the complementary intervals of the first $n-1$ sets have been subtracted from (0, 1). Divide (c, d) into $n+1$ equal parts, and, with each point of subdivision as centre, describe an interval of length $1/\{2^{n+1}(n+1)!\}$. These are complementary intervals of the n-th set.

The perfect set P is the set that remains when the complementary intervals have been subtracted from (0, 1).

It is easily seen that there are $n \,.\, n!$ intervals of the n-th set, so that

$$\begin{aligned} mP &= 1 - \sum_{n=1}^{\infty} \frac{n \,.\, n!}{2^{n+1}(n+1)!} \\ &> 1 - \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \\ &= \tfrac{1}{2}. \end{aligned}$$

10. We shall denote by x_0 a point of P which does not coincide with an end point of a complementary interval. We need to consider the complementary intervals of P in relation to x_0.

We shall use δ_n or (a_n, b_n) to denote any complementary interval of the n-th set.

After the complementary intervals of the first n sets have been subtracted from (0, 1), x_0 is contained between two intervals δ_r and δ_s for

some values of r and s not greater than n, either r or s being equal to n (unless it is contained between one interval δ_n and one of the points 0 or 1, and this case is excluded for sufficiently large values of n). We shall denote the middle points of these intervals δ_r and δ_s by p_r and q_s.

It is clear that either p_n or q_n exists for every value of n, and that the sequences $\{p_r\}$, $\{q_s\}$ increase and decrease respectively to the limit x_0.

Also no interval δ_{r-k}, where $k>0$, is contained between p_r and x_0, and

$$x_0 - p_r < \frac{1}{(r+1)!}, \tag{10.1}$$

since, in obtaining the intervals of the r-th set, the distance between two points of subdivision is less than $1/(r+1)!$.

In the same way, no interval δ_{s-k}, where $k>0$, is contained between x_0 and q_s, and

$$q_s - x_0 < \frac{1}{(s+1)!}. \tag{10.2}$$

11. We first construct a function $f(x)$ for which the two upper derivates are zero almost everywhere in P, while the lower derivates are not finite, showing that $CD^*f(x)<\infty$ does not imply the existence of a finite C-derivative.

Define
$$f\{a_n+\tfrac{1}{4}(b_n-a_n)\} = -(1/n),$$
$$f\{a_n+\tfrac{1}{2}(b_n-a_n)\} = 0,$$
$$f\{a_n+\tfrac{3}{4}(b_n-a_n)\} = +(1/n),$$

let $f(x)$ vanish at points of P and be linear in each quarter of an interval (a_n, b_n).

Then

$$\int_{a_n}^{b_n} f(t)\,dt = 0, \tag{11.1}$$

while, if $a_n<x<b_n$,

$$\int_{a_n}^{x} f(t)\,dt < 0, \tag{11.2}$$

$$\int_{x}^{b_n} f(t)\,dt > 0. \tag{11.3}$$

Let x_0 be any point of P which does not coincide with an end point of one of the complementary intervals.

Since $f(x)$ vanishes at points of P, it follows from (11.1), (11.2), and (11.3) that

$$\frac{2}{h^2}\int_{x_0}^{x_0+h} [f(t)-f(x_0)]\,dt \leqslant 0$$

for all values of h, while

$$\frac{2}{h^2}\int_{x_0}^{x_0+h}[f(t)-f(x_0)]\,dt=0$$

if x_0+h is a point of P.

Hence $CD^+f(x_0)$ and $CD^-f(x_0)$ are zero.

On the other hand, with the notation of § 10,

$$\frac{2}{(q_s-x_0)^2}\int_{x_0}^{q_s}[f(t)-f(x_0)]\,dt=\frac{2}{(q_s-x_0)^2}\int_{a_s}^{q_s}f(t)\,dt$$

$$=-\frac{2}{(q_s-x_0)^2}\,\frac{1}{4s}\,\frac{1}{2^{s+1}(s+1)!},$$

so that, in view of (10.2),

$$\frac{2}{(q_s-x_0)^2}\int_{x_0}^{q_s}[f(t)-f(x_0)]\,dt<-\frac{1}{2s}\,\frac{(s+1)!}{2^{s+1}},$$

and hence $$CD_+f(x_0)=-\infty.$$

By a similar method it can be shown that

$$\frac{2}{(p_r-x_0)^2}\int_{x_0}^{p_r}[f(t)-f(x_0)]\,dt<-\frac{1}{2r}\,\frac{(r+1)!}{2^{r+1}},$$

so that $$CD_-f(x_0)=-\infty.$$

12. The previous construction can easily be modified to yield a function $f(x)$ for which $CD^+f(x)=CD^-f(x)=+\infty$, $CD_+f(x)=CD_-f(x)=-\infty$ at almost all points of P.

It is sufficient to divide each interval δ_n into four equal parts, and take $f(x)$ to be a continuous function which vanishes at points of P, is negative in the first and last quarters of δ_n, positive in the two middle quarters, and such that

$$\int_{\lambda_n}^{\mu_n}|f(t)|\,dt=\frac{1}{4n}\,\frac{1}{2^{n+1}(n+1)!},$$

where (λ_n, μ_n) denotes any quarter of δ_n.

13. The construction of § 11 can also be modified to yield a function with infinite right- and left-hand derivatives almost everywhere in P.

Define $$f\{a_n+\tfrac{1}{2}(b_n-a_n)\}=-(1/n),$$

and let $f(x)$ vanish at points of P and be linear in each half of an interval δ_n.

Suppose that $0<h<1-x_0$. Then x_0+h is contained in an interval $(q_\sigma, q_{\sigma-\kappa})$, where $q_{\sigma-\kappa}$, q_σ are consecutive terms of the sequence $\{q_s\}$. Let α be such that

$$x_0+h = a_{\sigma-k}+\alpha(q_{\sigma-\kappa}-a_{\sigma-\kappa}). \tag{13.1}$$

Now
$$q_{\sigma-\kappa}-a_{\sigma-\kappa} = \frac{1}{2^{\sigma-\kappa+2}(\sigma-\kappa+1)!},$$

and
$$a_{\sigma-\kappa}-x_0 < \frac{1}{\sigma!},$$

since, in obtaining the complementary intervals of the $(\sigma-1)$-th set, the distance between two points of subdivision is less than $1/\sigma!$. It therefore follows from (13.1) that

$$h < 2\max\left\{\frac{1}{\sigma!}, \frac{\alpha}{2^{\sigma-\kappa+2}(\sigma-\kappa+1)!}\right\}. \tag{13.2}$$

First, suppose that

$$\frac{\alpha}{2^{\sigma-\kappa+2}(\sigma-\kappa+1)!} \leqslant \frac{1}{\sigma!}.$$

Then
$$\int_{x_0}^{x_0+h}[f(t)-f(x_0)]\,dt < \int_{a_\sigma}^{q_\sigma}[f(t)-f(x_0)]\,dt$$
$$= -\frac{1}{4\sigma}\,\frac{1}{2^{\sigma+1}(\sigma+1)!},$$

so that, in view of (13.2),

$$\frac{2}{h^2}\int_{x_0}^{x_0+h}[f(t)-f(x_0)]\,dt < -\frac{1}{8}\,\frac{(\sigma-1)!}{2^{\sigma+1}(\sigma+1)}. \tag{13.3}$$

On the other hand, if

$$\frac{\alpha}{2^{\sigma-\kappa+2}(\sigma-\kappa+1)!} > \frac{1}{\sigma!},$$

then
$$\int_{x_0}^{x_0+h}[f(t)-f(x_0)]\,dt < \int_{a_{\sigma-\kappa}}^{x_0+h}[f(t)-f(x_0)]\,dt$$
$$= \alpha^2\int_{a_{\sigma-\kappa}}^{q_{\sigma-\kappa}}[f(t)-f(x_0)]\,dt$$
$$= -\frac{\alpha^2}{4(\sigma-k)}\,\frac{1}{2^{\sigma-\kappa+1}(\sigma-\kappa+1)!},$$

so that, by (13.2),

$$(13.4)\qquad \frac{2}{h^2}\int_{x_0}^{x_0+h}[f(t)-f(x_0)]\,dt < -\frac{2^{\sigma-\kappa}}{\sigma-\kappa}\,(\sigma-\kappa+1)!.$$

It follows from (13.3) and (13.4) that

$$CD^+f(x_0) = CD_+f(x_0) = -\infty.$$

By a similar method it can be shown that

$$CD^-f(x_0) = CD_-f(x_0) = +\infty.$$

The function $f(x)$ thus has infinite left- and right-hand derivatives almost everywhere in P, and

$$CD_-f(x) > CD^+f(x)$$

at almost all points of P.

14. In order to construct a function with two opposed C-derivates finite, whilst the other two C-derivates are infinite, we require the following lemma.

LEMMA. *If n_1, n_2, n_3, ..., n_κ, ... is any given increasing sequence of positive integers, and if x_0 does not belong to a set E of measure zero, then the sequences $\{p_{r'}\}$, $\{q_{s'}\}$ of points p_r, q_s, which are interior to intervals δ_{n_1}, δ_{n_2}, δ_{n_3}, ..., δ_{n_κ}, ..., are each infinite and tend to x_0.*

Define

$$(14.1)\qquad f\{a_{n_\kappa}+\tfrac{1}{2}(b_{n_\kappa}-a_{n_\kappa})\} = \frac{1}{n_\kappa};$$

let $f(x)$ vanish except at points of the intervals δ_{n_κ} and be linear in each half of an interval δ_{n_κ}.

Suppose that there is only a finite number of points $p_{r'}$. Then $f(x)$ vanishes throughout an interval having x_0 as right-hand end point, and hence

$$D^-f(x_0) = D_-f(x_0) = 0.$$

On the other hand, there must be an infinite number of points $q_{s'}$, and in view of (10.2) and (14.1),

$$\frac{f(q_{s'})-f(x_0)}{q_{s'}-x_0} > \frac{(s'+1)!}{s'},$$

so that

$$D^+f(x_0) = +\infty.$$

It follows that x_0 must belong to a set E_1 of measure zero.

By a similar method, it can be shown that if there is only a finite number of points $q_{s'}$, then x_0 must belong to a set E_2 of measure zero.

The set E of the lemma is the sum of the sets E_1 and E_2.

Now let the sequence $\{n_\kappa\}$ be chosen in such a way that, for every integer κ,

$$\frac{[2^{(n_\kappa+2)}(n_\kappa+2)!]^2}{2^{n_{(\kappa+1)}}} < \frac{1}{2^\kappa}, \tag{14.2}$$

and let $f(x)$ be the function defined in the lemma. It will be shown that $CD_+ f(x)$ and $CD^- f(x)$ are zero almost everywhere in P, while the other C-derivates are not finite.

It is easily seen that

$$\frac{2}{(q_{s'}-x_0)^2}\int_{x_0}^{q_{s'}}[f(t)-f(x_0)]\,dt > \frac{1}{2s'}\,\frac{(s'+1)!}{2^{s'+1}},$$

$$\frac{2}{(p_{r'}-x_0)^2}\int_{x_0}^{p_{r'}}[f(t)-f(x_0)]\,dt < -\frac{1}{2r'}\,\frac{(r'+1)!}{2^{r'+1}},$$

the proofs being similar to those of § 11.

Hence, if x_0 does not belong to E,

$$CD^+ f(x_0) = +\infty,$$

$$CD_- f(x_0) = -\infty.$$

Let F be the set of measure zero determined by the lemma when the sequence $\{n_\kappa\}$ is replaced by the sequence $\{(n_\kappa+1)\}$, and suppose that x_0 does not belong to F.

If $q_{(n_\kappa+1)}$ exists, then $f(x)$ vanishes in $(x_0,\ q_{(n_\kappa+1)})$ except at points of intervals δ_{n_t}, where $t \geqslant \kappa+1$. Since the number of intervals of the n_t-th set is $n_t . n_t!$,

$$\begin{aligned}\int_{x_0}^{q_{(n_\kappa+1)}}[f(t)-f(x_0)]\,dt &\leqslant \sum_{t=\kappa+1}^{\infty} n_t . n_t! \int_{\delta_{n_t}} f(t)\,dt \\ &\leqslant \sum_{t=n_{(\kappa+1)}}^{\infty} t . t! \int_{\delta t} f(t)\,dt \\ &\leqslant \sum_{t=n_{(\kappa+1)}}^{\infty} t . t!\,\frac{1}{2t}\,\frac{1}{2^{t+1}(t+1)!} \\ &< \sum_{t=n_{(\kappa+1)}}^{\infty} \frac{1}{2^{t+1}} \\ &= \frac{1}{2^{n_{(\kappa+1)}}}.\end{aligned}$$

Since
$$q_{(n_\kappa+1)}-x_0 > \tfrac{1}{2}\{b_{(n_\kappa+1)}-a_{(n_\kappa+1)}\}$$
$$= \frac{1}{2^{(n_\kappa+2)}(n_\kappa+2)!},$$
it follows that
$$\frac{1}{(q_{(n_\kappa+1)}-x_0)^2}\int_{x_0}^{q_{(n_\kappa+1)}}[f(t)-f(x_0)]\,dt < \frac{[2^{n_\kappa+2}(n_\kappa+2)!]^2}{2^{n_{(\kappa+1)}}}$$
$$< \frac{1}{2^\kappa},$$
by (14.2).

Hence, since x_0 does not belong to F,
$$CD_+ f(x_0) = 0.$$

By a similar method, it can be shown that
$$CD^- f(x_0) = 0.$$

15. The previous construction can be modified to yield a function $f(x)$ such that $CD_+ f(x)$ is zero almost everywhere in P, while the other three derivates are infinite.

Define
$$f\{a_{n_\kappa}+\tfrac{1}{4}(b_{n_\kappa}-a_{n_\kappa})\} = 4/n_\kappa,$$
$$f\{a_{n_\kappa}+\tfrac{1}{2}(b_{n_\kappa}-a_{n_\kappa})\} = 0,$$
$$f\{a_{n_\kappa}+\tfrac{3}{4}(b_{n_\kappa}-a_{n_\kappa})\} = -(2/n_\kappa),$$
let $f(x)$ vanish except at points of the intervals δ_{n_κ} and be linear in each quarter of an interval δ_{n_κ}.

Then it can be shown that, if x_0 does not belong to E or F,
$$CD_+ f(x_0) = 0,$$
whilst
$$CD^+ f(x_0) = +\infty,$$
$$CD_- f(x_0) = -\infty,$$
the proofs being similar to those of § 14.

On the other hand, if p_{n_κ} exists,
$$\int_{p_{n_\kappa}}^{x_0} f(t)\,dt < \int_{p_{n_\kappa}}^{b_{n_\kappa}} f(t)\,dt + \sum_{t=\kappa+1}^{\infty} n_t . n_t! \int_{\delta_{n_t}} f(t)\,dt,$$
and, exactly as in § 14,
$$\sum_{t=\kappa+1}^{\infty} n_t . n_t! \int_{\delta_{n_t}} f(t)\,dt < \frac{1}{2^{n_{(\kappa+1)}}}.$$

Hence

$$\int_{x_0}^{p_{n_\kappa}} [f(t)-f(x_0)]\,dt > \frac{1}{2n_\kappa}\,\frac{1}{2^{(n_\kappa+1)}(n_\kappa+1)!} - \frac{1}{2^{n_{(\kappa+1)}}},$$

so that, in view of (10.1),

$$\frac{1}{(p_{n_\kappa}-x_0)^2}\int_{x_0}^{p_{n_\kappa}} [f(t)-f(x_0)]\,dt > (n_\kappa+1)!\,(n_\kappa+1)!\left[\frac{1}{2n_\kappa}\,\frac{1}{2^{(n_\kappa+1)}(n_\kappa+1)} - \frac{1}{2^{n_{(\kappa+1)}}}\right].$$

It follows from (14.2) that

$$CD^{-}f(x_0) = +\infty,$$

provided that x_0 does not belong to the set E determined by the lemma.

Westfield College,
Hampstead, London, N.W.3.

A THEOREM ON MEROMORPHIC FUNCTIONS

By J. M. WHITTAKER.

[Received and read 15 November, 1934.]

Introduction.

1. The fundamental inequality

$$|a_n|r^n \leqslant M(r) \leqslant \sum_{n=0}^{\infty} |a_n|r^n \quad (0 \leqslant r < R), \tag{1.1}$$

where $M(r)$ denotes the maximum modulus of a function $\Sigma a_n z^n$ regular in $|z| < R$, leads to very precise relations between the size of the function and the size of its Taylor coefficients. Thus the order ρ and type σ of an integral function can be estimated in terms of either, in virtue of the equations

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r} = \overline{\lim_{n\to\infty}} \frac{n\log n}{\log(1/|a_n|)} = \rho, \tag{1.2}$$

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^\rho} = \frac{1}{e\rho}\left(\overline{\lim_{n\to\infty}} n^{1/\rho}|a_n|^{1/n}\right)^\rho = \sigma. \tag{1.3}$$

Broadly speaking we may say that (1.1) is the basis of two important principles: (i) "some Taylor coefficients of a large function are large, all Taylor coefficients of a small function are small", and (ii) "a part of a Taylor series cannot increase more rapidly than the whole".

These advantages are not shared by the Mittag-Leffler expansion,

$$\sum_{n=0}^{\infty} c_n z^n + \Sigma\left\{\frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + \ldots + \frac{A_\lambda}{(z-a)^\lambda} + p(z)\right\}, \tag{1.4}$$

of a meromorphic function. If the function is of high order at least one of three conditions must be satisfied. The integral function $\Sigma c_n z^n$ must be

of high order, the A's must be large, or the poles must be numerous; and the third condition always indicates a function of high order. But we cannot say "the A's of a small function are small", since, in fact, they may be as large as we please. The function

$$\sum_{n=1}^{\infty} \frac{1}{(z-2^n)[z-2^n-\{1/\phi(n)\}]} \quad [\phi(n) > 0]$$

is of order zero, but its Mittag-Leffler expansion is

$$\sum_{n=0}^{\infty} c_n z^n + \sum_{n=1}^{\infty} \left\{-\frac{\phi(n)}{z-2^n} + p_n(z)\right\} + \sum_{n=1}^{\infty} \left\{\frac{\phi(n)}{z-2^n-\{1/\phi(n)\}} + q_n(z)\right\},$$

and $\phi(n)$ may increase arbitrarily rapidly. Nor is it true that the order of a part of a Mittag-Leffler expansion cannot exceed that of the whole. Indeed we can go further. Let

$$H_1(z) = \prod_{n=1}^{\infty} \left(1-\frac{z}{2^n}\right), \quad H_2(z) = \prod_{n=1}^{\infty} \left(1-\frac{z}{2^n+\{1/\phi(n)\}}\right),$$

where $\log\log\phi(n) = 2^n$. These are functions of order zero, so

$$F(z) = \{H_1(z)\, H_2(z)\}^{-1}$$

is a meromorphic function of order zero. $F(z)$ can be expressed in various ways (*e.g.* by splitting up the Mittag-Leffler expansion) as the sum of two functions $F_1(z)$, $F_2(z)$, the first with poles at the points 2^n, the second at the points $2^n+\{\phi(n)\}^{-1}$; but, however these functions are chosen, they will be of infinite order. For

$$F_1(z) = \frac{G_1(z)}{H_1(z)}, \quad F_2(z) = \frac{G_2(z)}{H_2(z)},$$

where $G_1(z)$, $G_2(z)$ are integral functions, and

$$G_1(z)\, H_2(z) + G_2(z)\, H_1(z) = H_1(z)\, H_2(z)\{F_1(z)+F_2(z)\} = 1.$$

Hence

$$G_1(2^m)\, \frac{1}{2^m\phi(m)+1} \prod_{n\neq m} \left(1-\frac{2^m}{2^n+\{1/\phi(n)\}}\right) = 1,$$

which gives

$$G_1(2^m) > 2^{-m^2}\phi(m),$$

and this implies that $G_1(z)$, and so also $F_1(z)$, is of infinite order.

Anomalies due to the same cause occur elsewhere. Thus it is always possible to find an integral function which takes given values E_1, E_2, ... at

a given set of points $e_1, e_2, \ldots$ ($|e_n| \to \infty$); but there may be no function of finite order which has this property, even if the E's are not very large and $|e_n| \to \infty$ very rapidly.

These difficulties evidently have their origin in the clustering together of the poles (or, in the last case, the e's) and may be expected to disappear if we restrict ourselves to functions in which the poles are more evenly spaced. The influence of spacing was discussed at length by Borel*, who distinguished between "ordinary distributions" of poles and "extraordinary distributions", the latter being those which give rise to difficulties; though, as Borel remarked, extraordinary distributions occur in the case of very simple functions such as $\operatorname{cosec} z \operatorname{cosec} \lambda z$ for certain values of λ. The recent work of Mursi and Winn and of Macintyre and Wilson on the interpolation problem is on similar lines. A result of the last named authors is particularly relevant. They proved that, if it is possible to surround the poles p_n of a meromorphic function of order ρ with circles of radius $|p_n|^{-h} (h > 0)$ which do not overlap, then the residue R_n at p_n satisfies the inequality

$$\overline{\lim_{|p_n| \to \infty}} \frac{\log^+ \log^+ |R_n|}{\log |p_n|} \leqslant \rho. \tag{1.5}$$

As has been seen, no upper limit can be assigned to the size of the residues unless the minimum distance between the poles is restricted in some way.

Extraordinary distributions of poles give rise to difficulties which are to some extent inevitable. If, as in the Mittag-Leffler expansion, each pole is considered separately, we cannot prevent the coefficients from becoming far too large. The remedy is to prohibit the splitting apart of clusters of poles which are very close together, and the object of the present paper is to show that, if the terms in the Mittag-Leffler expansion corresponding to each such cluster are bracketed together, the order of the function can be evaluated in terms of the resulting coefficients; and, moreover, that no part of the bracketed series can exceed the whole in order. If the bracketed series is

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \Sigma \left\{ \frac{P_0 + P_1 z + \ldots + P_{u-1} z^{u-1}}{(z-b_0)^{\lambda_0} \ldots (z-b_k)^{\lambda_k}} + Q(z) \right\},$$

it is shown that the order of $f(z)$ is

$$\rho = \max(\sigma, \tau, \kappa),$$

* References will be found at the end.

where σ is the order of the integral function $\Sigma a_n z^n$,

$$\tau = \overline{\lim_{|b_0|\to\infty}} \frac{\log^+\log^+ P}{\log|b_0|} \quad \left(P = \max_{0\leqslant i<\mu} |P_i|\right), \tag{1.6}$$

and κ is the exponent of convergence of the poles, *i.e.*

$$\kappa = \overline{\lim_{r\to\infty}} \frac{\log p(r)}{\log r}, \tag{1.7}$$

where

$$p(r) = \text{number of poles of } f(z) \text{ in } |z|\leqslant r. \tag{1.8}$$

This result is used to prove a dissection theorem, to remove a restriction in a theorem concerning the "sum" of a function, and to determine an upper bound for the order of the smallest meromorphic function with given poles and principal parts.

Expression of a meromorphic function in normal form.

2. LEMMA 1. *If $P(z)$ is a polynomial of degree greater than one and C is a simple closed contour surrounding its zeros, then*

$$\int_C \frac{dz}{P(z)} = 0.$$

Proof. If Γ is a "large" circle of radius R,

$$\left|\int_C \frac{dz}{P(z)}\right| = \left|\int_\Gamma \frac{dz}{P(z)}\right| \leqslant \int_0^{2\pi} \frac{R\,d\theta}{O(R^2)} = O\left(\frac{1}{R}\right).$$

LEMMA 2. *Let $f(z)$ be a meromorphic function of order ρ, and let h, ϵ be fixed positive numbers. If about each pole $p_n(\neq 0)$ there is drawn a circle of radius $|p_n|^{-h}$, then*

$$\log|f(z)| < r^{\rho+\epsilon},$$

if $|z| = r$ is sufficiently large and z is not inside any of the circles.

This is the second lemma of the paper of Macintyre and Wilson. To prove it, express $f(z)$ in the form $g_1(z)/g(z)$, where $g_1(z)$, $g(z)$ are integral functions of order not exceeding ρ, and use the well-known results of Hadamard and Borel that

$$\log|g_1(z)| < r^{\rho+\frac{1}{2}\epsilon},$$

and

$$\log|g(z)| > -r^{\rho+\frac{1}{2}\epsilon}$$

except possibly at points inside the circles.

3. We are now ready to group the poles together in clusters. Let $f(z)$ be a meromorphic function of order ρ and let κ be the exponent of convergence of its poles $p_1, p_2, \ldots$ (counted multiply and excluding zero, if it is a pole). The results are true but trivial if $\kappa = \infty$, so we suppose that κ is finite. Let $h > \kappa$ be a given positive number and let

$$\sum_{n=1}^{\infty} \frac{1}{|p_n|^h} = H.$$

Surround each pole p_n with a circle C_n of radius $|p_n|^{-h}$. Associate with C_1 any circles which touch or intersect it, add to the group any circles which touch or intersect these new circles, and so on. The group so formed will consist of a finite number of circles. For none of their centres can be more than

$$|p_1|^{-h} + 2\sum_{2}^{\infty} |p_n|^{-h} < 2H$$

distant from p_1, and if the circle of centre p_1, radius $2H$, contained an infinite number of poles they would have a limit point, *i.e.* the function would have an essential singularity.

Now take the first pole not in the first group and form a second group, and so on. The groups of circles may be called "nebulae" and the associated groups of poles "pole-clusters". The diameter $d(N)$ of a nebula N is always less than $2H$ and tends to zero as l, the distance of N from the origin, tends to infinity. For

$$(3.1) \qquad d(N) \leqslant \sum_{|p_n| \geqslant l} |p_n|^{-h} \to 0 \quad \text{as} \quad l \to \infty.$$

Let the poles of a typical cluster be $b_0, b_1, \ldots, b_k$, of orders $\lambda_0, \lambda_1, \ldots, \lambda_k$, and let

$$\lambda_0 + \lambda_1 + \ldots + \lambda_k = \mu,$$

so that

$$(3.2) \qquad f(z) = \frac{P(z)}{(z-b_0)^{\lambda_0}(z-b_1)^{\lambda_1}\ldots(z-b_k)^{\lambda_k}} + G(z),$$

where $P(z)$ is a polynomial of degree less than μ and $G(z)$ is regular inside and on the nebula associated with the b's.

$P(z)$ can be expressed in the form

$$(3.3) \quad B_\mu + B_{\mu-1}(z-b_k) + \ldots + B_{\mu-\lambda_k}(z-b_k)^{\lambda_k} + B_{\mu-\lambda_k-1}(z-b_k)^{\lambda_k}(z-b_{k-1}) + \ldots$$
$$+ B_1(z-b_k)^{\lambda_k}(z-b_{k-1})^{\lambda_{k-1}}\ldots(z-b_1)^{\lambda_1}(z-b_0)^{\lambda_0-1},$$

since, by induction, any polynomial of degree less than μ can be expressed as a linear combination of

$$1,\ z-b_k,\ (z-b_k)^2,\ \ldots,\ (z-b_k)^{\lambda_k}(z-b_{k-1}),\ \ldots.$$

On substitution for $P(z)$, (3.2) becomes

$$f(z)=\frac{B_1}{z-b_0}+\frac{B_2}{(z-b_0)^2}+\ldots+\frac{B_{\lambda_0}}{(z-b_0)^{\lambda_0}}$$
$$+\frac{B_{\lambda_0+1}}{(z-b_0)^{\lambda_0}(z-b_1)}+\ldots+\frac{B_\mu}{(z-b_0)^{\lambda_0}\ldots(z-b_k)^{\lambda_k}}+G(z).$$

Write

$$B=\max_{1\leqslant i\leqslant\mu}|B_i|,\quad R=\max_{0\leqslant i\leqslant k}|b_i|. \tag{3.4}$$

Then, by Lemma 1,

$$B_i=\frac{1}{2\pi i}\int_N\{f(z)-G(z)\}(z-b_0)^{\lambda_0}(z-b_1)^{\lambda_1}\ldots(z-b_s)^{\lambda_s}(z-b_{s+1})^{\nu}\,dz$$
$$(i=1,\ 2,\ \ldots,\ \mu),$$

where N is the boundary of the nebula and

$$\lambda_0+\lambda_1+\ldots+\lambda_s+\nu=i-1\quad(\nu\leqslant\lambda_{s+1}).$$

Since $G(z)$ is regular inside and on N this reduces to

$$B_i=\frac{1}{2\pi i}\int_N f(z)(z-b_0)^{\lambda_0}\ldots(z-b_{s+1})^{\nu}\,dz,$$

so that, by Lemma 2,

$$|B_i|\leqslant\frac{1}{2\pi}\int_N e^{(2R)^{\rho+\epsilon}}\{d(N)\}^{\lambda_0+\lambda_1+\ldots+\nu}\,ds\quad(R>R_0)$$
$$\leqslant\frac{1}{2\pi}\,e^{(2R)^{\rho+\epsilon}}\{d(N)\}^{i-1}\,2\pi d(N)$$
$$<e^{(2R)^{\rho+\epsilon}}\qquad(R>R_1),$$

using (3.1). Hence

$$\overline{\lim_{R\to\infty}}\ \frac{\log^+\log^+B}{\log R}\leqslant\rho. \tag{3.5}$$

This result should be compared with (1.5). It will be seen that the process of forming pole-clusters has yielded an inequality of the same nature as that of Macintyre and Wilson, but valid for all meromorphic functions.

4. A further lemma will be needed in the subsequent work.

LEMMA 3. *Let R_1, R_2, ... be a sequence of positive numbers tending to infinity, and let*

$$(4.1)\quad P_n(z) = D_{n,0} + D_{n,1}(z-d_{n,1}) + D_{n,2}(z-d_{n,1})(z-d_{n,2}) + \ldots + D_{n,\nu(n)}(z-d_{n,1})\ldots(z-d_{n,\nu(n)}),$$

where $d_{n,1}$, $d_{n,2}$, ..., $d_{n,\nu(n)}$, not necessarily distinct, are all in the circle $|z| \leqslant R_n$. Write

$$\delta_n(r) = \max_{|z|=r} |P_n(z)|, \quad D_n = \max_{0 \leqslant i \leqslant \nu(n)} |D_{n,i}|,$$

$$\alpha_1 = \overline{\lim_{R_n \to \infty}} \frac{\log^+ \log^+ \delta_n(R_n)}{\log R_n}, \quad \alpha_2 = \overline{\lim_{R_n \to \infty}} \frac{\log^+ \log^+ D_n}{\log R_n},$$

and denote by β any number not less than

$$\overline{\lim_{n \to \infty}} \frac{\log \nu(n)}{\log R_n}.$$

Then $$\max(\alpha_1, \beta) = \max(\alpha_2, \beta).$$

Proof. In the first place

$$\nu(n) < R_n^{\beta+\epsilon}, \quad \log D_n < R_n^{\alpha_2+\epsilon} \quad (n > n_\epsilon),$$

and
$$\begin{aligned} \delta_n(R_n) &\leqslant D_n\{1 + (2R_n) + (2R_n)^2 + \ldots + (2R_n)^{\nu(n)}\} \\ &\leqslant D_n(2R_n)^{\nu(n)+1} \qquad (R_n \geqslant 1), \end{aligned}$$

so that, for $n > n_\epsilon$,

$$\begin{aligned} \log \delta_n(R_n) &\leqslant \log D_n + \{\nu(n)+1\} \log(2R_n) \\ &< R_n^{\alpha_2+\epsilon} + \{R_n^{\beta+\epsilon}+1\} \log(2R_n). \end{aligned}$$

This gives $$\alpha_1 \leqslant \max(\alpha_2, \beta).$$

On the other hand, using Lemma 1, we have

$$D_{n,i} = \frac{1}{2\pi i} \int_C \frac{P_n(z)\,dz}{(z-d_{n,1})\ldots(z-d_{n,i+1})} \quad [i = 0, 1, \ldots, \nu(n)],$$

where $d_{n,\nu(n)+1} = 0$ and C is the circle $|z| = 2R_n$, so that

$$\begin{aligned} D_n &\leqslant \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta_n(2R_n)\,2R_n\,d\theta}{R_n^{i+1}} \\ &\leqslant 2\delta_n(2R_n) \quad (R_n \geqslant 1), \end{aligned}$$

and this gives $$\alpha_2 \leqslant \alpha_1.$$

5. The singular part of $f(z)$ at a pole-cluster has been denoted in (3.2) by

$$\frac{P(z)}{(z-b_0)^{\lambda_0}(z-b_1)^{\lambda_1}\dots(z-b_k)^{\lambda_k}}, \tag{5.1}$$

where $P(z)$ is a polynomial of degree less than μ, say

$$P(z) = P_0+P_1 z+\dots+P_{\mu-1}z^{\mu-1}. \tag{5.2}$$

The sum of the expressions (5.1) for all pole-clusters will not in general converge, but, as in Mittag-Leffler's theorem, we can overcome this difficulty by adding on to each singular part a suitable polynomial; in point of fact by subtracting from each singular part a certain number of the initial terms of its Taylor series. Define $-Q(z)$ to be the sum of the first n terms, so that

$$\frac{P(z)}{(z-b_0)^{\lambda_0}\dots(z-b_k)^{\lambda_k}}+Q(z) = R^{(n)}(z),$$

the remainder after n terms. In Mittag-Leffler's theorem n must be large enough to secure convergence, but is not restricted in the opposite sense. In the present case, however, it is necessary to assign limits to n in both directions. For, if we take too many terms of the Taylor series, it will be necessary to subtract the extra ones from the residual integral function $\Sigma a_n z^n$, and this may have the effect of making the order of the integral function too high. We shall choose n to be any integer satisfying the inequality

$$\log^+ P+Ap(R) \leqslant n \leqslant \log^+ P+Cp(R), \tag{5.3}$$

where P, $p(r)$, R are the numbers defined in (1.6), (1.8), (3.4), and $0<A<C$. A, C are the same for all pole-clusters, but n will, of course, be different for different clusters.

$R^{(n)}(z)$ can be expressed as a contour integral,

$$R^{(n)}(z) = \frac{1}{2\pi i}\int_C \frac{P(\zeta)\,d\zeta\, z^n}{(\zeta-b_0)^{\lambda_0}\dots(\zeta-b_k)^{\lambda_k}\zeta^n(\zeta-z)}, \tag{5.4}$$

where C is any simple closed contour which includes z but none of $b_0, b_1, \dots, b_k$.

Moreover, it is clear that

$$\left|\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{P(\zeta)\,d\zeta\, z^n}{(\zeta-b_0)^{\lambda_0}\dots(\zeta-b_k)^{\lambda_k}\zeta^n(\zeta-z)}\right| \leqslant \frac{\delta(r)|z|^n}{\{D(r,N)\}^{\mu} r^{n-1}\left|r-|z|\right|}, \tag{5.5}$$

where

$$\delta(r) = \max_{|\zeta|=r} |P(\zeta)|, \tag{5.6}$$

(5.7) $D(r, N) =$ distance of $|\zeta| = r$ from the nearest pole in N.

It will now be shown that the series is absolutely and uniformly convergent except in the neighbourhood of the poles.

Take a fixed value of z and consider a nebula N which satisfies the condition

$$|z| \leqslant \frac{1}{e^2} D(0, N) = \frac{D}{e^2} \quad \text{(say)}.$$

In (5.4) take C to be the circle $|\zeta| = D/e$. Then (as is evident from a diagram)

$$D(r, N) > \frac{D}{e}, \quad r - |z| > \frac{D}{e^2},$$

and (5.5) gives

$$\begin{aligned} |R^{(n)}(z)| &< \delta \left(\frac{D}{e}\right) \left(\frac{D}{e^2}\right)^n \left(\frac{e}{D}\right)^{\mu+n-1} \frac{e^2}{D} \\ &\leqslant P \frac{(D/e)^\mu - 1}{(D/e) - 1} D^{-\mu} e^{\mu-n+1} \\ &< P \left(\frac{D}{e}\right)^\mu D^{-\mu} e^{\mu-n+1} \quad (D > 2e) \\ &= P e^{-n+1} \\ &\leqslant e^{-A p(R)}. \end{aligned} \tag{5.8}$$

Suppose now that $R_1, R_2, \ldots, R_t$ are the values of R corresponding to t different nebulae, arranged in order of increasing magnitude. Since each nebula contains at least one pole it is evident that $p(R_1) \geqslant 1, p(R_2) \geqslant 2, \ldots,$ and so

$$\Sigma e^{-A p(R)},$$

summed over all nebulae, converges. This establishes the convergence of $\Sigma R^{(n)}(z)$. Let its sum be denoted by $h(z)$.

6. LEMMA 4. *The order of $h(z)$ is* $\max(\tau, \kappa)$.

Proof. τ, κ are defined by (1.6), (1.7). Write

$$\tau_1 = \overline{\lim_{R\to\infty}} \frac{\log^+ \log^+ \delta(R)}{\log R}, \quad \tau_2 = \overline{\lim_{R\to\infty}} \frac{\log^+ \log^+ B}{\log R}, \tag{6.1}$$

the numbers concerned being those defined in (3.4), (5.6). Evidently

$$\log|b_0| \sim \log R,$$

so

$$\tau = \overline{\lim_{R\to\infty}} \frac{\log^+\log^+ P}{\log R}. \tag{6.2}$$

Now apply Lemma 3 to the set of polynomials $P(z)$ considered first in the form (3.3) and then in the form (5.2). In the latter case the d's are all zero. The conclusions are that

$$\max(\tau_1, \kappa) = \max(\tau_2, \kappa), \quad \max(\tau_1, \kappa) = \max(\tau, \kappa).$$

Again, (3.5) may be applied to $h(z)$, and gives

$$\tau_2 \leqslant \rho_1 = \text{order of } h(z),$$

and, by definition,

$$\kappa \leqslant \rho_1.$$

These inequalities may be combined together and give

$$\max(\tau, \kappa) = \max(\tau_1, \kappa) = \max(\tau_2, \kappa) \leqslant \rho_1. \tag{6.3}$$

To prove the contrary inequality we proceed as follows.

With the origin as centre, draw circles through all points of all the nebulae. These circles will cut off on the real positive axis a set of measure not greater than $2H$. There are, therefore, arbitrarily large circles, of centre the origin, which do not intersect any of the nebulae. Let z be a point on one of these circles Γ_0.

The nebulae may be divided into three classes:

(i) those which are inside Γ_0;

(ii) those which are outside Γ_0 but do not satisfy the condition

$$D \geqslant e^2|z|;$$

(iii) those which satisfy this condition.

If N belongs to the first class,

$$R^{(n)}(z) = \frac{1}{2\pi i}\left\{\int_{\Gamma_2} - \int_N\right\} \frac{P(\zeta)\,d\zeta\, z^n}{(\zeta-b_0)^{\lambda_0}\ldots(\zeta-b_k)^{\lambda_k}\zeta^n(\zeta-z)},$$

where Γ_2 is any simple closed contour surrounding Γ_0. Take Γ_2 to be the circle, centre the origin, radius $e|z|$. Then

$$|R^{(n)}(z)| \leqslant \left|\frac{1}{2\pi i}\int_{\Gamma_2}\right| + \left|\frac{1}{2\pi i}\int_N\right|,$$

and, using (5.5), we have

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\Gamma_2}\right| &\leqslant \frac{\delta(e|z|)\,|z|^n}{|z|^\mu (e|z|)^{n-1}|z|}\\ &< P.(e|z|)^\mu |z|^{-\mu} e^{-n+1} = Pe^{\mu-n+1}\\ &\leqslant e^{\mu+1},\end{aligned}$$

by (5.3). Again, since, if ζ is in the boundary of N, $|\zeta| \geqslant D$, $|\zeta - z| \geqslant R^{-h}$ and the length of the boundary does not exceed that of all the circles C_n, *i.e.* $4\pi H$,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_N\right| &\leqslant \frac{1}{2\pi}\int_N \frac{|P(\zeta)|\,|z|^n}{R^{-h\mu} D^n R^{-h}}\,ds\\ &\leqslant \frac{1}{2\pi}\,\delta(|z|)\,|z|^n\, 4\pi H\, D^{-n}\, R^{h(\mu+1)}\\ &\leqslant 2H\,\delta(|z|)\,|z|^n\,|z|^{h(\mu+1)},\end{aligned}$$

assuming that $D \geqslant 1$. This condition is verified for all nebulae except, possibly, a finite number. If there are any such exceptional nebulae we may subtract the corresponding terms from $h(z)$ without affecting the order.

Combining these results we have

$$|R^{(n)}(z)| \leqslant e^{\mu+1} + 2H\delta(|z|)\,|z|^{n+h(\mu+1)} \tag{6.4}$$

for a nebula of the first class.

If N belongs to the second class,

$$R^{(n)}(z) = \frac{1}{2\pi i}\int_{\Gamma_3} \frac{P(\zeta)\,d\zeta\, z^n}{(\zeta-b_0)^{\lambda_0}\ldots(\zeta-b_k)^{\lambda_k}\,\zeta^n(\zeta-z)},$$

where Γ_3 may be taken to be the circle, centre the origin, which passes through the mid-point of the line joining z to the nearest pole in N. (5.5) gives

$$|R^{(n)}(z)| \leqslant \frac{\delta(e^2|z|)\,|z|^n}{\{D(r_3, N)\}^\mu\, r_3^{\,n}\,(r_3-|z|)},$$

when r_3 is the radius of Γ_3. Now it is evident from a diagram that

$$D(r_3, N) > \tfrac{1}{2}R^{-h}, \quad r_3 - |z| > \tfrac{1}{2}R^{-h},$$

so that

$$|R^{(n)}(z)| < \delta(e^2|z|)(2R^h)^{\mu+1}$$
$$< \delta(e^2|z|)(2e^2|z|)^{(1+h)(\mu+1)}.$$

This result, combined with (6.4), shows that

$$|R^{(n)}(z) < (1+2H)\,\delta(e^2|z|)(2e^2|z|)^{n+(1+h)(\mu+1)}, \tag{6.5}$$

if N belongs to the first or second class.

For nebulae of the third class we have the inequality (5.8).

Let m denote the number of nebulae in the first two classes. Evidently

$$m \leqslant p(e^2|z|),$$

so the maximum modulus of $h(z)$, $M(|z|)$ say, satisfies the inequality

$$M(|z|) \leqslant \sum_1^m |R^{(n)}(z)| + \sum_{m+1}^{\infty} |R^{(n)}(z)| \tag{6.6}$$
$$< p(e^2|z|)(1+2H)\,\delta'(2e^2|z|)^{n'+(1+h)(\mu'+1)} + \sum_{i=1}^{\infty} e^{-Ai},$$

where δ', n', μ' denote the largest possible values of $\delta(e^2|z|)$, n, μ for nebulae of the first two classes. Now, for such nebulae,

$$\mu \leqslant p(e^2|z|), \quad n \leqslant \log^+ P + Cp(R) \leqslant \log^+ P' + Cp(e^2|z|),$$

where P' is the largest value of P for nebulae of the first two classes, *i.e.*

$$\log^+ P' = \log^+\left\{\max_{R \leqslant e^2|z|} P\right\} < |z|^{\tau+\epsilon} \quad (|z| \geqslant R'_\epsilon),$$

by (6.2). Again,

$$p(|z|) < |z|^{\kappa+\epsilon} \quad (|z| \geqslant R''_\epsilon),$$

so that

$$n' + (1+h)(\mu'+1) < |z|^{\kappa+\epsilon} + |z|^{\tau+\epsilon} \quad (|z| \geqslant R'''_\epsilon)$$

The only other term of importance in (6.6) is δ' and, by (6.1), this does not exceed $\exp|z|^{\tau_1+\epsilon}$. Hence, for the values of z under consideration,

$$M(|z|) < \exp|z|^{\gamma+\epsilon} \quad (|z| \geqslant R_\epsilon),$$

where

$$\gamma = \max(\kappa, \tau, \tau_1) = \max(\tau, \kappa),$$

by (6.3). This gives

$$m(|z|) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| h(|z|e^{i\theta}) \right| d\theta$$

$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} \log^+ M(|z|)\, d\theta$$

$$< |z|^{\gamma+\epsilon} \qquad (|z| \geqslant R_\epsilon),$$

and the Nevanlinna function is

$$T(|z|) = m(|z|) + \int_0^{|z|} \frac{p(t)-p(o)}{t}\, dt + p(o)\log|z|$$

$$< |z|^{\gamma+\epsilon} + |z|^{\kappa+\epsilon} \qquad (|z| \geqslant R_\epsilon)$$

$$\leqslant 2|z|^{\gamma+\epsilon}.$$

This inequality is valid for all values of $|z|$ except, possibly, a set of measure $2H$. If $|z_1|$ belongs to this exceptional set, there must be a number $|z_2|$ not belonging to the exceptional set such that

$$|z_2| \leqslant |z_1| + 2H,$$

and, since $T(r)$ is an increasing function,

$$T(|z_1|) \leqslant T(|z_2|) \leqslant 2\{|z_1|+2H\}^{\gamma+\epsilon}.$$

This is true for all sufficiently large values of $|z_1|$, so

$$\rho_1 = \overline{\lim_{|z|\to\infty}} \frac{\log T(|z|)}{\log|z|} \leqslant \gamma = \max(\tau, \kappa),$$

an inequality in the sense opposite to (6.3). This completes the proof of Lemma 4.

7. $f(z)-h(z)$ is a meromorphic function with no singularities, *i.e.* an integral function. Denote it by $g(z) = \Sigma a_n z^n$, so that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \Sigma \left\{ \frac{P(z)}{(z-b_0)^{\lambda_0} \ldots (z-b_k)^{\lambda_k}} + Q(z) \right\}. \tag{7.1}$$

The expression on the right will be called a *normal form* of $f(z)$. It is not unique, since different values of h will lead to different systems of pole-clusters, and, moreover, there is a certain amount of latitude in the choice

of the integer n corresponding to a given pole-cluster, *i.e.* in the choice of $Q(z)$.

Let ρ denote the order of $f(z)$ and σ that of $g(z)$. It has been seen that $h(z)$ is of order $\gamma = \max(\tau, \kappa)$.

$h(z)$ can be expressed as the quotient, $h_1(z)/h_2(z)$, of two integral functions, $h_2(z)$ being of order κ and $h_1(z)$ of order γ if $\kappa < \gamma$ and of order not exceeding γ if $\kappa = \gamma$. Let us use the notation (κ), $(<\gamma)$, ... to denote "a function (either integral or meromorphic) of order κ", "a function of order less than γ", etc.

There are five cases to be considered:

(i) $\sigma < \gamma, \quad \kappa < \gamma$:

$$f(z) = \frac{g(z)\,h_2(z)+h_1(z)}{h_2(z)} = \frac{(<\gamma)+(\gamma)}{(\kappa)} = (\gamma),$$

since it is evident that the numerator and the denominator have no factors in common.

(ii) $\sigma < \gamma, \ \kappa = \gamma$:

$$f(z) = \frac{(\gamma)+(\leqslant\gamma)}{(\gamma)} = (\gamma).$$

(iii) $\sigma = \gamma, \quad \kappa < \gamma$:

$$f(z) = g(z)+h(z) = (\gamma)+(\gamma) = (\leqslant\gamma).$$

On the other hand (6.3) gives

$$\max(\tau_2, \kappa) = \max(\tau, \kappa) = \gamma,$$

or

$$\tau_2 = \gamma,$$

since $\kappa < \gamma$, and (3.5) gives

$$\rho \geqslant \tau_2.$$

Hence

$$f(z) = (\geqslant\gamma),$$

and so

$$f(z) = (\gamma).$$

(iv) $\sigma = \gamma = \kappa$:

$$f(z) = \frac{(\gamma)+(\leqslant\gamma)}{(\gamma)} = (\gamma).$$

(v) $\sigma > \gamma$:

$$f(z) = \frac{(\sigma)+(\leqslant\gamma)}{(\kappa)} = (\sigma)$$

It will be seen that in every case

$$\rho = \max(\sigma, \gamma) = \max(\sigma, \tau, \kappa).$$

Let

$$(7.2) \qquad f_1(z) = \Sigma' a_n z^n + \Sigma' \left\{ \frac{P(z)}{(z-b_0)^{\lambda_0} \dots (z-b_k)^{\lambda_k}} + Q(z) \right\}$$

be a part of the series (7.1), *i.e.* a series obtained by omitting certain terms in (7.1), without breaking up individual terms. It is evident from the preceding argument that the order of $f_1(z)$ cannot exceed that of $f(z)$. It should be added that $f_1(z)$ is not necessarily in normal form, since (5.3) may become false when $p_1(r)$ is substituted for $p(r)$. To obtain a normal form it may be necessary to transfer some terms from the $Q(z)$'s to $\Sigma' a_n z^n$.

The results may be summed up as follows.

THEOREM. *Let*

$$(7.3) \qquad f(z) = g(z) + \Sigma \left\{ \frac{P_0 + P_1 z + \dots + P_{\mu-1} z^{\mu-1}}{(z-b_0)^{\lambda_0} \dots (z-b_k)^{\lambda_k}} + Q(z) \right\}$$

be a meromorphic function of order ρ expressed in normal form. Then

$$(7.4) \qquad \rho = \max(\sigma, \tau, \kappa),$$

where σ is the order of $g(z)$, κ is the exponent of convergence of the poles, and

$$\tau = \overline{\lim_{|b_0| \to \infty}} \frac{\log^+ \log^+ P}{\log |b_0|} \qquad \left(P = \max_{0 \leqslant i < \mu} |P_i| \right).$$

The order of a part of the series (7.3) cannot exceed ρ.

We have assumed during the proof that ρ, σ, τ, κ are all finite, but it is easy to see that the result remains true if any of these numbers is infinite.

Applications.

8. Let the plane be subdivided into a finite or infinite number of regions $D_1, D_2, \dots$ (certain points being on the boundary of two or more regions), in any way such that each pole-cluster lies entirely within or on the boundary of a single region. A region may consist of disconnected parts. Then the series (7.3) can be broken up into parts $f_1(z), f_2(z), \dots$, each part containing the poles in one region (poles on the boundary of two regions being assigned to one or other of them); and these functions are all of order not exceeding ρ. The following result has therefore been established.

If $f(z)$ is a meromorphic function of order ρ and the plane is subdivided into regions D_1, D_2, ... in such a way that each pole-cluster lies entirely within or on the boundary of a single region, there are meromorphic functions $f_1(z)$, $f_2(z)$, ..., of order not exceeding ρ, such that the poles of $f_1(z)$ are in D_1, those of $f_2(z)$ in D_2, ..., and

$$f(z) = f_1(z) + f_2(z) + \dots . \tag{8.1}$$

If ρ is finite the projections of the nebulae on the real axis form a set of measure not exceeding $2H$. There is, therefore, a real number d not included in this set and the line $\Re z = d$ divides the plane into two regions of the type described. In this case the last result is as follows:

If $f(z)$ is a meromorphic function of finite order ρ, there is a real number d and meromorphic functions $f_1(z)$, $f_2(z)$, of order not exceeding ρ, such that the poles of $f_1(z)$ are in $\Re z < d$, those of $f_2(z)$ are in $\Re z > d$, and

$$f(z) = f_1(z) + f_2(z). \tag{8.2}$$

This result enables us to remove a restriction in a theorem concerning the "sum" of a meromorphic function, namely

If $f(z)$ is a meromorphic function of order ρ whose poles are all on the same side of a line $\Re z = d$, there is a meromorphic function $g(z)$, of order less than or equal to $\rho+1$, such that

$$g(z+1) - g(z) = f(z). \tag{8.3}$$

Apply this result to the functions $f_1(z)$, $f_2(z)$ of (8.2). The sum of the resulting functions $g_1(z)$, $g_2(z)$ will be a function $g(z)$ satisfying (8.3). Thus, if $f(z)$ is any meromorphic function of order ρ, there is a meromorphic function $g(z)$, or order less than or equal to $\rho+1$, which satisfies (8.3).

9. Macintyre and Wilson proved that, if p_1, p_2, ... and A_1, A_2, ... are complex numbers satisfying the relations

$$|p_n| \leqslant |p_{n+1}|, \quad |p_n| \to \infty, \tag{9.1}$$

$$\overline{\lim_{n\to\infty}} \frac{\log^+ \log^+ |A_n|}{\log |p_n|} = \nu, \tag{9.2}$$

$$\overline{\lim_{n\to\infty}} \frac{\log n}{\log |p_n|} = \kappa, \tag{9.3}$$

then there exists a meromorphic function of order not exceeding $\max(\nu, \kappa)$ with simple poles $p_1, p_2, \ldots$ and corresponding residues $A_1, A_2, \ldots$.

We can extend this result to the case where the poles are of any order.

Let there be given principal parts

$$\frac{E_1}{z-b}+\frac{E_2}{(z-b)^2}+\ldots+\frac{E_\lambda}{(z-b)^\lambda} \quad (|b|\to\infty), \tag{9.4}$$

and let κ be the exponent of convergence of the poles and let

$$\overline{\lim_{|b|\to\infty}} \frac{\log^+\log^+ E}{\log|b|}=\nu \quad \left(E=\max_{1\leqslant i\leqslant\lambda}|E_i|\right). \tag{9.5}$$

Then there is a meromorphic function of order not exceeding $\max(\nu, \kappa)$ *with the assigned principal parts.*

Proof. First form the poles into clusters in the manner described above. Let a typical cluster be $b_0, b_1, \ldots, b_k$, and let

$$\frac{E_1^{(i)}}{z-b_i}+\frac{E_2^{(i)}}{(z-b_i)^2}+\ldots+\frac{E_{\lambda_i}^{(i)}}{(z-b_i)^{\lambda_i}}=\frac{E_i(z)}{(z-b_i)^{\lambda_i}} \quad (i=0, 1, \ldots, k).$$

Add together the terms on the right for $i=0, 1, \ldots, k$, and bring to a common denominator. The numerator will be

$$P(z)=E_0(z)(z-b_1)^{\lambda_1}\ldots(z-b_k)^{\lambda_k}+\ldots.$$

Now
$$|E_i(z)|\leqslant E^{(i)}(|z|+|b_i|)^{\lambda_i} \quad (i=0, 1, \ldots, k),$$

where $E^{(i)}=\max_j|E_j^{(i)}|$, so that

$$\begin{aligned}\delta(|b_0|)&=\max_{|z|=|b_0|}|P(z)|\\ &\leqslant e^{|b_0|^{\nu+\epsilon}}(|b_0|+|b_1|)^{\lambda_1}\ldots(|b_0|+|b_k|)^{\lambda_k}+\ldots\\ &<e^{|b_0|^{\nu+\epsilon}}(3|b_0|)^{\mu}+\ldots\\ &=(k+1)\,e^{|b_0|^{\nu+\epsilon}}(3|b_0|)^{\mu},\end{aligned}$$

and, since
$$k, \mu\leqslant p(R)<R^{\kappa+\epsilon} \quad (|b_0|>K_\epsilon),$$

where $R=\max|b_i|$, this gives, for the number defined in (6.1),

$$\tau_1\leqslant\max(\nu, \kappa).$$

The normal series

$$\Sigma\left\{\frac{P(z)}{(z-b_0)^{\lambda_0}\dots(z-b_k)^{\lambda_k}}+Q(z)\right\}$$

is of order max (τ_1, κ), and so of order not exceeding $\max(\nu, \kappa)$, and has the assigned poles and principal parts.

References.

E. Borel, *Les fonctions méromorphes* (Paris, 1903), ch. 4.

A. J. Macintyre and R. Wilson, "On the order of interpolated integral functions and of meromorphic functions with given poles", *Quart. J. of Math.* (Oxford), 5 (1934), 211–220.

M. Mursi and C. E. Winn, "On an interpolated integral function of given order", *Quart. J. of Math.* (Oxford), 4 (1933), 173–179.

J. M. Whittaker, "The 'sum' of a meromorphic function", *Journal London Math. Soc.*, 8 (1933), 62–69.

The University,
Liverpool.

A LOCAL PROPERTY OF FOURIER SERIES

By L. S. BOSANQUET *and* A. C. OFFORD.

[Received 19 September, 1934.—Read 15 November, 1934.]

Let $f(t)$ be integrable L in the interval $(-\pi, \pi)$ and let its Fourier series be

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)=\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}A_n(t). \tag{1}$$

It is known that the summability (C, γ), $\gamma \geqslant 0$, of the series $\Sigma A_n = \Sigma A_n(x)$ depends only on the properties of $f(t)$ near the point $t = x$. But when $\gamma < 0$ this is no longer so. One reason for the failure of the property is not hard to find. We know only that $A_n = o(1)$, and no more need be true whatever the behaviour of the function near the point. The object of this note is to show that this is the only reason for the failure of the property. We show in fact that, if $A_n = o(n^\gamma)$*, $-1 < \gamma < 0$, then the summability (C, γ) of the Fourier series depends only on the local properties of the function. There are analogous results for boundedness (C).

It is known† that a necessary condition for summability to s by a Cesàro mean of negative order is that

$$\Phi(t)=\int_0^t\phi(u)\,du=\int_0^t\tfrac{1}{2}\{f(x+u)+f(x-u)-2s\}\,du=o(t) \tag{2}$$

as $t \to 0$, and so there is no loss of generality in supposing that this condition is fulfilled. The problem as stated above can be solved without using (2) explicitly‡, but its use enables us to obtain conditions in a simple form.

---

* It may be remarked that this in itself does not imply $a_n = o(n^\gamma)$ and $b_n = o(n^\gamma)$. As an example, let

$$A_n(t)=\frac{\cos nt}{n^{\frac{3}{4}}\log n}+\frac{\sin nt}{n^{\frac{3}{4}}}.$$

Then $A_n(t) = o(n^{-\frac{3}{4}})$ for $t = 0$ and $t = \pi$, but not for $0 < t < \pi$. Also $\Sigma A_n(t)$ is summable $(C, -\frac{3}{4})$ for $t = \pi$, but not for $t = 0$.

† G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 28 (1928), 301–311.

‡ By omitting the discussion of J_1, J_2, and J_4, below, and considering only the part of J_3 taken over the range (δ, π).

THEOREM 1. *Necessary and sufficient conditions for the Fourier series of* $f(t)$ *to be summable* $(C, -\alpha)$, $0 < \alpha < 1$, *to sum* s *for* $t = x$ *are that* (i) $A_n = o(n^{-\alpha})$, (ii) *condition* (2) *is satisfied, and* (iii)

$$n^{\alpha} \int_0^{\delta} \phi(t) \left(1 - \frac{t}{\delta}\right) \frac{\sin(n\,;\,t)}{t^{1-\alpha}}\, dt = o(1) \tag{3}$$

as $n \to \infty$, *where* $0 < \delta < \pi$, *but* δ *is otherwise arbitrary, and* $(n\,;\,t)$ *is written for* $(n + \frac{1}{2} - \frac{1}{2}\alpha)t + \frac{1}{2}\alpha\pi$.

Alternatively, (iii) *can be replaced by* (iii)'*

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} \left| n^{\alpha} \int_{1/n}^{\delta} \phi(t) \left(1 - \frac{t}{\delta}\right) \frac{\sin(n\,;\,t)}{t^{1-\alpha}}\, dt \right| = 0. \tag{4}$$

Proof. A necessary and sufficient condition for summability $(C, -\alpha)$ to s, in its unreduced form†, is that

$$\int_0^{\pi} \phi(t) \kappa^{-\alpha}(n, t)\, dt = o(1) \tag{5}$$

as $n \to \infty$, where $\kappa^{-\alpha}(n, t)$ is the n-th Cesàro mean of order $-\alpha$ of

$$\frac{1}{\pi} + \frac{2}{\pi} \sum_{\nu=1}^{n} \cos \nu t.$$

It is known‡ that, for $0 < t < \pi$,

$$|\kappa^{-\alpha}(n, t)| \leqslant An \tag{6}$$

$$\left| \frac{d}{dt} \kappa^{-\alpha}(n, t) \right| \leqslant An^2, \tag{7}$$

$$|\kappa^{-\alpha}(n, t) - K(n, t)| \leqslant An^{-1}t^{-2}, \tag{8}$$

$$\left| \frac{d}{dt} \{\kappa^{-\alpha}(n, t) - K(n, t)\} \right| \leqslant An^{-1}t^{-3}, \tag{9}$$

where

$$K(n, t) = \frac{2}{\pi} \frac{\Gamma(n+1)\,\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)} \frac{\sin(n\,;\,t)}{(2 \sin \frac{1}{2}t)^{1-\alpha}} \tag{10}$$

and A denotes a number independent of n and t§. It follows that, whenever

---

* A third form is $\lim_{\delta \to 0} \overline{\lim_{\tau \to \infty}}\, \overline{\lim_{n \to \infty}} \left| \int_{\tau/n}^{\delta} \right| = 0.$

† See J. J. Gergen, *Quart. J. of Math.* (Oxford), 1 (1930), 252–275, where other references are given.

‡ See J. J. Gergen, *loc. cit.*, 264.

§ *I.e.*, A is a constant depending only on α. Later we shall use B to denote a number depending only on δ and α, and C to denote an absolute constant. These are not necessarily the same at each occurrence.

(2) is satisfied, (5) can be replaced by

$$\int_0^\pi \phi(t)\, K(n, t)\, dt = o(1). \tag{11}$$

To prove this we write

$$\int_0^\pi \phi(t)\{\kappa^{-\alpha}(n, t) - K(n, t)\}\, dt = \Big[\Phi(t)\{\kappa^{-\alpha}(n, t) - K(n, t)\}\Big]_0^\pi - \int_0^\pi \Phi(t)\, \frac{d}{dt}\{\kappa^{-\alpha}(n, t) - K(n, t)\}\, dt.$$

By (2), (6), (8), and (10) the integrated term is $O(n^{-1})$. Now write

$$\begin{aligned}\int_0^\pi \Phi(t)\, \frac{d}{dt}\{\kappa^{-\alpha}(n, t) - K(n, t)\}\, dt &= \int_0^{1/n} \Phi(t)\, \frac{d}{dt}\, \kappa^{-\alpha}(n, t)\, dt - \int_0^{1/n} \Phi(t)\, \frac{d}{dt}\, K(n, t)\, dt \\ &\quad + \int_{1/n}^\pi \Phi(t)\, \frac{d}{dt}\{\kappa^{-\alpha}(n, t) - K(n, t)\}\, dt \\ &= J_1 + J_2 + J_3.\end{aligned}$$

Then, by (2) and (7),

$$J_1 = \int_0^{1/n} o(t)\, O(n^2)\, dt = o(1);$$

by (2) and (10),

$$J_2 = \int_0^{1/n} o(t)\{O(n^\alpha t^{\alpha-2}) + O(n^{\alpha+1} t^{\alpha-1})\}\, dt = o(1);$$

and, by (2) and (9),

$$J_3 = \int_{1/n}^\pi o(t)\, O(n^{-1} t^{-3})\, dt = o(1).$$

We next show that (11) can be replaced by

$$n^\alpha \int_0^\pi \phi(t)\, \frac{\sin(n\,;\, t)}{t^{1-\alpha}}\, dt = o(1). \tag{12}$$

Let $$h(t) = \frac{1}{(2 \sin \frac{1}{2}t)^{1-\alpha}} - \frac{1}{t^{1-\alpha}} \quad (0 < t \leqslant \pi), \qquad h(0) = 0.$$

Then we have to show that

$$I=\int_0^\pi \phi(t)\,h(t)\,\sin(n\,;\ t)\,dt=o(n^{-\alpha}) \tag{13}$$

as $n\to\infty$. It is easily seen that, for $0\leqslant t\leqslant\pi$, $h(t)$ is continuous, $h'(t)$ is bounded, and $h''(t)$ is integrable L. In particular, $h(t)$ is of bounded variation, and so, by a well-known theorem*, we have, writing $A_{-\nu}=A_\nu$,

$$\begin{aligned}I&=\tfrac{1}{2}A_0\int_0^\pi h(t)\,\sin(n\,;\ t)\,dt+\sum_{\nu=1}^{\infty}A_\nu\int_0^\pi h(t)\cos\nu t\,\sin(n\,;\ t)\,dt\\&=\tfrac{1}{2}\sum_{\nu=-\infty}^{\infty}A_\nu\int_0^\pi h(t)\,\sin(n-\nu\,;\ t)\,dt\\&=-\tfrac{1}{2}\sum_{\nu=-\infty}^{\infty}A_\nu\left[h(t)\,\frac{\cos(n-\nu\,;\ t)}{n-\nu+\frac{1}{2}-\frac{1}{2}\alpha}\right]_0^\pi+\tfrac{1}{2}\sum_{\nu=-\infty}^{\infty}A_\nu\int_0^\pi h'(t)\,\frac{\cos(n-\nu\,;\ t)}{n-\nu+\frac{1}{2}-\frac{1}{2}\alpha}\,dt.\end{aligned}$$

Since $h(0)=0$ and $\cos(n-\nu+\frac{1}{2})\pi=0$, the integrated terms vanish, and so, integrating by parts once more, we obtain

$$I=\tfrac{1}{2}\sum_{\nu=-\infty}^{\infty}A_\nu\left[h'(t)\,\frac{\sin(n-\nu\,;\ t)}{(n-\nu+\frac{1}{2}-\frac{1}{2}\alpha)^2}\right]_0^\pi-\tfrac{1}{2}\sum_{\nu=-\infty}^{\infty}A_\nu\int_0^\pi h''(t)\,\frac{\sin(n-\nu\,;\ t)}{(n-\nu+\frac{1}{2}-\frac{1}{2}\alpha)^2}\,dt.$$

Since $h'(t)$ is bounded and $h''(t)$ is integrable L, we have

$$|I|\leqslant A\,\Sigma'\,\frac{|A_\nu|}{(n-\nu)^2}+A\,|A_n|, \tag{14}$$

where Σ' denotes summation over $-\infty<\nu\leqslant n-1$, $n+1\leqslant\nu<\infty$. Now let $\epsilon>0$ be given, and choose $N=N(\epsilon)$ so that $|A_\nu|\leqslant\epsilon\,|\nu|^{-\alpha}$ for $|\nu|>N$. Then, since $n+\nu\geqslant n-\nu$ for $\nu\geqslant 0$, we have

$$\begin{aligned}|I|&\leqslant A\sum_{\nu=0}^{N}\frac{1}{n^2}+A\epsilon\left\{\sum_{\nu=N+1}^{n-1}+\sum_{\nu=n+1}^{\infty}\right\}\frac{\nu^{-\alpha}}{(n-\nu)^2}+o(n^{-\alpha})\\&=O(n^{-2})+\Sigma_1+\Sigma_2+o(n^{-\alpha}).\end{aligned}$$

Then

$$|\Sigma_2|\leqslant A\epsilon n^{-\alpha}\sum_{\nu=n+1}^{\infty}(n-\nu)^{-2}\leqslant A\epsilon n^{-\alpha},$$

and

$$\begin{aligned}|\Sigma_1|&\leqslant A\epsilon\left\{\sum_{1\leqslant\nu\leqslant\frac{1}{2}n}+\sum_{\frac{1}{2}n<\nu\leqslant n-1}\right\}\frac{\nu^{-\alpha}}{(n-\nu)^2}\\&\leqslant A\epsilon n^{-2}\sum_{1\leqslant\nu\leqslant\frac{1}{2}n}\nu^{-\alpha}+A\epsilon n^{-\alpha}\sum_{\frac{1}{2}n<\nu\leqslant n-1}(n-\nu)^{-2}\\&\leqslant A\epsilon\{n^{-1-\alpha}+n^{-\alpha}\}\leqslant A\epsilon n^{-\alpha}.\end{aligned}$$

Since ϵ is arbitrary, it follows that $I=o(n^{-\alpha})$.

---

* E. W. Hobson, *Functions of a real variable*, 2 (1926), 567.

We show now that, if $0 < \delta < \pi$,

$$n^\alpha \int_0^\delta \phi(t) \frac{t}{\delta} \frac{\sin(n\,;\,t)}{t^{1-\alpha}}\,dt + n^\alpha \int_\delta^\pi \phi(t) \frac{\sin(n\,;\,t)}{t^{1-\alpha}}\,dt = o(1) \qquad (15)$$

as $n \to \infty$. Let

$$\chi(t) = \begin{cases} t^{\alpha-1} & (\delta \leqslant t \leqslant \pi), \\ \delta^{-1} t^\alpha & (0 \leqslant t \leqslant \delta). \end{cases} \qquad (16)$$

Then we have to show that

$$I' = \int_0^\pi \phi(t)\chi(t) \sin(n\,;\,t)\,dt = o(n^{-\alpha}). \qquad (17)$$

Let Σ' be defined as in (14), and let $\mu = \min(|n-\nu|^{-1}, \delta)$. Then, proceeding as in the proof of (13), we obtain

$$I' = \tfrac{1}{2}\Sigma' A_\nu \int_0^\pi \chi'(t) \frac{\cos(n-\nu\,;\,t)}{n-\nu+\frac{1}{2}-\frac{1}{2}\alpha}\,dt + o(n^{-\alpha})$$

$$= \tfrac{1}{2}\Sigma' A_\nu \int_0^\mu \chi'(t) \frac{\cos(n-\nu\,;\,t)}{n-\nu+\frac{1}{2}-\frac{1}{2}\alpha}\,dt + \tfrac{1}{2}\Sigma' A_\nu \left[\chi'(t) \frac{\sin(n-\nu\,;\,t)}{(n-\nu+\frac{1}{2}-\frac{1}{2}\alpha)^2}\right]_{\mu,\,\delta+0}^{\delta-0,\,\pi}$$

$$- \Sigma' A_\nu \int_\mu^\pi \chi''(t) \frac{\sin(n-\nu\,;\,t)}{(n-\nu+\frac{1}{2}-\frac{1}{2}\alpha)^2}\,dt + o(n^{-\alpha}),$$

where the symbolism denotes that the integrations by parts are taken separately over the ranges (μ, δ) and (δ, π). Hence, substituting for $\chi(t)$ from (16), we find that

$$|I'| \leqslant B\Sigma' \frac{|A_\nu|}{|n-\nu|^{\alpha+1}} + o(n^{-\alpha}).$$

This inequality can be treated in the same manner as (14), and we thus obtain (17).

On combining (12) and (15) we obtain (3).

Again, since the integral in (5) is independent of δ, it is necessary and sufficient for the truth of (5) that its modulus should be annihilated by the operation $\lim_{\delta\to 0} \overline{\lim_{n\to\infty}}$. But the integrals in (11), (12), and (15) are also annihilated by $\lim_{\delta\to 0} \overline{\lim_{n\to\infty}}$. Hence (4) will be established if we show that

$$J_4 = \int_0^{1/n} \phi(t)\left(1 - \frac{t}{\delta}\right) \frac{\sin(n\,;\,t)}{t^{1-\alpha}}\,dt = o(n^{-\alpha}).$$

But, on integrating by parts, we obtain, in virtue of (2),

$$J_4 = \left[\Phi(t)\left(1-\frac{t}{\delta}\right)\frac{\sin(n\,;\,t)}{t^{1-\alpha}}\right]_0^{1/n} - \int_0^{1/n}\Phi(t)\,\frac{d}{dt}\left\{\left(1-\frac{t}{\delta}\right)\frac{\sin(n\,;\,t)}{t^{1-\alpha}}\right\}dt$$

$$= o\left(\frac{1}{n}\right)O(n^{1-\alpha}) + \int_0^{1/n} o(t)\{O(t^{\alpha-1}) + O(nt^{\alpha-1}) + O(t^{\alpha-2})\}\,dt$$

$$= o(n^{-\alpha}).$$

This completes the proof.

In the next theorem, without further use of (2), we establish a sufficient condition for the truth of (4). This is less obvious than one might expect, owing to the presence of the repeated limit, and the proof seems to be of independent interest.

THEOREM 2. *If*

$$\lim_{\delta\to 0}\ \overline{\lim_{n\to\infty}}\left|n^{\alpha}\int_{1/n}^{\delta}\phi(t)\,\frac{\sin(n\,;\,t)}{t^{1-\alpha}}\,dt\right| = 0, \tag{18}$$

then (4) *is true**.

Proof. Let

$$H(t) = \begin{cases} 1-\delta^{-1}t & (0\leqslant t\leqslant\delta), \\ 0 & (\delta\leqslant t\leqslant\pi), \end{cases} \tag{19}$$

and let the Fourier cosine series of $H(t)$ be $\frac{1}{2}c_0+\Sigma c_\nu\cos\nu t$. Then it is easily verified that

$$c_0 = \frac{\delta}{\pi}, \quad c_\nu = \frac{2}{\pi}\,\frac{1-\cos\nu\delta}{\delta\nu^2} \quad (\nu>0).$$

In particular we notice that $|c_\nu|\leqslant C\delta$ and $|c_\nu|\leqslant C\delta^{-1}\nu^{-2}$. Hence

$$\sum_{\nu=0}^{\infty}|c_\nu| \leqslant C\sum_{\nu\leqslant\delta^{-1}}\delta + C\sum_{\nu>\delta^{-1}}\delta^{-1}\nu^{-2} \leqslant C,$$

and

$$\left|\sum_{\nu>n} c_\nu\cos\nu t\right| \leqslant C\sum_{\nu>n}\delta^{-1}\nu^{-2} \leqslant C\delta^{-1}n^{-1} \leqslant Bn^{-1}.$$

---

* There is an analogous theorem with $\lim_{\delta\to 0}\overline{\lim_{\tau\to\infty}}\ \overline{\lim_{n\to\infty}}\left|\int_{\tau/n}^{\delta}\right|$. In the proof of this L_3 does not occur. When (2) is satisfied the theorems are equivalent. It may also be remarked that, when (2) is satisfied, (18) is equivalent to the corresponding condition with

$$\lim_{\delta\to 0}\ \overline{\lim_{n\to\infty}}\left|\int_0^{\delta}\right|,$$

We have to show that

$$\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\left|n^{\alpha}\int_{1/n}^{\delta}\phi(t)\,H(t)\,\frac{\sin(n;\,t)}{t^{1-\alpha}}\,dt\right|=0, \tag{20}$$

and we know that, given $\epsilon>0$, we can choose $\delta_0=\delta_0(\epsilon)$ and $n_0=n_0(\delta,\,\epsilon)$ such that, if $0<\delta<\delta_0$,

$$\left|\int_{1/n}^{\delta}\phi(t)\,\frac{\sin(n;\,t)}{t^{1-\alpha}}\,dt\right|<\epsilon n^{-\alpha}$$

for $n>n_0$. Now, if $c_{-\nu}=c_{\nu}$, we can write

$$H(t)=\tfrac{1}{2}\sum_{-\frac{1}{2}n\leqslant\nu\leqslant\frac{1}{2}n}c_{\nu}\cos\nu t+\sum_{\nu>\frac{1}{2}n}c_{\nu}\cos\nu t.$$

Substituting this expression for $H(t)$ in the integral of (20) and rearranging the result, we obtain

$$\begin{aligned}L&=\int_{1/n}^{\delta}\phi(t)\,H(t)\,\frac{\sin(n;\,t)}{t^{1-\alpha}}\,dt\\&=\tfrac{1}{2}\sum_{-\frac{1}{2}n\leqslant\nu\leqslant\frac{1}{2}n}c_{\nu}\int_{1/n}^{\delta}\phi(t)\,\frac{\sin(n+\nu;\,t)}{t^{1-\alpha}}\,dt+\int_{1/n}^{\delta}\phi(t)\sum_{\nu>\frac{1}{2}n}c_{\nu}\cos\nu t\,\frac{\sin(n;\,t)}{t^{1-\alpha}}\,dt\\&=L_1+L_2.\end{aligned}$$

Then, if $0<\delta<\delta_0$, and $n>n_0$, we have

$$|L_1|\leqslant\tfrac{1}{2}\sum_{-\frac{1}{2}n\leqslant\nu\leqslant\frac{1}{2}n}|c_{\nu}|\,\epsilon(n+\nu)^{-\alpha}\leqslant C\epsilon n^{-\alpha}\sum_{\nu=-\infty}^{\infty}|c_{\nu}|\leqslant C\epsilon n^{-\alpha}.$$

Now write

$$L_2=\int_{1/n}^{\tau/n}+\int_{\tau/n}^{\delta}=L_3+L_4.$$

Then, for a fixed δ, and a given $\epsilon'>0$, chosen after δ, we have

$$\begin{aligned}|L_4|&\leqslant Bn^{-1}\int_{\tau/n}^{\delta}|\phi(t)|\,\frac{dt}{t^{1-\alpha}}\\&\leqslant Bn^{-1}\left(\frac{\tau}{n}\right)^{\alpha-1}\int_{0}^{\delta}|\phi(t)|\,dt\\&\leqslant B\tau^{\alpha-1}n^{-\alpha}\leqslant B\epsilon' n^{-\alpha}\end{aligned}$$

if τ is chosen large enough. Now let δ and τ be fixed. Then

$$\begin{aligned}|L_3|&\leqslant Bn^{-1}\int_{1/n}^{\tau/n}|\phi(t)|\,\frac{dt}{t^{1-\alpha}}\\&\leqslant Bn^{-\alpha}\int_{1/n}^{\tau/n}|\phi(t)|\,dt=o(n^{-\alpha}).\end{aligned}$$

Thus, since ϵ' is arbitrary, $L_2 = o(n^{-\alpha})$, and, since ϵ is arbitrary, (20) follows immediately.

We end by stating a theorem which may be obtained from our results.

THEOREM 3. *If* $0 < \alpha < 1$, $\eta > 0$, *and* (i) $A_n = o(n^{-\alpha})$, (ii)

$$\int_0^t |d\{u\phi(u)\}| = O(t) \tag{21}$$

in the interval $(0, \eta)$, *then the Fourier series is summable* $(C, -\alpha)$ *if it is summable* (A).

This is an extension of results given with different degrees of generality by Young, Pollard, and Hardy and Littlewood* in the case $\eta = \pi$. In the latter case $\phi(t)$ is restricted by (21) to be of bounded variation in every interval (ϵ, π), $\epsilon > 0$.

L. S. Bosanquet,
University College, London.

A. C. Offord,
St. John's College, Cambridge.

* See G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 28 (1928), 301–311, where other references are given. For further extensions of these results, and an application of our conditions, see L. S. Bosanquet, *Quart. J. of Math.*, 6 (1935), 113–123.

ON FOURIER TRANSFORMS (II)

By A. C. OFFORD.

[Received 9 October, 1934.—Read 15 November, 1934.]

The object of this paper is to complete some of the results given in a preceding paper*. The following theorem will be proved.

THEOREM 1. *If $f(u)$ is integrable L in every finite range and if*

$$\left|\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\omega}\left(1-\frac{u}{\omega}\right)f(u)\cos xu\,du\right|\leqslant A, \tag{1}$$

where $\omega\geqslant 0$, x is arbitrary and A is an absolute constant†; then the integral

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}f(u)\cos xu\,du \tag{2}$$

is boundedly summable $(C, 1)$ almost everywhere to a function $F(x)$ and

$$f(u)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}F(x)\cos ux\,dx \quad (C, 1) \tag{3}$$

almost everywhere.

Functions which satisfy (1) are described in (I) as functions of the class H. The above theorem differs from Theorem 1 of (I) only in respect of the relation (3). I was then only able to prove that (3) holds with $(C, 2)$ summability. The new theorem is more symmetrical than the previous result.

The proofs of these two theorems fall into two parts. The first part, which is the same in each, consists in showing that if (1) is true, then (2) is summable $(C, 1)$ almost everywhere. I have given two proofs of this result. The first proof is given in (I) and there I consider only cosine

* "On Fourier transforms", *Proc. London Math. Soc.* (2), 38 (1934), 197–216. I shall refer to this paper as (I).

† Throughout the paper A will be used to denote an absolute constant. Later M will be used to denote a number which may depend on the range of values of one of the arguments, *e.g.* if the argument is t and $0\leqslant t\leqslant T$ then M may depend on T. A's and M's are not necessarily the same at each occurrence.

transforms. The second proof is given in another paper*, in which I deal with the more general Hankel transform. The two proofs are based on essentially different ideas.

The first part of the theorem being thus established, it only remains to prove that the integral in (3) is summable $(C, 1)$ almost everywhere.

We use the following notation. Write

$$f_1(x, \omega) = \int_0^\omega f(u) \cos xu \, du, \tag{4}$$

$$f_2(x, \omega) = \int_0^\omega f_1(x, u) \, du = \int_0^\omega (\omega - u) f(u) \cos xu \, du, \tag{5}$$

$$f_1(\omega) = f_1(0, \omega), \quad f_2(\omega) = f_2(0, \omega). \tag{6}$$

The following lemmas are required.

LEMMA 1. *If $f(u)$ satisfies* (1); *then*

$$|f_2(x, \omega)| \leqslant A\omega \tag{7}$$

and, in particular,

$$|f_2(\omega)| \leqslant A\omega. \tag{8}$$

The inequality (7) is an immediate consequence of (1) and (5), while (8) is obtained by putting $x = 0$ in (7).

LEMMA 2. *If $f(u)$ satisfies* (1) *and if $x \geqslant 2$, $\omega \geqslant 1$, then*

$$\left| \int_0^\omega (\omega - u) f(u) \sin xu \, du \right| \leqslant A\omega \log x\omega.$$

Proof. Write

$$g_1(u) = \begin{cases} (\omega - u) \sin xu & (0 \leqslant u \leqslant \omega), \\ 0 & (\omega \leqslant u), \end{cases}$$

and let $G_1(t)$ be the cosine transform of $g_1(u)$, *i.e.*

$$G_1(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty g_1(u) \cos tu \, du = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\omega (\omega - u) \cos tu \sin xu \, du \tag{9}$$

and

$$g_1(u) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty G_1(t) \cos ut \, dt. \tag{10}$$

* "On Hankel transforms", *Proc. London Math. Soc.* (2), 39 (1935), 49–67.

We have

$$G_1(t) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_0^\omega (\omega - u) \{\sin(t+x)u - \sin(t-x)u\} du \tag{11}$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \left\{ -\frac{2\omega x}{t^2-x^2} - \frac{\sin(t+x)\omega}{(t+x)^2} + \frac{\sin(t-x)\omega}{(t-x)^2} \right\}.$$

Our first step is to obtain an inequality for the integral $\int_0^\infty |G_1(t)|\,dt$. We have

$$\int_0^\infty |G_1(t)|\,dt = \left\{ \int_0^{x-1/\omega} + \int_{x-1/\omega}^{x+1/\omega} + \int_{x+1/\omega}^\infty \right\} |G_1(t)|\,dt$$

$$= I_1 + I_2 + I_3.$$

From (11),

$$I_3 \leqslant x\omega \int_{x+1/\omega}^\infty \frac{dt}{t^2-x^2} + \int_{x+1/\omega}^\infty \left\{ \frac{1}{(t+x)^2} + \frac{1}{(t-x)^2} \right\} dt$$

$$\leqslant \tfrac{1}{2}\omega \left[\log \frac{t-x}{t+x} \right]_{x+1/\omega}^\infty + 2\omega$$

$$\leqslant A\omega \log x\omega.$$

Similarly $I_1 \leqslant A\omega \log x\omega$. Again, by (9), $|G_1(t)| \leqslant \omega^2/(2\pi)^{\frac{1}{2}}$ and therefore

$$I_2 \leqslant \frac{\omega^2}{(2\pi)^{\frac{1}{2}}} \int_{x-1/\omega}^{x+1/\omega} dt = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \omega.$$

Hence

$$\int_0^\infty |G_1(t)|\,dt \leqslant A\omega \log x\omega. \tag{12}$$

This is the desired inequality.

We have shown (I, 203) that (2) is boundedly summable $(C, 1)$ almost everywhere to $F(x)$. Hence, in virtue of (12),

$$\int_0^\infty F(t)\,G_1(t)\,dt = \lim_{U\to\infty} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty G_1(t)\,dt \int_0^U \left(1 - \frac{u}{U}\right) f(u) \cos tu\,du$$

$$= \lim_{U\to\infty} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^U \left(1 - \frac{u}{U}\right) f(u)\,du \int_0^\infty G_1(t) \cos ut\,dt.$$

Therefore, by (10) and the consistency theorem for Cesàro summation,

$$\int_0^\infty F(t)\,G_1(t)\,dt = \lim_{U\to\infty} \int_0^U \left(1 - \frac{u}{U}\right) f(u)\,g_1(u)\,du \tag{13}$$

$$= \int_0^\omega (\omega - u) f(u) \sin xu\,du.$$

Now $F(t)$ is bounded, being the limit of a boundedly convergent sequence. Therefore, from (12) and (13), we get

$$\left| \int_0^\omega (\omega - u) f(u) \sin xu \, du \right| \leqslant A \int_0^\infty |G_1(t)| \, dt \leqslant A\omega \log x\omega.$$

This is the desired result.

Proof of Theorem 1. It only remains to show that the inverse relation (3) holds almost everywhere.

Write

$$G_2(x) = \begin{cases} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(1 - \frac{x}{X}\right) \cos tx & (0 \leqslant x \leqslant X), \\ 0 & (X \leqslant x), \end{cases}$$

and let $g_2(u)$ be the cosine transform of $G_2(x)$, *i.e.*

$$(14) \qquad g_2(u) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty G_2(x) \cos ux \, dx$$

$$= \frac{2}{\pi} \int_0^X \left(1 - \frac{x}{X}\right) \cos ux \cos tx \, dx$$

$$= \frac{1}{\pi X} \left\{ \frac{1 - \cos X(u+t)}{(u+t)^2} + \frac{1 - \cos X(u-t)}{(u-t)^2} \right\}.$$

Write $$\Delta(u, t) = (u+t)^{-2} + (u-t)^{-2}.$$

Then

$$(15) \quad g_2(u) = \frac{1}{\pi X} \left\{ \Delta(u, t) - \Delta(u, t) \cos Xu \cos Xt - \frac{4ut \sin Xu \sin Xt}{(u^2 - t^2)^2} \right\}.$$

Now (2) is boundedly summable $(C, 1)$ to $F(x)$. Hence, as in the proof of (13),

$$(16) \quad \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty F(x) G_2(x) \, dx$$

$$= \lim_{\omega \to \infty} \frac{2}{\pi} \int_0^X \left(1 - \frac{x}{X}\right) \cos tx \, dx \int_0^\omega \left(1 - \frac{u}{\omega}\right) f(u) \cos xu \, du$$

$$= \lim_{\omega \to \infty} \frac{2}{\pi} \int_0^\omega \left(1 - \frac{u}{\omega}\right) f(u) \, du \int_0^X \left(1 - \frac{x}{X}\right) \cos tx \cos ux \, dx$$

$$= \lim_{\omega \to \infty} \int_0^\omega \left(1 - \frac{u}{\omega}\right) f(u) g_2(u) \, du.$$

Write

$$J=\int_0^\omega\left(1-\frac{u}{\omega}\right)f(u)\,g_2(u)\,du=\left\{\int_0^{t+1}+\int_{t+1}^\omega\right\}\left(1-\frac{u}{\omega}\right)f(u)\,g_2(u)\,du$$

$$=J_1+J_2.$$

We now show that J_2 tends to zero as X tends to infinity. From (15), we have

$$J_2=\frac{1}{\pi X}\int_{t+1}^\omega\left(1-\frac{u}{\omega}\right)\Delta(u,t)\,f(u)\,du$$

$$-\frac{\cos tX}{\pi X}\int_{t+1}^\omega\left(1-\frac{u}{\omega}\right)\Delta(u,t)\,f(u)\cos Xu\,du$$

$$-\frac{4t\sin tX}{\pi X}\int_{t+1}^\omega\left(1-\frac{u}{\omega}\right)\frac{u}{(u^2-t^2)^2}f(u)\sin Xu\,du$$

$$=\frac{1}{\pi X}J_3-\frac{\cos tX}{\pi X}J_4-\frac{4t\sin tX}{\pi X}J_5.$$

Now, on using the notation of (4), (5), and (6), we have

$$J_4=-\left(1-\frac{t+1}{\omega}\right)\Delta(t+1,t)\,f_1(X,t+1)-\left[\frac{d}{du}\left\{\left(1-\frac{u}{\omega}\right)\Delta(u,t)\right\}f_2(X,u)\right]_{t+1}^\omega$$

$$+\int_{t+1}^\omega\left(\frac{d}{du}\right)^2\left\{\left(1-\frac{u}{\omega}\right)\Delta(u,t)\right\}f_2(X,u)\,du$$

$$=J_6+J_7+J_8.$$

Let $0\leqslant t\leqslant T$, $\omega\geqslant T+1$. The constants denoted by M are independent of t, ω, and X, but may depend on T. We have

$$|J_6|\leqslant|\Delta(t+1,t)||f_1(X,t+1)|\leqslant 2\int_0^{T+1}|f(u)|\,du=M.$$

Again, using Lemma 1,

$$|J_7|\leqslant\omega^{-1}|\Delta(\omega,t)||f_2(X,\omega)|+\omega^{-1}|\Delta(t+1,t)||f_2(X,t+1)|$$

$$+\left|\left[\frac{d}{du}\Delta(u,t)\right]_{u=t+1}\right||f_2(X,t+1)|$$

$$\leqslant A+A\omega^{-1}(t+1)+A(t+1)$$

$$\leqslant A(T+1)=M.$$

Further

$$|J_8| \leqslant \int_{t+1}^{\omega} \left| \left(\frac{d}{du}\right)^2 \left\{ \left(1-\frac{u}{\omega}\right) \Delta(u, t) \right\} \right| |f_2(X, u)|\, du$$

$$\leqslant A \int_{t+1}^{\omega} \frac{u\, du}{\omega(u-t)^3} + A \int_{t+1}^{\omega} \frac{u\, du}{(u-t)^4}$$

$$\leqslant A(T+1) = M.$$

Hence $$|J_4| \leqslant M,$$

provided that $$0 \leqslant t \leqslant T.$$

If, in the proof of this inequality, we replace $f_1(X, u)$ by $f_1(u)$ and $f_2(X, u)$ by $f_2(u)$, we obtain

$$|J_3| \leqslant M$$

under the same conditions as before.

Write $$\phi_1(x, \omega) = \int_0^{\omega} f(u) \sin xu\, du,$$

$$\phi_2(x, \omega) = \int_0^{\omega} \phi_1(x, u)\, du = \int_0^{\omega} (\omega-u) f(u) \sin xu\, du.$$

Then

$$J_5 = -\left(1-\frac{t+1}{\omega}\right) \frac{t+1}{(2t+1)^2} \phi_1(X, t+1) + \frac{1}{(w^2-t^2)^2} \phi_2(X, \omega)$$

$$+\left(1-\frac{t+1}{\omega}\right) \left[\frac{d}{du} \left\{ \frac{u}{(u^2-t^2)^2} \right\}\right]_{u=t+1}^{\omega} \phi_2(X, t+1) - \frac{t+1}{\omega(2t+1)^2} \phi_2(X, t+1)$$

$$+\int_{t+1}^{\omega} \left(\frac{d}{du}\right)^2 \left\{ \left(1-\frac{u}{\omega}\right) \frac{u}{(u^2-t^2)^2} \right\} \phi_2(X, u)\, du$$

$$= J_9 + J_{10} + J_{11} + J_{12} + J_{13}.$$

Let $0 \leqslant t \leqslant T$ and $\omega \geqslant T+1$. Then

$$|J_9| \leqslant \int_0^{T+1} |f(u)|\, du = M.$$

Again, from Lemma 2,

$$|\phi_2(X, \omega)| \leqslant A\omega \log X\omega,$$

so that

$$|J_{10}| \leqslant (\omega^2-t^2)^{-2}|\phi_2(X,\omega)| \leqslant M\log X,$$

$$|J_{11}| \leqslant A\{(2t+1)^{-2}+(2t+1)^{-1}\}(t+1)\log X(t+1) \leqslant M\log X,$$

$$|J_{12}| \leqslant A\omega^{-1}(2t+1)^{-1}(t+1)\log X(t+1) \leqslant M\log X,$$

$$|J_{13}| \leqslant A\int_{t+1}^{\omega}\frac{u^2\log Xu}{\omega(u^2-t^2)^2}\,du+A\int_{t+1}^{\omega}\frac{u^2\log Xu}{(u^2-t^2)^3}\,du+A\int_{t+1}^{\omega}\frac{u^3\log Xu}{(u^2-t^2)^4}\,du$$
$$\leqslant M\log X.$$

Hence $$|J_5| \leqslant M\log X,$$

provided that $$0 \leqslant t \leqslant T, \quad \omega \geqslant T+1, \quad X \geqslant 2.$$

From the inequalities for J_3, J_4, and J_5, we obtain

$$|J_2| \leqslant MX^{-1}\log X.$$

Therefore

$$J = \int_0^{t+1}\left(1-\frac{u}{\omega}\right)f(u)\,g_2(u)\,du+O(X^{-1}\log X),$$

where the constants implied by the symbol O are independent of ω and t, provided that $0 \leqslant t \leqslant T$. Therefore, from (16),

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}F(x)\,G_2(x)\,dx = \int_0^{t+1}f(u)\,g_2(u)\,du+O(X^{-1}\log X). \tag{17}$$

Let δ be a positive number such that $0<\delta<1$ and, in case $t<\delta$, let $f(-u)=f(u)$. Then, by (14), we have

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}F(x)\,G_2(x)\,dx = \frac{1}{\pi X}\int_{t-\delta}^{t+\delta}f(u)\,\frac{1-\cos(u-t)X}{(u-t)^2}\,du+O(X^{-1}\log X)$$
$$= \frac{2}{\pi X}\int_0^{\delta}\{f(t+u)+f(t-u)\}\frac{\sin^2\frac{1}{2}uX}{u^2}\,du+O(X^{-1}\log X).$$

Now it is well known that the first term of the second member tends to $f(t)$ almost everywhere as X tends to infinity, *i.e.*

$$f(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}F(x)\cos tx\,dx \quad (C,\,1),$$

almost everywhere. This is the desired result.

We have, however, proved a little more. We have shown that the convergence to zero of the term $O(X^{-1}\log X)$ in (17) is uniform in t for $0 \leqslant t \leqslant T$. It follows that the integral (3) behaves as regards summability

or uniform summability $(C, 1)$ just like the Fourier series of a function which is equal to $f(t)$ in an interval $(t-\delta,\ t+\delta)$.

The following theorem gives the corresponding result for the sine transform.

THEOREM 2. *If $h(u)$ is integrable L in every finite range, and if*

$$\left|\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\omega}\left(1-\frac{u}{\omega}\right)h(u)\sin xu\,du\right|\leqslant A, \tag{18}$$

where $\omega\geqslant 0$, x is arbitrary and A is an absolute constant; then the integral

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}h(u)\sin xu\,du \tag{19}$$

is boundedly summable $(C, 1)$ almost everywhere to a function $H(x)$, and

$$h(u)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}H(x)\sin ux\,dx \quad (C, 1) \tag{20}$$

almost everywhere.

The proof of this theorem is similar to that of Theorem 1 and we give it in outline only. It is, however, important to notice that there is one essential difference between the two proofs.

Write

$$h_1(x,\ \omega)=\int_0^{\omega}h(u)\sin xu\,du,$$

$$h_2(x,\ \omega)=\int_0^{\omega}h_1(x,\ u)\,du=\int_0^{\omega}(\omega-u)\,h(u)\sin xu\,du,$$

$$h_1(\omega)=\int_0^{\omega}h(u)\,du,$$

$$h_2(\omega)=\int_0^{\omega}h_1(u)\,du=\int_0^{\omega}(\omega-u)\,h(u)\,du.$$

Observe that, in this case, $h_2(0,\ \omega)=0$ and not $h_2(\omega)$ as in the case of cosine transforms. This is the point which gives rise to the difference between the proofs to which we referred.

It has been shown* that (19) is boundedly summable $(C, 1)$ to $H(x)$ and so it only remains to prove the relation (20). We use the following lemmas which are the analogues of Lemmas 1 and 2 respectively.

---

* "On Hankel transforms", *Proc. London Math. Soc.* (2), 39 (1935), 49–67.

LEMMA 3. *With the hypotheses of Theorem 2,*

$$|h_2(x, \omega)| \leqslant A\omega$$

for all x.

LEMMA 4. *Let $h(u)$ satisfy the hypotheses of Theorem 2. Let $\omega \geqslant 2$ and $\xi = \max(x, 1)$. Then*

$$\left| \int_0^\omega (\omega - u) h(u) \cos xu \, du \right| \leqslant A \omega \log \xi\omega, \tag{21}$$

and, in particular,

$$|h_2(\omega)| = \left| \int_0^\omega (\omega - u) h(u) \, du \right| \leqslant A\omega \log \omega. \tag{22}$$

Lemma 3 follows at once from the hypothesis (19). The proof of Lemma 4 is strictly analogous to that of Lemma 2. The only point where the proof of Theorem 2 differs from that of Theorem 1 is in the discussion of the terms corresponding to J_3. At this point the inequality (8) is replaced by (22). The presence of the factor $\log \omega$ in the latter does not affect the proof.

Combining Theorems 1 and 2, we deduce the following theorem.

THEOREM 3. *If $f(u)$ and $h(u)$ are integrable L in every finite range and if*

$$\left| \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\omega \left(1 - \frac{u}{\omega}\right) \{f(u) \cos xu + h(u) \sin xu\} \, du \right| \leqslant A, \tag{23}$$

where $\omega \geqslant 0$, x is positive or negative and A is an absolute constant, then the integral

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\omega \{f(u) \cos xu + h(u) \sin xu\} \, du$$

is boundedly summable $(C, 1)$ almost everywhere to a function $F(x)$ and

$$f(u) = \lim_{X \to \infty} \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{-X}^{X} \left(1 - \frac{|x|}{X}\right) F(x) \cos ux \, dx,$$

$$h(u) = \lim_{X \to \infty} \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{-X}^{X} \left(1 - \frac{|x|}{X}\right) F(x) \sin ux \, dx,$$

almost everywhere.

Condition (23) is satisfied for all x positive and negative. Hence it is easy to see that $f(u)$ satisfies (1) and $h(u)$ satisfies (21). Therefore Theorem 3 follows from Theorems 1 and 2.

St. John's College,
Cambridge.

ON POSITIVE HARMONIC FUNCTIONS

(*Second Paper*)

By S. VERBLUNSKY.

[Received 28 December, 1934.—Read 17 January, 1935.]

1. In a previous paper* I considered the class of functions

$$1+\sum_1^\infty (a_n \cos nx+b_n \sin nx)\, r^n \qquad \text{(i)}$$

which are positive for $0 \leqslant r < 1$ and obtained a parametric representation of all the sequences c_n, $n = 1, 2, \ldots$, where $2c_n = a_n - ib_n$, in terms of a sequence of independent complex parameters $z_n = \rho_n e^{i\phi_n}$, subject to the condition $\rho_n{}^2 \leqslant 1$. The function (i) is the real part of

$$1+2\Sigma c_n z^n \quad (z = re^{ix}).$$

We can write

$$1 = (1+2\Sigma \gamma_n z^n)(1+2\Sigma c_n z^n), \qquad (a)$$

where $2\gamma_n = \alpha_n - i\beta_n$, and then

$$1+\Sigma (\alpha_n \cos nx+\beta_n \sin nx)\, r^n \qquad \text{(ii)}$$

is also a p.h.f.(positive harmonic function). Let ζ_n be the parameters of (ii). In § 4, we show that $z_n+\zeta_n = 0$ for all n.

In P.H. I, the class P_n was defined as the class of functions (i) for which $z_\nu = 0$, $\nu > n$. If a parameter z_i, $i \leqslant n$, is of unit modulus, then

$$1+\Sigma (a_n \cos nx+b_n \sin nx) \qquad \text{(iii)}$$

is obtained by the formal differentiation of a series which converges to a step

* *Proc. London Math. Soc.* (2), 38 (1934), 125–157. Referred to as P.H. I. A knowledge of this will be assumed.

function*. Consequently (iii) is not a Fourier series. If, however, none of the parameters is of unit modulus, then we show that (iii) is the reciprocal of a positive trigonometric polynomial of order less than or equal to n. The order is the suffix of the last non-zero parameter. Conversely, if $T(x)$ is a positive trigonometric polynomial of order n, and if we choose $c > 0$ so that the constant term of the Fourier series of $f(x) = c/T(x)$ is 1, then the Poisson function (i) of f is of class P_n. This result gives a new parametric representation of positive trigonometric polynomials.

If $\phi(x)$ is integrable in $(0, 2\pi)$, we write

$$A(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \phi\, dx. \tag{iv}$$

We examine the class P_2, and show that, if $T(x)$ is a positive trigonometric polynomial of order 2 for which the value of $A(1/T)$ is 1, then the Fourier constants of $1/T$ can be obtained by rational operations from the coefficients of T.

If ϕ is positive and integrable in $(0, 2\pi)$, we write

$$G(\phi) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\phi\, dx\right), \tag{v}$$

where the second member is zero if $\log\phi$ is not integrable.

If (i) is positive, then

$$F(x) = x - \Sigma(b_n \cos nx - a_n \sin nx)/n$$

is non-decreasing. We show that

$$G(F') = \prod_{r=1}^{\infty} (1-\rho_r^2). \tag{vi}$$

We again examine the class P_2 and give a complete solution of the following problem. Given

$$P(x) = \alpha_1 \cos x + \beta_1 \sin x + \alpha_2 \cos 2x + \beta_2 \sin 2x,$$

to find necessary and sufficient conditions that the integral equation

$$\int_0^{2\pi} \log\{t + P(x)\}\, dx = 0$$

* This follows from P.H. I. Theorem C and Lemma 2.

shall have a solution (it can only have one), and to solve it when it has a solution. It appears that the solution is a simple rational function of α_1, β_1, α_2, β_2. The integral equation (when it has a solution) is thus equivalent to a linear equation in t. On the other hand, the equation

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{dx}{t+P(x)}=1,$$

which always has a solution, will be shown to be equivalent, in general, to an equation of degree 8 in t.

Great help has been derived from a paper of Szegö*, references to which will be given at the appropriate points.

2. We call 1, c_1, c_2, ..., where the c's are defined as in § 1, the Fourier constants of the p.h.f. (i).

THEOREM 1. *A necessary and sufficient condition that* 1, $c_1\gamma_1$, $c_2\gamma_2$, ... *shall be the Fourier constants of a p.h.f. whenever* 1, c_1, c_2, ... *are the Fourier constants of a p.h.f. is that* 1, γ_1, γ_2, ... *shall be the Fourier constants of a p.h.f.*

If we write $\gamma_n=\lambda_n+i\mu_n$, the theorem is equivalent to the following lemma.

LEMMA 1. *A necessary and sufficient condition that*

$$1+\Sigma\{(\lambda_n a_n+\mu_n b_n)\cos nx+(\lambda_n b_n-\mu_n a_n)\sin nx\}r^n \qquad (1)$$

shall be positive whenever (i) *is positive is that*

$$1+2\Sigma(\lambda_n\cos nx-\mu_n\sin nx)r^n \qquad (2)$$

shall be positive.

To prove that the condition is necessary, we observe that

$$1+2\Sigma r^n\cos nx$$

is positive. Hence, if we take $a_n=2$ and $b_n=0$ in (i), (1) must be positive. But (1) is then identical with (2).

---

* Szego, *Math. Zeitschrift*, 6 (1920), 167–202.

To prove that the condition is sufficient, we observe that, if (i) is positive, we can write

$$g(x) = x - \Sigma(b_n \cos nx - a_n \sin nx)/n,$$

where $g(x)$ is non-decreasing. Hence

$$-\frac{b_n}{n} = \frac{1}{\pi}\int_0^{2\pi} \{g(x)-x\} \cos nx\, dx, \quad \frac{a_n}{n} = \frac{1}{\pi}\int_0^{2\pi} \{g(x)-x\} \sin nx\, dx,$$

and so

$$\frac{1}{\pi}\int_0^{2\pi} (\lambda_n \cos nt + \mu_n \sin nt)\, d[g(x+t)-(x+t)]$$
$$= (\lambda_n a_n + \mu_n b_n) \cos nx + (\lambda_n b_n - \mu_n a_n) \sin nx.$$

The first member of this equation is

$$\frac{1}{\pi}\int_0^{2\pi} (\lambda_n \cos nt + \mu_n \sin nt)\, dg(x+t).$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi} dg(x+t) = 1,$$

we have, for the value of (1),

$$\frac{1}{2\pi}\int_0^{2\pi} \{1 + 2\Sigma(\lambda_n \cos nt + \mu_n \sin nt)\, r^n\}\, dg(x+t).$$

Hence, if (2) is positive, or, what is the same, if

$$1 + 2\Sigma\{\lambda_n \cos nx + \mu_n \sin nx\} r^n$$

is positive, so is (1).

3. Let $0 < a < 1$. Consider the p.h.f.

$$1 + 2a\Sigma r^n \cos nx.$$

With the notation of P.H. I, we have

$$D_n = \begin{vmatrix} 1 & a & \dots & a \\ a & 1 & a & \dots & a \\ \dots & \dots & \dots & \dots \\ a & \dots & a & 1 \end{vmatrix}.$$

It is easy to show that $D_n = (1-a)^n(1+na)$. To determine the parameters z_n, we have from P.H.I, equation (22),

$$z_n D_{n-1} = c_n D_{n-2} - (-1)^n \bar{\delta}_n,$$

where the second member is $(-1)^{n+1}$ times the minor of the element in the bottom left-hand corner of D_n. This minor is

$$\Lambda_n = \begin{vmatrix} a & & \dots & a \\ 1 & a & \dots & a \\ a & 1 & \dots & a \\ \dots & \dots & \dots & \\ a & \dots & a & 1 \end{vmatrix}.$$

We find that $\Lambda_n = (-1)^{n-1}(1-a)^{n-1}a$. Thus $z_n D_{n-1} = (1-a)^{n-1}a$, whence

$$z_n = \rho_n = \frac{a}{1+(n-1)a}. \tag{3}$$

4. We consider the two p.h.f., (i) and (ii), related as in § 1.

THEOREM II. *The parameters z_n of* (i) *and ζ_n of* (ii) *satisfy $z_n+\zeta_n = 0$ for all n.*

From equation (*a*) of § 1, we have

$$1 = 1+2\Sigma(\gamma_n+2\gamma_{n-1}c_1+\dots+2\gamma_1 c_{n-1}+c_n)z^n.$$

Hence $\gamma_1+c_1 = 0$, and

$$\gamma_n+c_n = -2(\gamma_{n-1}c_1+\dots+\gamma_1 c_{n-1}) \quad (n>1). \tag{4}$$

It is known (P.H.I, Theorem I) that $c_1 = z_1$, $\gamma_1 = \zeta_1$, so that $z_1+\zeta_1 = 0$. Assume that

$$\zeta_i+z_i = 0 \quad (i = 1, \dots, n-1). \tag{5}$$

The expression for c_ν in terms of the parameters z_ν is of the form

$$c_\nu = z_\nu \prod_{r=1}^{\nu-1}(1-z_r\bar{z}_r)+q_\nu(z_1, \bar{z}_1, \dots, z_{\nu-1}, \bar{z}_{\nu-1}), \tag{6}$$

where q_ν is a polynomial in the arguments indicated. Likewise,

$$\gamma_\nu = \zeta_\nu \prod_{r=1}^{\nu-1} (1-\zeta_r \bar{\zeta}_r) + q_\nu(\zeta_1, \bar{\zeta}_1, \ldots, \zeta_{\nu-1}, \bar{\zeta}_{\nu-1}). \tag{7}$$

By (5), (6), (7), the second member of (4) is a polynomial in

$$z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}.$$

By (6), (7), with $\nu = n$, (4) becomes

$$(\zeta_n + z_n) \prod_{r=1}^{\nu-1} (1-z_r \bar{z}_r) = P_n(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}), \tag{8}$$

where P_n is a polynomial in its arguments.

If P_n is not identically zero, there is a set of values of z_i $(i = 1, \ldots, n-1)$, with $|z_i| < 1$, such that

$$P_n(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}) = ke^{i\alpha} \quad (k > 0).$$

Take $z_n = -e^{i\alpha}$. Then, by (8), $|\zeta_n| > 1$. This contradicts the definition of the parameter ζ_n, and shows that $P_n \equiv 0$. By (8), $\zeta_n + z_n = 0$, and this completes the proof by induction.

If we apply the theorem to the function of § 3, we find that, for $0 < \alpha < 1$, the parameters of the p.h.f.

$$1 - 2\alpha \Sigma (1-2\alpha)^{n-1} \cos nx \, r^n$$

are

$$z_n = -\frac{\alpha}{1+(n-1)\alpha}. \tag{9}$$

5. If the p.h.f. (i) is of class P_ν, and the series (iii) is the Fourier series of a function $f(x)$, then we call $f(x)$ an integrable function of class P_ν.

The class P_1 (P.H. I, 141) consists of the harmonic functions

$$\frac{1-r^2\rho_1^{\,2}}{1-2r\rho_1 \cos(\phi_1+x)+r^2\rho_1^{\,2}} \quad (0 \leqslant \rho_1 \leqslant 1),$$

and the integrable functions of this class are the functions

$$\frac{1-\rho_1^{\,2}}{1-2\rho_1 \cos(\phi_1+x)+\rho_1^{\,2}} \quad (0 \leqslant \rho_1 < 1).$$

Consider the class P_ν for $\nu > 1$. If one of the parameters z_i $(i = 1, \ldots, \nu)$ is of unit modulus, let z_j be the first such parameter. We then know that

(iii) is obtained by the formal differentiation of a series which converges to a non-diminishing step function with precisely j non-equivalent discontinuities (P.H. I, Theorem C and Lemma 2), so that (iii) is not a Fourier series. We consider, then, the case in which $|z_i| < 1$ $(i = 1, \ldots \nu)$.

In the determinant $D_\nu = D(c_1, \ldots, c_\nu)$, let $\gamma_0, \ldots, \gamma_\nu$ denote the cofactors of the elements in the last row*.

Then $\gamma_\nu = D_{\nu-1}$. Now equation (22) of P.H. I is $\gamma_0 = -z_\nu D_{\nu-1}$, so that

$$\gamma_0/\gamma_\nu = -z_\nu. \tag{10}$$

Let $$H(z) = 1 + \frac{\gamma_{\nu-1}}{\gamma_\nu} z + \frac{\gamma_{\nu-2}}{\gamma_\nu} z^2 + \ldots + \frac{\gamma_0}{\gamma_\nu} z^\nu.$$

It is known (P.H. I, Theorem II) that the coefficients of $H(z)$ are polynomials in $z_1, \bar{z}_1, \ldots, z_{\nu-1}, \bar{z}_{\nu-1}, z_\nu$, with integer coefficients. For $n > \nu$, the values of c_n in P_ν are given by equation (37) of P.H. I, namely

$$c_n + \frac{\gamma_{\nu-1}}{\gamma_\nu} c_{n-1} + \ldots + \frac{\gamma_0}{\gamma_\nu} c_{n-\nu} = 0. \tag{11}$$

Hence, in $$H(z) \sum_1^\infty c_n z^{n-1} \tag{12}$$

the coefficient of z^{n-1} is zero for $n > \nu$. Thus (12) is a polynomial $g(z)$ of degree less than or equal to $\nu - 1$. Write

$$G(x) = H(z) + 2z\, g(z),$$

so that $G(x)$ is of degree less than or equal to ν. Then

$$G/H = 1 + 2\Sigma c_n z^n$$

for all sufficiently small z.

We observe that, when $z_1 = \ldots = z_{\nu-1} = 0$, D_ν becomes

$$\begin{vmatrix} 1 & 0 & \ldots & 0 & z_\nu \\ 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \bar{z}_\nu & 0 & \ldots & 0 & 1 \end{vmatrix},$$

* We use the numbers γ in a different sense from that of equation (a), § 1.

and hence $\gamma_0 = -z_\nu$, $\gamma_1 = \ldots = \gamma_{\nu-1} = 0$, $\gamma_\nu = 1$. For this special choice of the parameters, we have

$$H(z) = 1 - z_\nu z^\nu. \tag{13}$$

Write $$1 = (1 + 2\Sigma c_n z^n)(1 + 2\Sigma d_n z^n).$$

The parameters of the d's are the negatives of the parameters of the c's. Hence, denoting by $\mathfrak{H}(z)$, $\mathfrak{G}(z)$ what $H(z)$, $G(z)$ become on replacing the parameters by their negatives, we have, for all sufficiently small z,

$$H/G = \mathfrak{G}/\mathfrak{H}. \tag{14}$$

The polynomial $H(z)$ is irreducible in $R(z_1, \bar{z}_1, \ldots, z_{\nu-1}, \bar{z}_{\nu-1}, z_\nu)$, *i.e.* in the field of the expressions obtained by rational operations on the $2n-1$ letters. For, if we take $z_1 = \bar{z}_1 = \ldots = z_{\nu-1} = \bar{z}_{\nu-1} = 0$, $H(z)$ is given by (13), which is irreducible in $R(z_\nu)$. The equation $H\mathfrak{H} = G\mathfrak{G}$ holds for all z. Since H is irreducible, it must be a factor of G or of $\mathfrak{G}$. Now G is of degree less than or equal to ν; and the constant term in H and in G is 1. If H was a factor of G, we should have $H = G$, which is impossible. Hence H is a factor of $\mathfrak{G}$. Now $\mathfrak{G}$ is of degree less than or equal to ν, and its constant term is 1. Hence $H = \mathfrak{G}$, $\mathfrak{H} = G$. We thus have, for all sufficiently small z,

$$\mathfrak{H}(z)/H(z) = 1 + 2\Sigma c_n z^n. \tag{15}$$

The series in (15) converges for $|z| < 1$. Hence the equation (15) could cease to be valid only for a z with $|z| < 1$, if H had a zero in $|z| < 1$. In that case, by considerations of continuity, $\mathfrak{H}(z)$ would have a zero at the same point. Then the highest common factor of H and $\mathfrak{H}$ would be of positive degree. Since both are irreducible, and have the constant term 1, we should have $H = \mathfrak{H}$, which is impossible. Thus H has no zeros in $|z| < 1$, and (15) holds for $|z| < 1$.

LEMMA 2. *If* $|z_i| < 1$ $(i = 1, \ldots, \nu)$, *then* $H(z)$ *has no zeros on* $|z| = 1$.

The coefficients of $H(z)$ do not involve $\bar{z}_\nu$, and they are of the first degree in z_ν (P.H.I, 144). We can, therefore, write

$$H(e^{i\theta}) = Az_\nu + B,$$

where A and B are polynomials in $z_1, \bar{z}_1, \ldots, z_{\nu-1}, \bar{z}_{\nu-1}, e^{i\theta}$. If, for given $z_1, \ldots, z_{\nu-1}$, the equation $|A| = |B|$ is not satisfied for all θ, then it can

be satisfied only for at most 2ν non-equivalent values of θ. When $z_\nu = e^{i\phi_\nu}$, we know (P.H.I, Theorem C) that the equation

$$Ae^{i\phi_\nu} + B = 0 \tag{16}$$

is satisfied by ν incongruent values of θ. Hence the number of different sets of solutions in $e^{i\theta}$ of (16) for all ϕ_ν is at most $2\nu!/\nu!$. Each set of solutions, together with the values of $c_1, \ldots, c_{\nu-1}$, determines the remaining c's, and, in particular, c_ν uniquely. This implies that, for varying ϕ_ν, c_ν takes only a finite set of values. But (P.H.I, Theorem II)

$$c_\nu = q_\nu(z_1, \bar{z}_1, \ldots, z_{\nu-1}, \bar{z}_{\nu-1}) + e^{i\phi_\nu} \prod_{r=1}^{\nu-1} (1-\rho_r^2),$$

and so takes an infinite set of different values for varying ϕ_ν. Hence $|A| = |B|$ for all θ. Thus the equation $H(e^{i\theta}) = 0$ can be satisfied only if $|z_\nu| = 1$. This proves the lemma.

6. By Lemma 2 it follows from (15) that

$$\mathfrak{Re}\,(e^{i\theta})/H(e^{i\theta}) = \lim_{r\to 1} 1 + 2\Sigma c_n z^n \quad (z = re^{i\theta}).$$

Thus
$$1 + \Sigma(a_n \cos n\theta + b_n \sin n\theta) \tag{17}$$

is summable by Poisson's method to

$$R\,\frac{\mathfrak{Re}\,(e^{i\theta})}{H(e^{i\theta})} = \frac{R[\mathfrak{Re}\,(e^{i\theta})\,\bar{H}(e^{i\theta})]}{H(e^{i\theta})\,\bar{H}(e^{i\theta})}, \tag{18}$$

where $\bar{H}(e^{i\theta})$ denotes the conjugate of $H(e^{i\theta})$.

The denominator is a positive trigonometric polynomial of order ν. The numerator is a trigonometric polynomial of order less than or equal to ν, and is non-negative, since (18) is the boundary function of a p.h.f. Thus (17) converges to (18), and is therefore a Fourier series. We now have

LEMMA 3. $$R[\mathfrak{Re}\,(e^{i\theta})\,\bar{H}(e^{i\theta})] = \prod_{r=1}^{\nu} (1-\rho_\nu^2).$$

From (15) we infer that

$$R[\mathfrak{Re}\,(z)\,\bar{H}(z)] = H(z)\,\bar{H}(z)\,[1 + \Sigma(a_n \cos n\theta + b_n \sin n\theta)\,r^n] \quad (z = re^{i\theta}).$$

We here let $z_j \to e^{i\phi_j}$, where j is one of the numbers $1, \ldots, \nu$. We then obtain an equation

$$R[\mathfrak{A}^j(z)\bar{H}^j(z)] = H^j(z)\bar{H}^j(z)\,[1+\Sigma'(a_n{}^j\cos n\theta + b_n{}^j \sin n\theta) r^n], \qquad (19)$$

where the expressions indexed with j denote the limiting values of the corresponding unindexed expressions. We know that, in (19), the expression in square brackets tends to zero as $r \to 1$, except for j incongruent values of θ. Hence

$$R[\mathfrak{A}^j(e^{i\theta}),\ \bar{H}^j(e^{i\theta})] = 0$$

except for a finite set of values of θ, and hence for all θ. Thus

$$R[\mathfrak{A}(e^{i\theta})\bar{H}(e^{i\theta})]$$

has the factor $1-\rho_j{}^2$, and hence it has the factor $\prod_{r=1}^{\nu}(1-\rho_r{}^2)$.

We shall complete the proof of the lemma by induction, and therefore introduce a suffix ν in H and $\mathfrak{A}$. We suppose that $|z_i| < 1$ for $i = 1, \ldots, \nu$. Then, in (15), the real part of the power series can be any p.h.f. which arises from an integrable function of class P_ν. Such power series we call Fourier power series of class P_ν. If we put $z_\nu = 0$, we obtain the Fourier power series of class $P_{\nu-1}$. These, however, are equal to $\mathfrak{A}_{\nu-1}(z)/H_{\nu-1}(z)$. If, then, we denote by $H_\nu{}^0(z)$, $\mathfrak{A}_\nu{}^0(z)$, the result of putting $z_\nu = 0$ in $H_\nu(z)$, $\mathfrak{A}_\nu(z)$ respectively, we obtain

$$\mathfrak{A}_{\nu-1}(z)/H_{\nu-1}(z) = \mathfrak{A}_\nu{}^0(z)/H_\nu{}^0(z).$$

Here $H_{\nu-1}$, $\mathfrak{A}_{\nu-1}$ are irreducible polynomials of degree $\nu-1$ and with the constant term 1. From the observation that the coefficients of z^ν in $H_\nu(z)$ and $\mathfrak{A}_\nu(z)$ are z_ν, $-z_\nu$ respectively, it follows that $H_\nu{}^0(z)$, $\mathfrak{A}_\nu{}^0(z)$ are of degree less than or equal to $\nu-1$. Further, the constant term in each is 1. Hence

$$\mathfrak{A}_{\nu-1} = \mathfrak{A}_\nu{}^0, \quad H_{\nu-1} = H_\nu{}^0.$$

We now proceed with the proof of the lemma. The function

$$R_\nu = R[\mathfrak{A}_\nu(e^{i\theta})\bar{H}_\nu(e^{i\theta})]$$

is divisible by $\Pi_\nu = \prod_{r=1}^{\nu}(1-\rho_r{}^2)$.

Since Π_ν, H_ν are linear functions of z_ν and do not involve $\bar{z}_\nu$, R_ν/Π_ν involves only $\rho_1, \ldots, \rho_{\nu-1}$ and sines and cosines of $\phi_1, \ldots, \phi_\nu$. If we put $\rho_\nu = 0$, we get

$$R_\nu^0/\Pi_\nu^0 \doteq R_{\nu-1}/\Pi_{\nu-1} = 1$$

by the lemma for $\nu-1$. Hence $R_\nu = \Pi_\nu$. This proves the lemma.

The term of highest order in $H(e^{i\theta})\,\bar{H}(e^{i\theta})$ is

$$z_\nu e^{i\nu\theta} + \bar{z}_\nu e^{-i\nu\theta} = 2\rho_\nu \cos(\nu\theta + \phi_\nu),$$

and is not zero if $z_\nu \neq 0$. If $z_\nu = 0$, the p.h.f. of class P_ν is also of class $P_{\nu-1}$. We thus have

THEOREM III. *If* (i) *is a p.h.f. of class* P_ν, *then*

(*a*) *if* z_j *is the first parameter of unit modulus, the series*

$$1 + \Sigma(a_n \cos nx + b_n \sin nx) \tag{20}$$

is obtained by the formal differentiation of a series which converges to a non-diminishing step function with j *non-equivalent discontinuities*;

(*b*) *if* $|z_i| < 1$ $(i = 1, \ldots, \nu)$, *then* (20) *is the Fourier series of the function*

$$\prod_{r=1}^{\nu} (1-\rho_r^2)/H(e^{i\theta})\,\bar{H}(e^{i\theta}).$$

The denominator is a positive trigonometric polynomial of order ν *when* $z_\nu \neq 0$.

7. If f is an integrable function of class P_ν, and not of class $P_{\nu-1}$, we say that it is precisely of class P_ν. We shall call a positive trigonometric polynomial $T(x)$ *normalized*, if $A(1/T) = 1$ [*cf.* § 1(iv)]. We have

THEOREM IV. *If* $T_\nu(x)$ *is a normalized positive trigonometric polynomial of order* ν, *then* $f = 1/T$ *is an integrable function which is precisely of class* P_ν.

Let $1, c_1, \ldots$ be the Fourier constants of f, and $z_n (= \rho_n e^{i\phi_n})$ its parameters. We first observe that f cannot be an integrable function of class P_μ with $\mu < \nu$; for then, by Theorem III, we should have $f = 1/T_\mu$. It is, therefore, sufficient to show that $\rho_n = 0$ for $n > \nu$. Writing $D_n = D(c_1, \ldots, c_n)$, we know (P.H.I, Lemma 4) that

$$D_{n+1}/D_n = \prod_{r=1}^{n+1} (1-\rho_r^2).$$

We must, therefore, show that D_n/D_{n-1} is constant for $n \geqslant \nu$. This has been proved by Szegö (*loc. cit.*, Theorem XVI). We shall, however, complete the proof by the use of the relevant arguments of Szegö's paper.

Consider the positive definite Hermitian form

$$H = \sum_{i,\,j=1}^{n} a_{ij} y_i \bar{y}_j \quad (a_{ij} = \bar{a}_{ji}).$$

The first step in Kronecker's method of reduction consists in replacing this form by

$$\sum_{i,\,j=1}^{n-1} a_{ij} \eta_i \bar{\eta}_j + \frac{A}{A_{nn}} y_n \bar{y}_n, \tag{21}$$

where A is the determinant $|a_{ij}|$ $(i, j = 1, \ldots, n)$ and A_{nn} is the minor of a_{nn} in that determinant. The first term in (21) being positive definite, we can say that

$$\min_{y_n = 1} H = A/A_{nn}. \tag{22}$$

Apply this to the positive definite Laurentian form

$$L_n = \sum_{i,\,j=0}^{n} c_{j-k} x_i \bar{x}_j.$$

The determinant is

$$D(c_1, \ldots, c_n) = \begin{vmatrix} 1 & c_1 & \ldots & c_n \\ c_{-1} & 1 & \ldots & c_{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ c_{-n} & \ldots & \ldots & 1 \end{vmatrix}. \tag{23}$$

Write $y_j = x_{n-j}$. Then the form

$$\mathfrak{L}_n = \sum_{j,\,k=0}^{n} c_{j-k} y_{n-j} \bar{y}_{n-k}$$

has the determinant $D(c_{-1}, \ldots, c_{-n})$; and the minor of the element in the bottom right-hand corner of that determinant is $D(c_{-1}, \ldots, c_{-(n-1)})$, which, by the structure of a Laurentian determinant, is D_{n-1}. Thus

$$\min_{x_0 = 1} L_n = \min_{y_n = 1} \mathfrak{L}_n = D_n/D_{n-1}.$$

But $$L_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,|x_0+x_1e^{it}+\ldots+x_ne^{nit}|^2\,dt.$$

Hence, writing $$Q(e^{it}) = x_0+x_1e^{it}+\ldots+x_ne^{nit},$$

we have $$\frac{D_n}{D_{n-1}} = \min\frac{1}{2\pi}\int f(t)\,|Q(e^{it})|^2\,dt. \tag{24}$$

Since $T_\nu(x)$ is a positive trigonometric polynomial of order ν, we can, by a theorem of F. Riesz and Fejér*, write

$$T_\nu(x) = c\,|1+a_1e^{ix}+\ldots+a_\nu e^{i\nu x}|^2 = c\,|P(e^{ix})|^2,$$

where $c>0$. We can arrange that $P(z)\neq 0$ for $|z|<1$. For, if z_0 is a zero of $P(z)$ in $|z|<1$, we can replace $P(z)$ by $P(z)(1-\bar{z}_0z)/(z-z_0)$. Also, $P(e^{i\theta})\neq 0$ since $T_\nu(x)>0$ for all x. Hence, by Cauchy's theorem, $A(e^{it}Q/P) = x_0$, and so $A(|Q/P|^2)\geqslant[A(|Q/P|)]^2\geqslant x_0{}^2$. By (24), $D_n/D_{n-1}\geqslant 1/c$ for $n=1,\ 2,\ \ldots$.

For $n\geqslant\nu$, we can give $x_1,\ \ldots,\ x_n$ the special values $a_1,\ \ldots,\ a_\nu,\ 0,\ \ldots,\ 0$; and then $A(f|Q|^2) = 1/c$. By (24), $D_n/D_{n-1}\leqslant 1/c$ for $n\geqslant\nu$. Hence $D_n/D_{n-1} = 1/c$ for $n\geqslant\nu$, which proves the theorem.

8. We showed in P.H. I (150) that, if $1,\ c_1,\ c_2,\ \ldots$ are the Fourier constants of a p.h.f. of class P_2, and $z_1,\ z_2$ are its parameters, then

$$\frac{z_1+z_2z}{1+(\bar{z}_1z_2-z_1)z-z_2z^2} = \sum_{n=1}^{\infty} c_nz^{n-1}.$$

Hence $$F(z) = \frac{1+(\bar{z}_1z_2+z_1)z+z_2z^2}{1+(\bar{z}_1z_2-z_1)z-z_2z^2} = 1+2\Sigma\,c_nz^n.$$

The integrable functions of class P_2 are characterised by $0\leqslant\rho_1<1$, $0\leqslant\rho_2<1$. They are the functions $RF(e^{i\theta})$, namely (replacing θ by x)

$$\frac{(1-\rho_1{}^2)(1-\rho_2{}^2)}{[1+(\bar{z}_1z_2-z_1)e^{ix}-z_2e^{2ix}][1+(\bar{z}_1z_2-\bar{z}_1)e^{-ix}-\bar{z}_2e^{-2ix}]}. \tag{25}$$

For the purposes of the present discussion of the class P_2, we can simplify the notation and write

$$z_1 = \rho e^{i\theta},\ z_2 = \sigma e^{i\phi}\quad(0\leqslant\rho<1,\quad 0\leqslant\sigma<1).$$

---

* *Cf.* Szasz, *Math. Zeitschrift*, 4 (1919), 163–176.

The denominator in (25) is

$$k(x) = (1+\rho^2)(1+\sigma^2) - 2\rho^2\sigma\cos(2\theta-\phi)$$
$$+2\rho\cos x[2\sigma\cos(\theta-\phi) - (1+\sigma^2)\cos\theta]$$
$$+2\rho\sin x[2\sigma\sin(\theta-\phi) + (1+\sigma^2)\sin\theta]$$
$$-2\sigma\cos 2x\cos\phi + 2\sigma\sin 2x\sin\phi.$$

With the notation $\Pi = (1-\rho^2)(1-\sigma^2)$

we have

THEOREM V. *Every normalized positive trigonometric polynomial of order 2 is of the form* $k(x)/\Pi$.

Let $$\alpha_0+\alpha_1\cos x+\beta_1\sin x+\alpha_2\cos 2x+\beta_2\sin 2x \tag{26}$$

be a normalized positive trigonometric polynomial of order 2. We then have the five equations

$$\Pi\alpha_0 = (1+\rho^2)(1+\sigma^2) - 2\rho^2\sigma\cos(2\theta-\phi), \tag{27}$$

$$\Pi\alpha_1 = 2\rho[2\sigma\cos(\theta-\phi) - (1+\sigma^2)\cos\theta], \tag{28}$$

$$\Pi\beta_1 = 2\rho[2\sigma\sin(\theta-\phi) + (1+\sigma^2)\sin\theta], \tag{29}$$

$$\Pi\alpha_2 = -2\sigma\cos\phi, \tag{30}$$

$$\Pi\beta_2 = 2\sigma\sin\phi. \tag{31}$$

We shall now show that the four unknowns $\rho\cos\theta$, $\rho\sin\theta$, $\sigma\cos\phi$, $\sigma\sin\phi$, are members of the field $R(\alpha_0, \ldots, \beta_2)$, *i.e.* they can be obtained by rational operations on $\alpha_0, \ldots, \beta_2$.

If we put

$$h^2 = \alpha_2{}^2+\beta_2{}^2, \quad \delta = (1+\sigma^2)/(1-\sigma^2), \tag{32}$$

we find, from (30) and (31), that

$$\delta^2 = 1+h^2(1-\rho^2)^2. \tag{33}$$

We find, on eliminating ϕ from (27) by (30) and (31), that

$$\alpha_0 = \frac{1+\rho^2}{1-\rho^2}\delta - \rho^2[-\alpha_2\cos 2\theta + \beta_2\sin 2\theta]. \tag{34}$$

We may suppose that $\rho > 0$. For if $\rho = 0$, we have $\alpha_1 = \beta_1 = 0$, and then (26) is normalized if and only if the same is true of $\alpha_0 + \alpha_2 \cos x + \beta_2 \sin x$. This case is quite easy, and may be left to the reader.

If we square and add (28) and (29), and write $g^2 = \alpha_1^2 + \beta_1^2$, we obtain

$$\frac{g^2}{4\rho^2} = h^2 + \frac{\delta^2}{(1-\rho^2)^2} + \frac{2\delta}{(1-\rho^2)} [\alpha_2 \cos 2\theta - \beta_2 \sin 2\theta].$$

Substitute in (34) for the expression in square brackets. On multiplying the resulting equation by δ, and using (33), we get

$$\delta\alpha_0 = \frac{2+\rho^2}{2(1-\rho^2)} + (h^2 + \tfrac{1}{8}g^2)(1-\rho^2). \tag{35}$$

We write $$f^2 = h^2 + \tfrac{1}{8}g^2, \quad \lambda = 1-\rho^2.$$

On squaring (35), we obtain for λ the quartic equation

$$4\lambda^4(\alpha_0^2 h^2 - f^4) + 4f^2\lambda^3 + \lambda^2(4\alpha_0^2 - 12f^2 - 1) + 6\lambda - 9 = 0. \tag{36}$$

Write $$j = \alpha_1\alpha_2 + \beta_1\beta_2, \quad k = \alpha_1\beta_2 - \alpha_2\beta_1.$$

On eliminating ϕ from (28), (29) by (30), (31), and solving the resulting equations for $\sin\theta$, $\cos\theta$, we obtain

$$\lambda^2 j - \alpha_1\delta\lambda = 2\rho\cos\theta, \quad \lambda^2 k - \beta_1\delta\lambda = -2\rho\sin\theta. \tag{37}$$

On squaring and adding, we obtain

$$\lambda^4(j^2+k^2) - 2\delta\lambda^3(\alpha_1 j + \beta_1 k) + g^2\delta^2\lambda^2 = 4(1-\lambda).$$

Now $j^2 + k^2 = g^2 h^2$. Write

$$l = \alpha_1 j + \beta_1 k + (\alpha_1^2 - \beta_1^2)\alpha_2 + 2\alpha_1\beta_1\beta_2.$$

Substitute for δ^2 from (33), and for δ from (35). We get

$$\lambda^4(2\alpha_0 g^2 h^2 - 2f^2 l) + \lambda^3 l + \lambda^2(\alpha_0 g^2 - 3l) + 4\lambda\alpha_0 - 4\alpha_0 = 0. \tag{38}$$

The equations (36), (38) have a common root, namely $\lambda = 1-\rho^2$. If we suppose for the moment that this root can be obtained by rational operations from the two equations, it will follow from (35) that δ is in $R(\alpha_0, \ldots, \beta_2) \equiv R$. By (32), σ^2 is in that field. So then is Π. By (30) and (31), $\sigma\cos\phi$, $\sigma\sin\phi$ are in R; and, by (37), so are $\rho\cos\theta$, $\rho\sin\theta$.

9. We must, therefore, prove that the common root $\lambda = 1-\rho^2$ of (36), (38) can be obtained by rational operations on the coefficients of those equations. We first observe that, given any four numbers α_1, α_2, β_1, β_2, there is one and only one value of α which makes

$$T(\alpha) = \alpha+\alpha_1 \cos x+\ldots+\beta_2 \sin 2x$$

a normalized trigonometric polynomial. For, if α makes $T(\alpha)$ positive, then

$$\frac{\partial}{\partial\alpha}\int_0^{2\pi}\frac{dx}{T(\alpha)} = -\int_0^{2\pi}\frac{dx}{T^2} < 0.$$

This unique value of α, α_0 say, is thus a function of the four variables α_1, β_1, α_2, β_2. We denote by E the Euclidean space of four dimensions whose points are $(\alpha_1, \beta_1, \alpha_2, \beta_2)$. We are concerned only with points for which $h^2 = \alpha_2{}^2+\beta_2{}^2 > 0$.

In the following discussion it will be necessary to know that certain polynomials in the coefficients (36), (38) do not vanish identically in α_1, β_1, α_2, β_2. For this purpose it will be sufficient to know their values in a special case. We take

$$\rho = \sigma = \tfrac{1}{2}, \quad \theta = \phi = 0. \tag{39}$$

We find that equation (36) becomes

$$(4\lambda-3)(4\,.\,11\,.\,13\,\lambda^3+3\,.\,229\,\lambda^2+2\,.\,9^2\lambda+3\,.\,9^2) = 0, \tag{40}$$

and that the equation (38) becomes

$$(4\lambda-3)(4^3\,.\,11\,\lambda^3+4^2\,.\,3\,.\,10\lambda^2+9^2\,.\,7\lambda+3\,.\,9^2\,.\,7) = 0. \tag{41}$$

The equations (40), (41) have only one root in common. To prove this, it is sufficient to show that the resultant* of the two cubic factors in (40), (41) respectively is not zero. This is easily verified, since, if we replace every element in the resultant (which is a determinant of order 6) by its residue mod 5, the determinant so obtained differs from zero.

We write $\lambda\mu = 1$. Then (36) becomes

$$P(\mu) = 9\mu^4-6\mu^3-\mu^2(4\alpha_0{}^2-12f^2-1)-4f^2\mu-4(\alpha_0{}^2h^2-f^4) = 0,$$

* We use the definition of Bôcher, *Higher algebra* (1924), 195.

and (38) becomes

$$Q(\mu) = 4a_0\mu^4 - 4a_0\mu^3 - \mu^2(a_0g^2 - 3l) - \mu l - 2(a_0g^2h^2 - f^2l) = 0.$$

These equations have the root $\mu = (1-\rho^2)^{-1}$ in common. Hence the resultant $R_0(P, Q)$ is zero. On equating the resultant to zero, we obtain an equation of degree 12 in a_0, namely

$$R_0 = 16^3 a^{12} h^4[16h^2 + (4h^2 - g^2)^2] + \ldots = 0. \tag{42}$$

When (39) holds, the highest common factor of $P(\mu)$, $Q(\mu)$ is of the first degree. The first sub-resultant* R_1 of $P(\mu)$, $Q(\mu)$ is then different from zero. The first sub-resultant is a polynomial of degree 9 in a_0:

$$R_1 = -16a_0{}^9h^2 + \ldots. \tag{43}$$

Since R_0 is zero and $R_1 \neq 0$ when (39) holds, the resultant $R(R_0, R_1)$ of the two polynomials in a_0 does not vanish identically. Now $R(R_0, R_1)$ is a polynomial in $\alpha_1, \beta_1, \alpha_2, \beta_2$, so that $R = 0$ is an algebraic surface in E. For points not on that surface, R_0, R_1 have no common factor. Since $R_0 = 0$, we then have $R_1 \neq 0$. Hence the highest common factor of $P(\mu)$, $Q(\mu)$ is of degree 1. This highest common factor may be exhibited in the form

$$\begin{vmatrix} 9 & -6 & -(4a_0{}^2 - 12f^2 - 1) & -4f^2 & -4(a_0{}^2h^2 - f^4) & \mu^2P(\mu) \\ 0 & 9 & -6 & -(4a_0{}^2 - 12f^2 - 1) & -4f^2 & \mu P(\mu) \\ 0 & 0 & 9 & -6 & -(4a_0{}^2 - 12f^2 - 1) & P(\mu) \\ 0 & 0 & 4a_0 & -4a_0 & -(a_0g^2 - 3l) & Q(\mu) \\ 0 & 4a_0 & -4a_0 & -(a_0g^2 - 3l) & -l & \mu Q(\mu) \\ 4a_0 & -4a_0 & -(a_0g^2 - 3l) & -l & -2(a_0g^2h^2 - f^2l) & \mu^2 Q(\mu) \end{vmatrix}.$$

We observe that the term independent of μ is a polynomial in $\alpha_0, \ldots, \beta_2$, with integer coefficients, say $f(\alpha_0, \ldots, \beta_2)$. The coefficient of μ is $-R_1$, which is also a polynomial of the same kind. Thus, for points not on $R = 0$, we have $\mu = f/R_1$ for the common root of $P(\mu) = 0$, $Q(\mu) = 0$. Since $\mu = (1-\rho^2)^{-1} > 0$, we have

$$f \neq 0 \tag{44}$$

for points not on

$$R = 0. \tag{45}$$

* Bôcher, *loc. cit.*, 197.

The relation
$$1-\rho^2 = R_1/f \tag{46}$$

shows that ρ^2 can be obtained by rational operations on $\alpha_0, \ldots, \beta_2$, when $(\alpha_1, \ldots, \beta_2)$ does not lie on (45).

Suppose now that $(\alpha_1, \ldots, \beta_2)$ lies on (45). We can find four rational numbers r_i $(i = 1, \ldots, 4)$ such that, for all sufficiently small $d \neq 0$, the points $(\alpha_1+r_1 d, \ldots, \beta_2+r_4 d)$ do not lie on (45). Let $\alpha_0^{(d)}$ be the number which makes

$$T_d = \alpha_0^{(d)}+(\alpha_1+r_1 d)\cos x+\ldots+(\beta_2+r_4 d)\sin 2x$$

a normalized positive trigonometric polynomial. Let ρ_d be the corresponding value of ρ. Then $\lim \rho_d = \rho$ as $d \to 0$. Since (44) holds for a point not on (45), we have

$$1-\rho_d{}^2 = \frac{R_1(\alpha_0^{(d)}, \alpha_1+r_1 d, \ldots)}{f(\alpha^{(d)}, \alpha_1+r_1 d, \ldots)} \tag{47}$$

for all sufficiently small $d \neq 0$.

Consider the equation obtained from (42) on replacing α_0 by x:

$$Hx^{12}+\ldots = 0, \quad H = 16^3 h^4[16h^2+(4h^2-g^2)^2].$$

If $x = \alpha_0$ is a multiple root, we can, by a suitable number $\mu < 12$ of differentiations, obtain an equation

$$\mu!\binom{12}{\mu} Hx^{12-\mu}+\ldots = 0,$$

of which $x = \alpha_0$ is a simple root. Let $H_d, \ldots$ denote what $H, \ldots$ become when $\alpha_1, \ldots, \beta_2$ are replaced by $\alpha_1+r_1 d, \ldots, \beta_2+r_4 d$ respectively. Then, for all sufficiently small d, $H_d > 0$, since this holds for $d = 0$, and $\alpha_0^{(d)}$ is a simple root of

$$\mu!\binom{12}{\mu} H_d x^{12-\mu}+\ldots = 0.$$

Hence $\alpha_0^{(d)}-\alpha_0$ can be expanded in a power series in d; and the coefficients are rational functions of $\alpha_0, \ldots \beta_2$, whose coefficients are rational numbers. Thus the numerator and denominator in (47) are power series in d with coefficients of the afore-mentioned kind. It follows that $1-\rho^2$ is the quotient of the coefficients of the lowest (necessarily equal) powers of d in the expansions of R_1 and f respectively.

We have now proved that $\rho\cos\theta$, $\rho\sin\theta$, $\sigma\cos\phi$, $\sigma\sin\phi$ are rational functions (with integer coefficients) of $\alpha_0, \ldots, \beta_2$. Using the fact that the Fourier constants of a function of class P_2 are polynomials with integer coefficients in the parameters z_1, z_2, we can enunciate the following theorem.

THEOREM VI. *If* (26) *is a normalized positive trigonometric polynomial* $T(x)$, *then the Fourier constants of* $1/T$ *are rational functions of* $\alpha_0, \ldots, \beta_2$, *with integer coefficients.*

10. Given $$P(x) = \alpha_r \cos x + \ldots + \beta_2 \sin 2x,$$

to find the value of α_0 which makes $T = \alpha_0 + P$ normalized, *i.e.* to solve the integral equation

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{dx}{t+P(x)} = 1,$$

we have the equations (36), (38), which can be written as equations in α_0:

$$4\lambda^2(\lambda^2 h^2+1)\,\alpha_0{}^2 = (2\lambda^2 f^2-\lambda+3)^2, \tag{48}$$

$$(2\lambda^4 g^2 h^2+\lambda^2 g^2+4\lambda-4)\alpha_0 = l\lambda^2(2\lambda^2 f^2-\lambda+3). \tag{49}$$

We may suppose that $g^2 = \alpha_1{}^2+\beta_1{}^2 > 0$, so that $\rho^2 > 0$. For $\lambda = 1-\rho^2$, we have $1 > \lambda > 0$, and (48) shows that $2\lambda^2 f^2-\lambda+3 \neq 0$. We have three cases.

(α) $l = 0$. Equation (49) gives

$$2\lambda^4 g^2 h^2+\lambda^2 g^2+4\lambda-4 = 0.$$

The first member is an increasing function of λ for $\lambda > 0$. It is negative for $\lambda = 0$, positive for $\lambda = 1$. Hence it vanishes for precisely one value Λ which lies between 0 and 1. The equation (48) then gives the value of α_0.

(β) $l > 0$. By (35), we have

$$\sqrt{(1+h^2\lambda^2)}\,\alpha_0 = \frac{3-\lambda}{2\lambda}+f^2\lambda = \frac{3-\lambda+2f^2\lambda^2}{2\lambda},$$

i.e., by (49),

$$2l\lambda^3\sqrt{(1+h^2\lambda^2)} = 4-4\lambda-\lambda^2 g^2-2\lambda^4 g^2 h^2.$$

We write

$$Y = 2l\lambda^3\sqrt{(1+h^2\lambda^2)}, \quad Z = 2\lambda^4 g^2 h^2 + \lambda^2 g^2 + 4\lambda - 4.$$

We find from (48), (49),

$$(Y-Z)(Y+Z) \equiv 4l^2\lambda^6(1+\lambda^2 h^2) - (2\lambda^4 g^2 h^2 + \lambda^2 g^2 + 4\lambda - 4)^2 = 0. \qquad (50)$$

The functions Y, Z are increasing functions of λ for $\lambda > 0$; $Y+Z$ is negative for $\lambda = 0$ and positive for $\lambda = 1$. Hence $Y+Z$ has precisely one zero in $(0, \infty)$, and this zero Λ lies between 0 and 1. This is the required value of $1-\rho^2$. Since $Y(\Lambda) > 0$, we have $Z(\Lambda) < 0$. Since Z is an increasing function of λ for $\lambda > 0$, we have $Z < 0$ in $(0, \Lambda)$. Hence $Y-Z > 0$ in $(0, \Lambda)$. Hence Λ is the least positive root of (50). If this root is known, the value of α_0 is obtained from (49). It will be seen from our discussion of case (γ) that $Y-Z$ has precisely one zero in $(0, \infty)$. Thus (50) has two positive roots, and the smaller one gives the required value of $1-\rho^2$.

(γ) $l < 0$. We first prove that $Y+Z$ is an increasing function of λ for $\lambda > 0$. We observe that

$$l^2 - g^4 h^2 = -[\beta_2(\alpha_1^2 - \beta_1^2) - 2\alpha_2\alpha_1\beta_1]^2 \leqslant 0. \qquad (51)$$

We have

$$\frac{dY}{d\lambda} = \frac{6l\lambda^2 + 8lh^2\lambda^4}{\sqrt{(1+h^2\lambda^2)}}, \quad \frac{dZ}{d\lambda} > 8\lambda^3 g^2 h^2 + 2\lambda g^2.$$

It is sufficient to show that

$$\left(\frac{dZ}{d\lambda}\right)^2 > \left(\frac{dY}{d\lambda}\right)^2$$

for $\lambda > 0$; *i.e.* that

$$g^4(1+4h^2\lambda^2)^2(1+h^2\lambda^2) > l^2\lambda^2(3+4h^2\lambda^2)^2,$$

which will follow from (51) if we prove that

$$(1+4h^2\lambda^2)^2(1+h^2\lambda^2) - h^2\lambda^2(3+4h^2\lambda^2)^2 > 0.$$

This is true, the first member being 1.

The increasing function $Y+Z$ is negative for $\lambda = 0$ and positive for $\lambda = 1$. It has precisely one zero Λ in $(0, \infty)$. Again, $Y-Z$ is a decreasing

function of λ for $\lambda > 0$. It is positive for $\lambda = 0$, and negative for $\lambda = 1$, so that $Y - Z$ has also precisely one zero in $(0, \infty)$. But, since $Y(\Lambda) < 0$, we have $Z(\Lambda) > 0$ and $Y(\Lambda) - Z(\Lambda) < 0$. Hence the zero of $Y - Z$ is less than that of $Y + Z$. We can, therefore, say that the required value of $1 - \rho^2$ is the greatest positive root of (50). If this root is known, the value of α_0 is obtained from (49).

If we write $P(x)$ in the form $\gamma_1 \cos(x + \mu_1) + \gamma_2 \cos(2x + 2\mu_2)$ with $\gamma_1 > 0$, $\gamma_2 > 0$, then

$$\alpha_1 = \gamma_1 \cos \mu_1, \quad \beta_1 = -\gamma_1 \sin \mu_1, \quad \alpha_2 = \gamma_2 \cos 2\mu_2, \quad \beta_2 = -\gamma_2 \sin 2\mu_2,$$

and

$$l = \gamma_1^2 \gamma_2 \cos(2\mu_1 - 2\mu_2).$$

If $l = 0$, then $2(\mu_1 - \mu_2) = (2r + 1)\pi/2$ and

$$P(x) = \gamma_1 \cos(x + \mu_1) + (-1)^r \gamma_2 \sin(2x + 2\mu_2).$$

By the transformation $x + \mu_1 = x'$, or $x + \mu_1 = x' + \pi$, we obtain

$$\gamma_1 \cos x' + (-1)^r \gamma_2 \sin 2x' \quad \text{or} \quad -\gamma_1 \cos x' + (-1)^r \gamma_2 \sin 2x'.$$

We may therefore take $P(x) = \alpha_1 \cos x + \beta_2 \sin 2x$ as the normal form of $P(x)$ when $l = 0$. Here $g^2 = \alpha_1^2$, $h^2 = \beta_2^2$, so that λ and α_0 have the same values for $P(x)$ as for $-P(x)$; *i.e.* the equations $A[t + P] = 1$, $A[t - P] = 1$ are satisfied by the same value of t.

If $l > 0$, then, for $-P(x)$, the functions f^2, g^2, h^2 have the same values as for $P(x)$, while l becomes $-l$. It follows, from our discussion of the cases (β), (γ), that the greater positive root of (50) is the value of λ which corresponds to the solution of

$$A[t - P] = 1. \tag{52}$$

Similarly, when $l < 0$, and the greater positive root of (50) corresponds to the solution of the original integral equation, the smaller positive root corresponds to the solution of (52).

When $l^2 - g^4 h^2 = 0$, the equation (50) simplifies. By (51), we then have $\beta_2(\alpha_1^2 - \beta_1^2) = 2\alpha_2 \alpha_1 \beta_1$, which is, with the above notation,

$$\gamma_1^2 \gamma_2 \sin(2\mu_1 - 2\mu_2) = 0.$$

It follows that in this case we may take $P(x) = \alpha_1 \cos x + \alpha_2 \cos 2x$, and

$l = a_1^2 a_2$. The equation (50) is then

$$16\lambda^5 g^2 h^2 + \lambda^4 g^2(g^2 - 16h^2) + 8\lambda^3 g^2 + 8\lambda^2(2 - g^2) - 32\lambda + 16 = 0,$$

and $1-\rho^2$ is the smaller or the greater positive root according as $a_2 > 0$ or $a_2 < 0$.

11. In this section we obtain a result (Theorem VII) which is a generalization of a theorem of Szegö (*loc. cit.*, 185), whose methods we employ. We use the notation (iv), (v) of § 1.

We denote by B the class of functions $f(x)$ which are defined in $(0, 2\pi)$, and are bounded and B-measurable in that interval. Let $F(x)$ be a non-diminishing function, defined in $(0, 2\pi)$. We consider the greatest lower bound of

$$\frac{1}{2\pi}\int_0^{2\pi} e^{f(x)}\, dF \tag{53}$$

for all functions $f(x)$ of B, subject to

$$A(f) = 0. \tag{54}$$

LEMMA 5. *The greatest lower bound of* (53) *for all* $f \subset B$ *subject to* (54) *is* $G(F')$.

Let E denote the set of points in $0 < x < 2\pi$ at which $F'(x) = \infty$. Let n denote a positive integer. Let $\phi = f$ for $x \subset CE$, $\phi = -n$ for $x \subset E$, where $f \subset B$ satisfies (54). Then $\phi \subset B$ and $A(\phi) = 0$. For all sufficiently large n,

$$\int_0^{2\pi} e^f\, dF \geqslant \int_0^{2\pi} e^\phi\, dF,$$

while

$$\lim_{n\to\infty} \int_0^{2\pi} e^\phi\, dF = \int_0^{2\pi} e^\phi\, F'(x)\, dx.$$

We therefore have to consider

$$I = \inf_{f \subset B} \frac{1}{2\pi}\int_0^{2\pi} e^{f(x)} F'(x)\, dx = \inf_{f \subset B} A(e^f F'),$$

subject to (54). We have, for any f which satisfies the stated conditions,

$$G(F') = \exp[A(\log F')] = \exp[A(f + \log F')].$$

The last expression is

$$G(e^f F') \leqslant A(e^f F')$$

by a known inequality*. Thus

$$I \geqslant G(F'). \tag{55}$$

Let $F_n' = \max [F'(x), 1/n], \quad \Phi_n' = \min(F_n', n).$

Write $$\int_0^{2\pi} \log \Phi_n' \, dx = 2\pi \log c.$$

Consider the function $f(x) = \log c - \log \Phi_n'$. Then f is bounded and B-measurable, and it satisfies (54). We have

$$A(e^f F') = A(cF'/\Phi_n') = G(\Phi_n') A(F'/\Phi_n').$$

The limit of the last expression as $n \to \infty$ is $G(F')$. Hence $I \leqslant G(F')$, and the lemma follows from (55).

If $f \subset B$ does not satisfy (54), then, if we write $A(f) = c$, the function $f - c \subset B$ satisfies (54). But

$$\frac{1}{2\pi}\int_0^{2\pi} e^f \, dF/G(e^f) = \frac{1}{2\pi}\int_0^{2\pi} e^{f-c} \, dF/G(e^{f-c}) = \frac{1}{2\pi}\int_0^{2\pi} e^{f-c} \, dF.$$

We therefore have

LEMMA 6. $$\inf_{f \subset B} \frac{1}{2\pi}\int_0^{2\pi} e^f \, dF/G(e^f) = G(F'). \tag{56}$$

If $f \subset B$, then e^f is bounded and B-measurable, and $\inf e^f > 0$. Conversely, if ϕ is bounded and B-measurable, and $\inf \phi > 0$, then $\log \phi \subset B$. We denote this class of functions ϕ by B'. Then (56) implies that

$$\inf_{f \subset B'} \frac{1}{2\pi}\int_0^{2\pi} f \, dF/G(f) = G(F').$$

Write $$F(f) = \frac{1}{2\pi}\int_0^{2\pi} f \, dF.$$

Let C denote the class of functions which are continuous and positive in

* Hardy, Littlewood, and Pólya, *Inequalities* (1934), 136.

$(0, 2\pi)$. We shall prove that

$$\inf_{f \subset B'} F(f)/G(f) = \inf_{\phi \subset C} F(\phi)/G(\phi). \tag{57}$$

In the first place, since $C \subset B'$,

$$J = \inf_{\phi \subset C} F(\phi)/G(\phi) \geqslant G(F'). \tag{58}$$

LEMMA 7. *Given a function $f \subset B'$, there is a function $\phi \subset C$ such that*

$$|F(f)/G(f) - F(\phi)/G(\phi)| < \epsilon, \tag{59}$$

where ϵ is any assigned positive number.

The lemma is true if $f \subset C$. If f is of Class 1, then

$$f = \lim \phi_n, \tag{60}$$

where ϕ_n is continuous, and

$$\inf f \leqslant \inf \phi_n \leqslant \sup \phi_n \leqslant \sup f. \tag{61}$$

Then $$\lim F(\phi_n)/G(\phi_n) = F(f)/G(f),$$

so that for the ϕ of the lemma we can choose a suitable ϕ_n.

Suppose that the lemma is true for any function $f \subset B'$ which is of class less than α. If f is of class α, we have (60), (61), where ϕ_n is of class less than α. For a suitable n,

$$|F(\phi_n)/G(\phi_n) - F(f)/G(f)| < \tfrac{1}{2}\epsilon;$$

and then, since the lemma holds for ϕ_n, there is a $\phi \subset C$ such that

$$|F(\phi_n)/G(\phi_n) - F(\phi)/G(\phi)| < \tfrac{1}{2}\epsilon.$$

Thus the lemma holds for functions of class α. By transfinite induction, it holds for all $f \subset B'$.

By (59), $$J \leqslant \inf_{f \subset B'} F(f)/G(f) = G(F'),$$

so that (57) follows from (58).

Let (T) denote the class of positive trigonometric polynomials, and let (T_n) denote the class of positive trigonometric polynomials of order n. Then $(T) \subset C$, and given $\phi \subset C$ and $\epsilon > 0$, there is a trigonometric polynomial $P \subset (T)$ such that $\max |\phi - P| < \epsilon$. Hence

$$\inf_{\phi \subset (T)} F(\phi)/G(\phi) = G(F'). \tag{62}$$

The sequence

$$\inf_{(T_n)} F(T_n)/G(T_n) \quad (n = 0, 1, \ldots),$$

where, for each n, the greatest lower bound is taken for all positive trigonometric polynomials T_n of order n, is non-increasing. The limit of the sequence is seen from (62) to be $G(F')$. We thus have

THEOREM VII. *If $F(x)$ is non-decreasing in $(0, 2\pi)$, then*

$$\lim_{n\to\infty} \inf_{(T_n)} \frac{1}{2\pi} \int_0^{2\pi} T_n \, dF/G(T_n) = G(F').$$

12. Theorems III and IV furnish a parametric representation of positive trigonometric polynomials of order ν. Any positive trigonometric polynomial of order ν, T_ν say, can therefore be uniquely expressed in the form

$$T_\nu = cH(e^{i\theta})\bar{H}(e^{i\theta})/\Pi \quad (c > 0), \tag{63}$$

where

$$\Pi = \prod_{r=1}^{\nu} (1 - \rho_r^2),$$

and $z_r = \rho_r e^{i\phi_r}$ $(r = 1, \ldots, \nu)$. We showed, in Lemma 2, that $H(z)$ has no zeros in $|z| \leqslant 1$. Thus $\log H(z)$ is regular for $|z| \leqslant 1$, and $\log|H(z)|$ is an harmonic function in that domain. Hence

$$\frac{1}{2\pi} \int_0^{2\pi} \log|H(e^{i\theta})|^2 \, d\theta = \log|H(0)|^2 = 0,$$

since the constant term in $H(z)$ is 1. Then $A(\log T_\nu) = \log(c/\Pi)$, and

$$G(T_\nu) = c/\Pi. \tag{64}$$

Let (i), § 1, be a p.h.f. We can then write

$$F(x) = x - \Sigma(b_n \cos nx - a_n \sin nx)/n,$$

where $F(x)$ is non-decreasing. By (63), (64), we have

$$\frac{1}{2\pi}\int_0^{2\pi} T_\nu dF/G(T_\nu) = \frac{1}{2\pi}\int_0^{2\pi} H.\bar{H}dF. \tag{65}$$

We observe that, since $H.\bar{H}$ is a trigonometric polynomial of order ν, we can, without altering the value of the integral last written, replace F by any function of bounded variation Φ, whose expansion

$$\Phi = x - \Sigma(f_n \cos nx - e_n \sin nx)/n$$

agrees with that of F in the first $\nu+1$ terms, *i.e.* $f_n = b_n$, $e_n = a_n$ for $n \leqslant \nu$. Let $\zeta_n (= \sigma_n e^{i\theta_n})$ be the parameters of (i). Suppose, in the first place, that $|\zeta_n| < 1$ for all n. Since $c_1, \ldots, c_\nu$ are determined by $\zeta_1, \ldots, \zeta_\nu$, we can choose Φ so that

$$1 + \Sigma(e_n \cos nx + f_n \sin nx) r^n$$

is of class P_ν, with parameters $\zeta_1, \ldots, \zeta_\nu, 0, 0, \ldots$. Then Φ is the integral of

$$\prod_{r=1}^{\nu} (1-\sigma_r^2)/J(e^{i\theta})\ \bar{J}(e^{i\theta}),$$

where J is what H becomes when $z_1, \ldots, z_\nu$ are replaced by $\zeta_1, \ldots, \zeta_\nu$. Thus, by (65),

$$\inf_{(T_\nu)} \frac{1}{2\pi}\int_0^{2\pi} T_\nu dF/G(T_\nu) = \prod_{r=1}^{\nu} (1-\sigma_r^2) \inf A(|H/J|^2), \tag{66}$$

where, in the second member, we take the greatest lower bound for all values of $z_1, \ldots, z_\nu$ subject to $|z_\nu| < 1$. Since we can take $H = J$, we have

$$\inf A(|H/J|^2) \leqslant 1. \tag{67}$$

On the other hand,

$$A(|H/J|^2) \geqslant [A(|H/J|)]^2 \geqslant |A(H/J)|^2.$$

But H/J is regular in $|z| \leqslant 1$, and its constant term is 1. Hence, by Cauchy's theorem, $A(e^{i\theta} H/J) = 1$, and $A(|H/J|^2) \geqslant 1$. This shows that the sign of inequality holds in (67). By Theorem VII and (66), we have

$$G(F') = \prod_{r=1}^{\infty} (1-\sigma_r^2).$$

If, for some n, $|\zeta_n| = 1$, then F is a step function with a finite set of discontinuities in $(0, 2\pi)$, and both sides of the preceding equation are

equal to zero. We thus have

THEOREM VIII. *If*

$$F(x) = x - \Sigma(b_n \cos nx - a_n \sin nx)/n$$

is non-decreasing, and z_n $(= \rho_n e^{i\phi_n})$ *are the parameters of* (i), *then*

$$G(F') = \prod_{r=1}^{\infty} (1-\rho_r^2).$$

13. Let $T_\nu(x)$ be a positive trigonometric polynomial of order ν. There is one positive number γ such that

$$\int_0^{2\pi} \log [\gamma T_\nu(x)]\, dx = 0.$$

The positive trigonometric polynomial γT_ν is uniquely expressible in the form

$$\gamma T_\nu(x) = \delta H(e^{ix}) \bar{H}(e^{ix})/\Pi \quad (\delta > 0).$$

We have seen that $\int_0^{2\pi} \log H . \bar{H}\, dx = 0,$

so that $\delta = \Pi$. Thus

$$\frac{1}{2\pi} \int_0^{2\pi} \log T_\nu(x)\, dx = -\log \gamma,$$

where γ is uniquely determined by the condition that, for suitable values of the parameters, $\gamma T_\nu = H\bar{H}$.

Let (26), which we denote by $T(x)$, be a positive trigonometric polynomial of order 2 which satisfies $A(\log T) = 0$. Then $T = H\bar{H}$, with $\nu = 2$, for suitable values z_i $(i = 1, 2)$ of modulus less than 1. As in § 8, we write $z_1 = \rho e^{i\theta}$, $z_2 = \sigma e^{i\phi}$. In § 8 we obtained an expression for $H\bar{H}$. We accordingly have the five equations

$$a_0 = (1+\rho^2)(1+\sigma^2) - 2\rho^2\sigma \cos(2\theta - \phi), \tag{68}$$

$$a_1 = 2\rho[2\sigma \cos(\theta - \phi) - (1+\sigma^2) \cos\theta], \tag{69}$$

$$\beta_1 = 2\rho[2\sigma \sin(\theta - \phi) + (1+\sigma^2) \sin\theta], \tag{70}$$

$$a_2 = -2\sigma \cos\phi, \tag{71}$$

$$\beta_2 = 2\sigma \sin\phi. \tag{72}$$

Write $$4\xi^2 = \alpha_1^2+\beta_1^2, \quad 4\eta^2 = \alpha_2^2+\beta_2^2.$$

Then $$\eta^2 = \sigma^2. \tag{73}$$

Eliminating ϕ from (69) and (70) by (71) and (72), and solving for $\cos\theta$, $\sin\theta$, we obtain

$$\alpha_1\alpha_2+\beta_1\beta_2-\alpha_1(1+\eta^2) = 2\rho\cos\theta(1-\eta^2)^2,$$

$$\alpha_1\beta_2-\beta_1\alpha_2-\beta_1(1+\eta^2) = -2\rho\sin\theta(1-\eta^2)^2.$$

Square and add. On writing

$$4\lambda = \alpha_1(\alpha_1\alpha_2+\beta_1\beta_2)+\beta_1(\alpha_1\beta_2-\beta_1\alpha_2),$$

we get

$$\rho^2(1-\eta^2)^4 = 4\xi^2\eta^2-2\lambda(1+\eta^2)+\xi^2(1+\eta^2)^2. \tag{74}$$

On the other hand, if we eliminate ϕ from (69) and (70) we easily derive the equations

$$\alpha_1\cos\theta+\beta_1\sin\theta = -2\rho\alpha_2-2\rho(1+\eta^2)\cos 2\theta,$$

$$\alpha_1\sin\theta-\beta_1\cos\theta = 2\rho\beta_2-2\rho(1+\eta^2)\sin 2\theta.$$

Square and add. We get

$$\xi^2 = 4\rho^2\eta^2+\rho^2(1+\eta^2)^2+2\rho^2(1+\eta^2)\,[\alpha_2\cos 2\theta-\beta_2\sin 2\theta]. \tag{75}$$

We now eliminate ϕ from (68) by (71) and (72), and then eliminate θ by (75). We get, on substituting for ρ^2 from (74),

$$\alpha_0 = 1+\eta^2+\frac{\xi^2(1+\eta^2)-\lambda}{(1-\eta^2)^2}. \tag{76}$$

Since $\sigma^2<1$, $\rho^2<1$, (73) and (74) give the conditions

$$\eta^2<1, \tag{77}$$

$$(1-\eta^2)^4 > 4\xi^2\eta^2-2\lambda(1+\eta^2)+\xi^2(1+\eta^2)^2. \tag{78}$$

Suppose now that we have five numbers $\alpha_0, \ldots, \beta_2$ which satisfy (76), (77), (78). By (73), (71), (72), we obtain a value σ such that $0\leqslant\sigma<1$, and, when $\sigma>0$, an angle ϕ. By (78), we obtain from (74) a value ρ such that $0\leqslant\rho<1$, and, when $\rho>0$, from the two equations following (73), *i.e.*

from (69) and (70), an angle θ. With these values, the second member of (68) is equal to the second member of (76), *i.e.* to α_0, so that (68) holds. Hence $T(x)$ is of the form $H\bar{H}$, and is therefore a positive trigonometric polynomial which satisfies

$$A(\log T) = 0. \tag{79}$$

We thus have

THEOREM IX. *Necessary and sufficient conditions that the polynomial* $T(x)$, *given by* (26), *shall be positive and satisfy* (79) *are that* (76), (77), (78) *shall be satisfied.*

14. Given four numbers $\alpha_1, \beta_1, \alpha_2, \beta_2$, there does not necessarily exist a number α_0 which makes (26) a positive trigonometric polynomial $T(x)$ satisfying (79). Let

$$-m = \min(\alpha_1 \cos x + \ldots + \beta_2 \sin 2x) = \min P(x).$$

Then $m+P(x) \geqslant 0$ for all x. Any number ξ which makes $\xi+P(x) > 0$ for all x satisfies $A[\log(\xi+P)] > A[\log(m+P)]$. If, then, $A[\log(m+P)] \geqslant 0$, there will be no number α_0 with the required properties. If, however,

$$A[\log(m+P)] < 0, \tag{80}$$

then, since $A[\log(\xi+P)]$ is, for $\xi > m$, an increasing function of ξ which takes positive values (*e.g.* for $\xi > 1+m$), there is precisely one value $\xi = \alpha_0$ which satisfies (79). This value of α_0 is the value given by (76). If (77) and (78) hold, then the preceding proof shows that α_0, given by (76), makes $T(x)$ positive and satisfies (79). Thus the condition (80) is equivalent to (77) and (78).

We now consider the cases in which

$$A[\log(m+P)] = 0. \tag{81}$$

If $0 < \lambda < 1$, then $$A[\log(\lambda m + \lambda P)] < 0. \tag{82}$$

Hence, by what precedes, there is an $\alpha = \alpha(\lambda)$ such that $\alpha, \lambda\alpha_1, \ldots, \lambda\beta_2$ are respectively equal to the second members of (68)–(72) for suitable $\rho, \sigma, \theta, \phi$ with $0 \leqslant \rho < 1$, $0 \leqslant \sigma < 1$. By the principle of selection, we can give λ a sequence of values such that $\rho, \sigma, \theta, \phi$ tend to limits as $\lambda \to 1$. Further,

$$A[\log(m+\lambda P)] = \log \lambda^{-1} + A[\log(m/\lambda + P)] > 0,$$

by (81). By (82), $\lambda m < a < m$, and so $a \to m$ as $\lambda \to 1$. Thus $a_0 = m$, $a_1, \ldots, \beta_2$ are given by (68)–(72) for suitable $\rho, \sigma, \theta, \phi$. But $\rho < 1$, $\sigma < 1$ cannot both be satisfied, since then we would have (80) and not (81). Hence $(1-\rho)(1-\sigma) = 0$. This gives rise to three cases.

(*a*) $\rho = 1$, $\sigma < 1$. Here $\eta^2 < 1$, and the equations (68)–(72) can be treated as before. We obtain a_0 by (76) and, instead of (78), we have

$$(1-\eta^2)^4 = 4\xi^2\eta^2 - 2\lambda(1+\eta^2) + \xi^2(1+\eta^2)^2.$$

(*b*) $\rho < 1$, $\sigma = 1$. Here $\eta^2 = 1$. The two equations which follow (73) simplify, and are equivalent to

$$a_1(a_2-2) + \beta_1\beta_2 = 0, \quad a_1\beta_2 - \beta_1(a_2+2) = 0. \tag{83}$$

If $a_1^2 + \beta_1^2 > 0$, we can, by subjecting $P(x)$ to the transformation $x = x' + t$, arrange that $\beta_1 = 0$, $a_1 \neq 0$. Then, by (83), $a_2 = 2$, $\beta_2 = 0$. By (71), $\cos\phi = -1$, and, by (69), $a_1 = -8\rho\cos\theta$. Further, (68) gives

$$a_0 = 2 + 4\rho^2\cos 2\theta = 2 + \tfrac{1}{16}a_1^2.$$

We thus obtain the class of polynomials

$$2 + \tfrac{1}{16}a_1^2 + a_1\cos(x+t) + 2\cos(2x+2t) \quad (|a_1| < 8). \tag{84}$$

If $a_1^2 + \beta_1^2 = 0$, then (69) and (70) give, on squaring and adding,

$$32\rho^2\{1 - \cos(2\theta - \phi)\} = 0.$$

If $\rho = 0$, then (68) gives $a_0 = 2$, and we have the polynomials

$$2 - 2\cos(2x+\phi), \tag{85}$$

which can be obtained from (84) by putting $a_1 = 0$, $t = (\phi+\pi)/2$. If $\rho > 0$, then $\cos(2\theta - \phi) = 1$; (68) gives $a_0 = 2$, and we again get (85).

(*c*) $\rho = 1$, $\sigma = 1$. This is treated as case (*b*). The only difference is that in (84) the possible values of a_1 are given by $|a_1| \leqslant 8$. We thus have

THEOREM X. *Given the trigonometric polynomial*

$$P(x) = a_1\cos x + \beta_1\sin x + a_2\cos 2x + \beta_2\sin 2x,$$

we write

$$4\xi^2 = a_1^2 + \beta_1^2, \quad 4\eta^2 = a_2^2 + \beta_2^2, \quad 4\lambda = a_2(a_1^2 - \beta_1^2) + 2a_1\beta_1\beta_2.$$

The integral equation in a_0, $A[\log(a_0+P)]=0$, *has a solution only in the following cases.*

(i) *If* $\eta^2<1$ *and* $(1-\eta^2)^4 \geqslant 4\xi^2\eta^2-2\lambda(1+\eta^2)+\xi^2(1+\eta^2)^2$. *The solution is*

$$a_0 = 1+\eta^2+\frac{\xi^2(1+\eta^2)-\lambda}{(1-\eta^2)^2}.$$

(ii) *If* $\eta^2=1$ *and* $P(x)$ *is of the form* $a_1\cos(x+t)+2\cos 2(x+t)$, *where* $|a_1|\leqslant 8$. *The solution is*

$$a_0 = 2+\tfrac{1}{16}a_1{}^2.$$

The University,
Manchester.

THE SUMMABILITY OF DOUBLE POWER SERIES

By B. E. LAWRENCE.

[Received 8 January, 1935.—Read 17 January, 1935.]

In its simplest form Fatou's theorem states that

If $a_n \to 0$, $f(x) = \Sigma a_n x^n$ *converges at every regular point of* $|x| = 1$.

In an attempt to extend the theorem to double series I have considered the more general problem of summability (C, β, γ). The method is an extension to functions of two variables of the method used in a paper on single power series by A. C. Offord*.

In the first five paragraphs I find (Theorem I) a condition for summability in the form of a double integral taken round contours in the neighbourhood of $x = 1$, $y = 1$. The condition for summability at $x = e^{i\theta}$, $y = e^{i\phi}$ can be deduced immediately.

In the following paragraphs the condition is used to prove an analogue (Theorem II, § 8) of the following theorem, due to P. Dienes:

If $a_n = o(n^\gamma)$, *where* $\gamma > 0$, *and if* $\Sigma a_n x^n \to S$ *as* $x \to 1$ *along any path within* $|x| = 1$, *then the series* Σa_n *is summable* (C, γ) *to* S.

Finally, Fatou's theorem is considered. Theorem III and the examples given with it (§ 10) are of interest in showing that $a_{mn} \to 0$ (in every sense) is not a sufficient condition for the truth of Fatou's theorem for functions of two variables.

I must acknowledge here the suggestions and criticisms of Dr. L. S. Bosanquet during the preparation of the paper and further suggestions by Dr. Bosanquet and Prof. Hardy after it had been written. In particular, the paragraphs on Fatou's theorem have been much improved by their help.

* *Proc. London Math. Soc.* (2), 33 (1932), 467–480.

1. *Introduction.* If $f(x, y) = \Sigma\Sigma a_{\mu\nu} x^\mu y^\nu$ is regular* for $|x|<1$, $|y|<1$, we have

$$(1) \qquad s_{mn} = \sum_0^m \sum_0^n a_{\mu\nu} = -\frac{1}{4\pi^2}\int_C\int_D \frac{f(x, y)}{x^{m+1}y^{n+1}(1-x)(1-y)}\,dx\,dy,$$

where C, D are contours enclosing the origin and lying entirely within $|x|=1$, $|y|=1$.

The double series $\Sigma\Sigma a_{\mu\nu}$ converges† to S if

$$(2) \qquad \int_C\int_D \frac{f(x, y)-S}{x^{m+1}y^{n+1}(1-x)(1-y)}\,dx\,dy = o(1).$$

On replacing a_{00} by $a_{00}-S$ and generalizing convergence to summability, we see that the series is summable† (C, β, γ) to zero if

$$(3) \qquad s_{mn}^{(\beta,\gamma)} = -\frac{1}{4\pi^2}\int_C\int_D \frac{f(x, y)}{x^{m+1}y^{n+1}(1-x)^{\beta+1}(1-y)^{\gamma+1}}\,dx\,dy = o(m^\beta n^\gamma),$$

where $\beta > -1$, $\gamma > -1$.

My object is to show that this integral may be replaced by a simpler one provided that the coefficients a_{mn} satisfy certain sufficient conditions (6). The reduction from (3) to the form (10) is analogous to Theorem I of the paper by A. C. Offord already referred to. The conditions imposed on the coefficients take a different form, however, and there are features of the work peculiar to double series.

2. *Notation.* Writing $(1-y)^{-1-\gamma} = \Sigma A_n^{(\gamma)} y^n$, let

$$(4) \qquad (1-y)^{-1-\gamma}\sum_n a_{mn} y^n = \sum_n p_{mn} y^n,$$

we also define p_{mn} and q_{mn} by

$$(5) \qquad p_{mn} = \sum_{\nu=0}^{n} A_\nu^{(\gamma)} a_{m, n-\nu}, \quad q_{mn} = \sum_{\mu=0}^{m} A_\mu^{(\gamma)} a_{m-\mu, n},$$

the two definitions of p_{mn} being equivalent. In what follows $\phi_{mn} = o(m^\beta n^\gamma)$ is used to mean that, given an arbitrary positive ϵ, m_0 and n_0 exist such that, if both $m > m_0$ and $n > n_0$, then

$$|\phi_{mn}| < \epsilon m^\beta n^\gamma.$$

The condition 6 (*c*) below means that, if $m > m_0$ *or* $n > n_0$, then

$$|a_{mn}| < \epsilon m^\beta n^\gamma.$$

---

* μ, ν range from 0 to ∞, but it is convenient to assume $a_{\mu 0} = 0$, $a_{0\nu} = 0$ if $\mu > 0$, $\nu > 0$.

† In the Pringsheim sense.

The conditions imposed on the coefficients are then

$$(6)\qquad \begin{aligned} &(a)\ p_{mn} = o(m^\beta n^\gamma), \quad q_{mn} = o(m^\beta n^\gamma),\\ &(c)\ a_{mn} = o(m^\beta n^\gamma) \quad \text{as} \quad m+n \to \infty,\\ &(b)\ |p_{mn}| < K m^\beta n^\gamma \quad \text{for all } m \text{ and } n > n_1,\\ &\quad |q_{mn}| < K m^\beta n^\gamma \quad \text{for all } n \text{ and } m > m_1. \end{aligned}$$

The condition (c) shows that

$$\overline{\lim_{(m+n)\to\infty}} \sqrt[m+n]{|a_{mn}|} \leqslant 1.$$

Hence, by a well-known result for double power series*, $f(x, y)$ is regular for $|x| < 1$, $|y| < 1$.

The reduction of (3) consists of two parts. In the first (§ 3) it is shown that the factor $(1-x)^{-1-\beta}(1-y)^{-1-\gamma}$ in (3) may be replaced by the product $P(x) \cdot Q(y)$, where†

$$(7)\qquad P(x) = (1-x)^{-1-\beta} + c_0 + c_1 x, \quad Q(y) = (1-y)^{-1-\gamma} + d_0 + d_1 y,$$

and the constants c_0, d_0, c_1, d_1 are independent of m and n.

In the second part (§§ 4, 5) we take C, D to be the circles $|x| = 1-m^{-1}$, $|y| = 1-n^{-1}$ and each is broken into two parts. Thus $C = C_1 + C_2$, where on C_1, $x = (1-m^{-1})e^{i\theta}$ and $-\eta < \theta < \eta$; C_2 is the remainder of C. D_1 and D_2 are similarly defined. The constants η, η' here introduced are quite arbitrary, but c_0, c_1 are afterwards connected with η and d_0, d_1 with η' by the relations

$$(8)\qquad P(e^{i\eta}) = P(e^{-i\eta}) = 0, \quad Q(e^{i\eta'}) = Q(e^{-i\eta'}) = 0.$$

3. We can now state

THEOREM I. *If the coefficients of* $f(x, y) = \Sigma\Sigma a_{\mu\nu} x^\mu y^\nu$ *satisfy the conditions* (6), *then a necessary and sufficient condition that* $\Sigma\Sigma a_{\mu\nu}$ *shall be summable* (C, β, γ) *to the sum* S *is*

$$(10)\qquad \int_{C_1}\int_{D_1} \{f(x, y) - S\} \frac{P(x)\,Q(y)}{x^{m+1} y^{n+1}}\, dx\, dy = o(m^\beta n^\gamma).$$

---

* F. Hartogs, "Beiträge zur elementaren Theorie der Potenzreihen und der eindeutigen analytischen Funktionen zweier Veränderlichen", *Münchener Inaugural-Dissertation* (1904), 15–16.

† It might appear that a better kernel would be

$$(1-x)^{-1-\beta}(1-y)^{-1-\gamma} + c_0 + c_1 x + c_2 y + c_3 xy,$$

but there are difficulties with this form, at any rate in the present approach to the problem. In particular, we use $P(x) = O(m^{-1})$, $Q(y) = O(n^{-1})$ in (21), but for the suggested kernel there seems to be no simple analogue of these results with a finite number of constants $c_0, c_1, \ldots$.

We shall make use of the following lemma. Its proof, which is omitted, follows closely pages 472–4 of the paper by A. C. Offord already referred to.

LEMMA. *If the coefficients of a power series* $g_n(x) = \Sigma a_m x^m$, *convergent for* $|x| < 1$, *are functions not only of* m *but also of a parameter* n, *and if* (*a*) $a_m = o(m^\beta n^\gamma)$ *and* (*b*) $|a_m| < K m^\beta n^\gamma$ *for all* $m > 0$ *and all sufficiently large* n, *then*

$$\int_{C_2} g_n(x)\, P(x)\, x^{-m-1}\, dx = o(m^\beta n^\gamma). \tag{11}$$

Proof of Theorem I. Considering now the integral (3) and introducing the expressions $P(x)$, $Q(y)$ from (7), we may write

$$\begin{aligned} -4\pi^2 s_{m,n}^{(\beta,\gamma)} &= \int_C \int_D \frac{f(x,y)}{x^{m+1} y^{n+1}} \{P(x)\,Q(y) - (1-x)^{-\beta-1}(d_0 + d_1 y) \\ &\qquad - (1-y)^{-\gamma-1}(c_0 + c_1 x) - (c_0 + c_1 x)(d_0 + d_1 y)\}\, dx\, dy \\ &= I_1 - I_2 - I_3 - I_4. \end{aligned} \tag{3'}$$

Now
$$I_4 = \int_C \int_D \Sigma\Sigma\, b_{\mu\nu} x^{\mu-m-1} y^{\nu-n-1}\, dx\, dy,$$

where $b_{\mu\nu} = c_0 d_0 a_{\mu\nu} + c_0 d_1 a_{\mu,\nu-1} + c_1 d_0 a_{\mu-1,\nu} + c_1 d_1 a_{\mu-1,\nu-1}$.

After the two integrations only b_{mn} survives and so, by (6 *c*),

$$I_4 = -4\pi^2 b_{mn} = o(m^\beta n^\gamma). \tag{13}$$

To deal with I_3 we may write (for $|x| < 1$, $|y| < 1$)

$$f(x,y)(1-y)^{-1-\gamma} = \Sigma\Sigma\, p_{\mu\nu} x^\mu y^\nu.$$

Hence
$$\begin{aligned} I_3 &= \int_C \int_D \Sigma\Sigma\, (c_0 p_{\mu\nu} + c_1 p_{\mu-1,\nu})\, x^{\mu-m-1} y^{\nu-n-1}\, dx\, dy \\ &= -4\pi^2 \{c_0 p_{mn} + c_1 p_{m-1,n}\}. \end{aligned}$$

Thus, by (6 *a*), we have

$$I_3 = o(m^\beta n^\gamma) \quad \text{and} \quad I_2 = o(m^\beta n^\gamma). \tag{14}$$

Hence, by (13) and (14), we may replace (3) by

$$-4\pi^2 s_{m,n}^{(\beta,\gamma)} = o(m^\beta n^\gamma) + \int_C \int_D f(x,y)\, P(x)\, Q(y)\, x^{-m-1} y^{-n-1}\, dx\, dy. \tag{16}$$

This completes the first part of the reduction.

4. For the second part, we write

$$\int_C \int_D = \int_{C_1} \int_{D_1} + \int_C \int_{D_2} + \int_{C_2} \int_D - \int_{C_2} \int_{D_2} \tag{17}$$
$$= J_1 + J_2 + J_3 - J_4.$$

We first prove that $J_3 = o(m^\beta n^\gamma)$. J_2 can then be treated similarly after changing the order of integration. In each case we use Cauchy's theorem and then the lemma already stated.

We have, for J_3,

$$\int_D f(x, y)\, Q(y)\, y^{-n-1}\, dy = \int_D \Sigma\Sigma(p_{\mu\nu} + d_0 a_{\mu\nu} + d_1 a_{\mu,\nu-1})\, x^\mu y^{\nu-n-1}\, dy$$
$$= 2\pi i\, \Sigma(p_{\mu n} + d_0 a_{\mu n} + d_1 a_{\mu,n-1})\, x^\mu$$
$$= \Sigma a_\mu x^\mu = g_n(x),$$

say. In virtue of the hypotheses (6), this function satisfies the conditions of the lemma. Hence

$$\int_{C_2} g_n(x)\, P(x)\, x^{-m-1}\, dx = o(m^\beta n^\gamma).$$

From this, and a similar argument for J_2, we have then

$$J_3 = o(m^\beta n^\gamma), \quad J_2 = o(m^\beta n^\gamma). \tag{18}$$

5. The method of dealing with J_4 is more tedious. We integrate by parts twice with respect to x, y and show that the various parts of the result are of the required order. Here the properties of $P(x)$, $Q(y)$ and the hypothesis (6 c) are used.

We introduce the function

$$F(x, y) = f(x, y)\, x^{-m-1} y^{-n-1}. \tag{19}$$

Let us denote its various first and second integrals with respect to x, y by

$$F_{10},\ F_{01},\ F_{11},\ F_{21},\ F_{12},\ F_{22}.$$

Ignoring limits for the moment, we have

$$\int FPQ\, dx = F_{10} PQ - \int F_{10} P' Q\, dx,$$

$$\iint FPQ\, dx\, dy = F_{11} PQ - \int F_{11} PQ'\, dy - \int F_{11} P' Q\, dx + \iint F_{11} P' Q'\, dx\, dy,$$

where the order of integration has been changed.

When we repeat the process, our final expression becomes

$$(20)\quad F_{11}PQ-F_{12}PQ'-F_{21}P'Q+F_{22}P'Q'+\int_{D_2}(F_{12}P-F_{22}P')\,Q''\,dy$$
$$+\int_{C_2}(F_{21}Q-F_{22}Q')\,P''\,dx+\int_{C_2}\int_{D_2}F_{22}P''\,Q''\,dx\,dy.$$

Let $x=a,\ b$ denote the end points of the path of integration C_2, $y=c,\ d$ those of D_2. Then each completely integrated term in (20) will give rise to four expressions and each partially integrated one will produce two. Thus from $F_{21}QP''$ we get

$$F_{21}(x,\ d)\,P''(x)\,Q(d)-F_{21}(x,\ c)\,P''(x)\,Q(c),$$

and from $F_{11}PQ$ we get

$$F_{11}(b,\ d)\,P(b)\,Q(d)-F_{11}(b,\ c)\,P(b)\,Q(c)-F_{11}(a,\ d)\,P(a)\,Q(d)$$
$$+F_{11}(a,\ b)\ P(a)\,Q(b).$$

Now we have*

$$(21)\qquad (a)\quad P(a),\quad P(b)=O(m^{-1});\quad Q(c),\quad Q(d)=O(n^{-1}),$$
$$(b)\quad P'(x),\quad P''(x)=O(1);\quad Q'(y),\quad Q''(y)=O(1),$$

when x is on C_2 and y on D_2. Further, it may be shown, by tedious but quite straightforward estimation of terms, that, when x is on C_2 and y on D_2,

$$(22)\qquad (a)\ F_{11}(x,\ y)=o(m^{\beta+1}n^{\gamma+1}),\quad (b)\ F_{21}(x,\ y)=o(m^{\beta}n^{\gamma+1}),$$
$$(c)\ F_{12}(x,\ y)=o(m^{\beta+1}n^{\gamma}),\qquad (d)\ F_{22}(x,\ y)=o(m^{\beta}n^{\gamma}).$$

On assuming these results, we see that every term arising from (20) is of order $o(m^{\beta}n^{\gamma})$. Hence the result

$$(23)\qquad J_4=o(m^{\beta}n^{\gamma}).$$

Thus, from (16) and (17),

$$s_{mn}^{(\beta,\ \gamma)}=o(m^{\beta}n^{\gamma})+J_1,$$

and the proof of (10) is complete.

6. *Applications of Theorem I.* Two applications of the reduced test for summability follow. They are the analogues of Fatou's theorem and

* Offord, *loc. cit.*, 471–2.

Dienes' theorem on single power series, referred to in § 1. In each case a proof of the original theorem is given first in order to make clear the proof of the double series theorem.

7. DIENES' THEOREM*. *If* $f(x)=\Sigma a_n x^n$, *where* $a_n=o(n^\gamma)$, $\gamma>0$, *and if* $f(x)\to 1$ *as* $x\to 1$ *along any path from within* $|x|=1$, *then* Σa_n *is summable* (C,γ) *to* S.

Proof. Take $S=0$. By Theorem I of Offord's paper it is sufficient to show that, if $x=(1-n^{-1})e^{i\theta}$, then

$$\int_{-\eta}^{\eta}\frac{f(x)}{x^{n+1}}\{(1-x)^{-1-\gamma}+c_0+c_1x\}\frac{dx}{d\theta}\,d\theta=o(n^\gamma). \tag{24}$$

Let I_1, I_2 denote the parts of this integral corresponding to $(1-x)^{-1-\gamma}$ and c_0+c_1x, and suppose that $|f(x)|<\epsilon$ if $|x|<1$ and $|1-x|<\delta$. Choose $\eta<\delta$ and $n>n_0$, so that the path of integration C_1 lies entirely within $|x|<1$ and $|1-x|<\delta$. Then we have

$$|I_2|<K(\epsilon)=o(n^\gamma). \tag{25}$$

Also, since $|1-x|^2=1-2r\cos\theta+r^2$, $r^{-n-1}=O(1)$,

$$|I_1|<\epsilon K\int_{-\eta}^{\eta}(1-2r\cos\theta+r^2)^{-\frac{1}{2}(\gamma+1)}d\theta. \tag{26}$$

Now, for $n>2$ and $\eta<\frac{1}{2}\pi$,

$$\begin{aligned}|1-2r\cos\theta+r^2| &= (1-r)^2+4r\sin^2\tfrac{1}{2}\theta\\ &\geqslant n^{-2}+4(1-n^{-1})(\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta)^2\\ &> n^{-2}+\tfrac{1}{2}\theta^2\cos^2\tfrac{1}{2}\eta\\ &> n^{-2}+\tfrac{1}{4}\theta^2.\end{aligned}$$

Hence
$$|I_1|<2\epsilon K\int_0^{\eta}\frac{n^{\gamma+1}}{(1+\frac{1}{4}n^2\theta^2)^{\frac{1}{2}(\gamma+1)}}\,d\theta.$$

Writing $\phi=\frac{1}{2}n\theta$, and $\kappa=\frac{1}{2}(\gamma+1)$, we have

$$|I_1|<4\epsilon Kn^\gamma\int_0^{\infty}\frac{d\phi}{(1+\phi^2)^\kappa}.$$

* Offord, *loc. cit.*, 474, Theorem 2. The proof given is a simplified version of Theorem 3, 475–6.

Thus, since $\kappa > \frac{1}{2}$, we have

$$I_1 = o(n^\gamma), \tag{27}$$

and (24) follows from (25) and (27).

8. *Extension of Dienes' theorem to double series.*

THEOREM II. *If* $f(x, y) = \Sigma\Sigma a_{mn} x^m y^n$, *where the coefficients satisfy the conditions* (6) *for* $\beta > 0$, $\gamma > 0$, *and if* $f(x, y) \to S$ *as* $x \to 1$, $y \to 1$ *along any path inside* $|x| < 1$, $|y| < 1$, *then* $\Sigma\Sigma a_{mn}$ *is summable* (C, β, γ) *to* S.

Proof. Take $S = 0$ and use Theorem I. Then we have to show that

$$I = \int_{C_1}\int_{D_1} \frac{f(x, y)}{x^{m+1} y^{n+1}} P(x) Q(y)\, dx\, dy = o(m^\beta n^\gamma). \tag{28}$$

This double integral may be broken into four, namely,

$$I_1 = \int_{C_1}\int_{D_1} \frac{f(x, y)}{x^{m+1} y^{n+1}} (c_0 + c_1 x)(d_0 + d_1 y)\, dx\, dy,$$

$$I_2 = \int_{C_1}\int_{D_1} \frac{c_0 + c_1 x}{x^{m+1}} \frac{f(x, y)}{y^{n+1}(1-y)^{\gamma+1}}\, dx\, dy,$$

I_3 of similar form,

$$I_4 = \int_{C_1}\int_{D_1} \frac{f(x, y)}{x^{m+1} y^{n+1}} (1-x)^{-\beta-1}(1-y)^{-\gamma-1}\, dx\, dy.$$

Now, by hypothesis, $|f(x, y)| < \epsilon$ provided that x, y are such that $|x| < 1$, $|x-1| < \delta$ and $|y| < 1$, $|y-1| < \delta'$. Choose $\eta < \delta$, $\eta' < \delta'$, and every "point" of C_1, D_1 satisfies these conditions. It follows at once that, since β and γ are positive,

$$I_1 = \epsilon O(1) = o(m^\beta n^\gamma). \tag{29}$$

On writing $y = se^{i\phi}$, $s = 1 - n^{-1}$ in I_2, we infer from the one variable argument that

$$\left| \int_{D_1} \frac{f(x, y)}{y^{n+1}(1-y)^{\gamma+1}}\, dy \right| < K \left| \int_{-\eta}^{\eta} \frac{\epsilon\, d\phi}{(1 - 2s\cos\phi + s^2)^{\frac{1}{2}(\gamma+1)}} \right| = o(n^\gamma).$$

Hence

$$I_2 = o(m^\beta n^\gamma).$$

Similarly, $$I_3 = o(m^\beta n^\gamma).$$

Also,

$$|I_4| < \epsilon \left| \int_{C_1} x^{-m-1}(1-x)^{-\beta-1}\,dx \right| \left| \int_{D_1} y^{-n-1}(1-y)^{-\gamma-1}\,dy \right|.$$

We now apply the one variable argument to each factor and get

$$|I_4| < \epsilon K m^\beta n^\gamma.$$

Hence, since $I = I_1+I_2+I_3+I_4$, we deduce the result (28), namely

$$I = o(m^\beta n^\gamma),$$

and the theorem is proved.

9. FATOU'S THEOREM. *If* $f(x) = \Sigma a_n x^n$ *for* $|x| < 1$, *where* $a_n \to 0$, *then* Σa_n *converges at any regular point on the circle of convergence* $|x| = 1$.

Proof. Take the regular point to be $x = 1$. Assuming that $f(1) = 0$, we have to show that

$$s_n = \sum_{\nu=0}^{n} a_\nu = o(1).$$

If we apply once more Offord's* Theorem I, with $x = (1-n^{-1})e^{i\theta}$, we find

$$s_n = o(1) + \int_{-\eta}^{\eta} \frac{f(x)\,Q(x)}{x^{n+1}} \frac{dx}{d\theta}\,d\theta. \tag{35}$$

We have to show that, by proper choice of η, we can make the modulus of the integral in (35) less than an arbitrary positive ϵ.

Now, since $Q(x)$ [which now denotes $(1-x)^{-1}+c_0+c_1x$] vanishes for $x = e^{i\eta}$, $x = e^{-i\eta}$, we may write

$$Q(x) = \frac{A(x-e^{i\eta})(x-e^{-i\eta})}{1-x},$$

and we find that

$$A = \tfrac{1}{2}(1-\cos\eta)^{-1}.$$

Also, since $x = 1$ is a regular point and $f(1) = 0$, we have, when x is near 1,

$$f(x) = O(|1-x|).$$

* *Loc. cit.*, 470.

Hence the modulus of the integral in (35) is less than

$$(36) \qquad K\int_{-\eta}^{\eta} \frac{|1-x|\,A\,|x^2-2x\cos\eta+1|}{|1-x|}\, r^{-n}\,d\theta$$

$$= O\left\{\frac{\eta}{1-\cos\eta}\,|r^2e^{2i\theta}-2re^{i\theta}\cos\eta+1|\right\}.$$

The modulus of the last expression readily reduces to

$$\{(r-1)^4+4r(r-1)^2(1-\cos\eta\,\cos\theta)+4r^2(\cos\eta-\cos\theta)^2\}^{\frac{1}{2}}.$$

Putting $r=1-n^{-1}$, we see that the required modulus is, for large enough values of n, less than $2(1-\cos\eta)$. Hence, by (36) and (35),

$$s_n = o(1)+O(\eta).$$

But η is arbitrary; so we can make $O(\eta)<\epsilon$. Hence $s_n\to 0$, as stated.

10. *Extension of Fatou's theorem to double series.*

THEOREM III. *If the function $f(x, y)$, defined for $|x|<1$, $|y|<1$ by the series $\Sigma\Sigma a_{mn}x^m y^n$, is regular for $x=1$, $y=1$, and if*

$$\text{I} \quad p_{mn}=\sum_{\nu=1}^{n} a_{m\nu}=o(1),$$

$$\text{II} \quad q_{mn}=\sum_{\mu=1}^{m} a_{\mu n}=o(1),$$

$$\text{III} \quad a_{mn}=o(1) \quad as \quad m+n\to\infty,$$

then the sum by rows, the sum by columns, and the Pringsheim sum of $\Sigma\Sigma a_{mn}$ all exist and are equal to $f(1, 1)$.

Before proving the theorem, we may observe that the conditions I, II are necessary conditions of convergence and that the Pringsheim limit $\lim a_{mn}=0$ follows from these. Moreover, it is easy to show, by an example, that condition III must be satisfied when m, n tend to infinity separately. Thus, the function

$$(1+x)^{-1}+(1+y)^{-1}$$

is regular for $x=1$, $y=1$, and in the power series $a_{mn}=0$ if $m>1$ and $n>1$, but the derived double series is not convergent.

Further, we can see that something more than the full condition III is necessary. Consider the example

$$f(x, y) = \sum_1^\infty (-1)^{m-1} \frac{x^m}{m} (1+y+y^2+\ldots+y^{m-1})$$

$$= \frac{\log(1+x)-\log(1+xy)}{1-y} = \frac{\phi(x, y)}{1-y},$$

say. The function $\phi(x, y)$ is regular for $x = 1$, $y = 1$, and $\phi(x, 1) \equiv 0$. It follows [*e.g.* by expanding $\phi(x, y)$ in powers of $x-1$, $y-1$] that the quotient $f(x, y)$ obtained by removing the factor $1-y$ is also regular for $x=1$, $y=1$.

Also, $a_{mn} = 0$ if $n \geqslant m$, $a_{mn} = (-1)^{m-1}/m$ if $n < m$. Thus $a_{mn} \to 0$ in all three senses.

However, $p_{mn} = (-1)^{m-1}\kappa$, where κ is either 1 or n/m according as $n \geqslant m$ or $n < m$.

But, since $p_{mn} = s_{mn} - s_{m-1, n}$, it is necessary for convergence that p_{mn} shall tend to zero.

Professor Hardy has pointed out that the conditions I, II, III do not ensure the convergence of the diagonal series. For suppose that we write

$$f(x, y) = \frac{\phi(x)-\phi(y)}{x-y} = \phi'(x) \quad \text{when} \quad y = x.$$

Suppose, further, that $\phi(x)$ is regular for $x = 1$ and is represented by $\Sigma a_m x^m$ for $|x| < 1$ and that $a_m \to 0$. We find that

$$f(x, y) = \Sigma a_m (x^{m-1}+x^{m-2}y+\ldots+y^{m-1}).$$

The diagonal series is thus $\Sigma m a_m$, which is not, in general, convergent. The conditions I, II, III are satisfied however, so that by Theorem III the other three series are proved to converge.

A simple (but more stringent) condition, which implies I, II, III and ensures the convergence of all four series in the general case, is

$$(m+n)\, a_{mn} \to 0 \quad \text{as} \quad m+n \to \infty.$$

When this condition is satisfied, we can deduce the convergence at every regular "point" on $|x| = 1$, $|y| = 1$, and not merely at $x = 1$, $y = 1$.

Proof of Theorem III. We deal first with the repeated series

$$\sum_{m=1}^\infty \left(\sum_{n=1}^\infty a_{mn} \right).$$

If we write $P_m(y) = \sum_{\nu=1}^{\infty} a_{m\nu} y^\nu$, and $f(x, y) = a_{00} + \sum_1^\infty P_m(y) x^m$, then we have

$$P_m(y) = \frac{1}{2\pi i} \int_C \frac{f(x, y)\, dx}{x^{m+1}},$$

where C is any contour within $|x| = 1$ and enclosing the origin.

Now $f(x, y)$ is regular* for $|x| < 1$, $y = 1$, and for $x = 1$, $|y| < 1$, so $P_m(y)$ is regular for $|y| < 1$ and $y = 1$. But, for $|y| < 1$, $P_m(y) = \Sigma a_{m\nu} y^\nu$ and $a_{m\nu} \to 0$. Hence, by Fatou's theorem, $\Sigma a_{m\nu}$ converges to $P_m(1)$ for every m. Therefore, from hypothesis I,

$$P_m(1) \to 0.$$

Also,
$$f(x, 1) = a_{00} + \Sigma P_m(1)\, x^m.$$

Since this function is regular for $|x| < 1$ and $x = 1$, and since $P_m(1) \to 0$, we have, by a second application of Fatou's theorem,

$$a_{00} + \Sigma P_m(1) \to f(1, 1),$$

i.e. $\Sigma\Sigma a_{mn}$ converges to $f(1, 1)$ when summed by rows. Similarly for the sum by columns.

It remains to show that there is a Pringsheim sum.

Now when $\beta = 0$, $\gamma = 0$, the conditions (6) of Theorem I reduce to I, II, III together with

$$|p_{mn}| < K, \quad |q_{mn}| < K.$$

These are satisfied since we know that the individual rows and columns are convergent. We may assume that $f(1, 1) = 0$.

Hence we deduce, from Theorem I,

$$s_{mn} = \int_{C_1} \int_{D_1} \frac{f(x, y)\, P(x)\, Q(y)}{x^{m+1} y^{n+1}}\, dx\, dy + o(1), \tag{37}$$

where
$$P(x) = c_0 + c_1 x + (1-x)^{-1},$$

$$Q(y) = d_0 + d_1 y + (1-y)^{-1},$$

and
$$P(e^{\pm i\eta}) = 0, \quad Q(e^{\pm i\eta'}) = 0.$$

* This is a particular case of a theorem proved on pages 58–62 in Hartogs' dissertation, already referred to. It is not essential for this that $f(x, y)$ should be regular for $x = 1$, $y = 1$. Regularity for $|x| < r = 1$, $|y| < s = 1$ is sufficient provided that r, s are not "maximal radii of convergence".

If we now put

$$\phi(x, y) = f(x, y) - f(x, 1) - f(1, y),$$

then, since $f(1, 1) = 0$, we have, for all values of y, x,

$$\phi(1, y) = 0, \quad \phi(x, 1) = 0.$$

Hence, when x, y are near 1,

$$\phi(x, y) = O\{|1-x||1-y|\}.$$

Therefore, from (37),

$$(38) \qquad |s_{mn}| = o(1) + O\left|\int_{C_1}\int_{D_1} \frac{|1-x|P(x)}{x^{m+1}} \frac{|1-y|Q(y)}{y^{n+1}}\,dx\,dy\right|$$
$$+\left|\int_{C_1}\int_{D_1} \frac{f(x, 1)P(x)Q(y)}{x^{m+1}y^{n+1}}\,dx\,dy\right|$$
$$+\left|\int_{C_1}\int_{D_1} \frac{f(1, y)P(x)Q(y)}{x^{m+1}y^{n+1}}\,dx\,dy\right|$$
$$= o(1) + I_1 + I_2 + I_3,$$

say. I_1 is less than a constant multiple of

$$\left|\int_{C_1} |1-x|P(x)x^{-m-1}\,dx\right| \left|\int_{D_1} |1-y|Q(y)y^{-n-1}\,dy\right|.$$

Hence, by the arguments for the one variable theorem,

$$(39) \qquad I_1 = O(\eta)\,O(\eta').$$

To deal with I_2, I_3 we find inequalities for $f(x, 1)$ and $f(1, y)$. If $|x-1| < \delta$, $|y-1| < \delta'$, we may write

$$f(x, y) = \Sigma\Sigma b_{mn}(x-1)^m (y-1)^n,$$

and so $$f(x, 1) = \sum_1^\infty b_{m0}(x-1)^m, \quad f(1, y) = \sum_1^\infty b_{0n}(y-1)^n.$$

Since $f(1, 1) = 0$, we have then, if x, y are near 1,

$$f(x, 1) = O\{|1-x|\}, \quad f(1, y) = O\{|1-y|\}.$$

Now $I_3 = JK$, where

$$J = \int_{C_1} \frac{P(x)}{x^{m+1}}\,dx, \quad K = \int_{D_1} \frac{f(1, y)Q(y)}{y^{n+1}}\,dy.$$

Using the inequality for $f(1, y)$, and proceeding as in (35) and (36), we have

$$|K| < 2\eta' \max \left| \frac{f(1, y)\, Q(y)}{y^n} \right| = O(\eta').$$

Again, dealing with J as with (35), we find, if $r = 1-m^{-1}$,

$$(40) \quad -2iJ(1-\cos\eta)$$
$$= \int_{-\eta}^{\eta} \frac{r^2 e^{2i\theta} - 2re^{i\theta}\cos\eta + 1}{r^m e^{mi\theta}(1-re^{i\theta})}\, d\theta$$
$$= r^{-m} \int_{-\eta}^{\eta} (e^{-mi\theta} - re^{(1-m)i\theta})\, d\theta + 2r^{1-m}(1-\cos\eta) \int_{-\eta}^{\eta} \frac{e^{(1-m)i\theta}\, d\theta}{1-re^{i\theta}}.$$

Therefore, on using the identity

$$(1-x)^{-1} = 1+x+\ldots+x^{m-1}+x^m(1-x)^{-1},$$

we have

$$(41) \quad J = O(m^{-1}) + r^{1-m} i \int_{-\eta}^{\eta} e^{(1-m)i\theta}\{1+re^{i\theta}+\ldots+(re^{i\theta})^{m-1}\}\, d\theta + \int_{-\eta}^{\eta} \frac{ire^{i\theta}}{1-re^{i\theta}}\, d\theta.$$

The last integral, L say, gives

$$\left[-\log(1-re^{i\theta})\right]_{-\eta}^{\eta} = 2i \arctan\left(\frac{r\sin\eta}{1-r\cos\eta}\right).$$

Hence,

$$(42) \quad |L| < 2 \arctan\left(\frac{\sin\eta}{1-\cos\eta}\right) < \pi.$$

Term by term integration of the remaining integral gives us $2i(\eta+P)$,

where $$P = \sum_{\lambda=1}^{m-1} \frac{1}{\lambda r^\lambda} \sin\lambda\eta.$$

Now, by Abel's inequality, for $\mu \leqslant m-1$,

$$|P| < r^{1-m} \max \left| \sum_{\lambda=1}^{\mu} \lambda^{-1} \sin\lambda\eta \right|$$
$$= (1-m^{-1})^{-m}\, O(1).$$

Thus

$$(43) \quad P = O(1),$$

and, by (41), (42), (43),

$$(44) \quad J = O(m^{-1}) + O(1) + O(\eta) = O(1) + O(\eta).$$

Hence $$I_3 = O(1)\,O(\eta')+O(\eta)\,O(\eta'),$$

and we can deal with I_2 in similar fashion. Therefore, from (38) and (39),

$$|s_{mn}| < \lambda\eta\eta'+\mu\eta+\nu\eta'+\rho,$$

where λ, μ, ν are independent of m, n, η, η' and ρ is arbitrarily small.

Thus, with η, η' small enough, we have, for all sufficiently large m, n,

$$|s_{mn}| < \epsilon,$$

i.e. s_{mn} converges to zero as m, $n \to \infty$, or the Pringsheim sum exists and is equal to $f(1, 1)$. Thus Theorem III is completely proved.

Goldsmiths' College,
London, S.E.14.

RELATIONS BETWEEN WELL-POISED HYPERGEOMETRIC SERIES OF THE TYPE ${}_7F_6$

By F. J. W. WHIPPLE.

[Received 8 March, 1935.—Read 14 March, 1935.]

1. In his paper* on "Transformations of well-poised hypergeometric series" Dr. W. N. Bailey found a pair of relations between well-poised series of the type ${}_7F_6$. He remarked, however, that he had not generalized all the relations known previously for terminating series of the same type. This remark is repeated in his recently published tract† "Generalized hypergeometric series". In the following paper relations, each involving three ${}_7F_6$'s, are deduced from Bailey's fundamental relation between a well-poised ${}_7F_6$ and two Saalschützian ${}_4F_3$'s. The new equations resemble the relations, originally discovered by Thomae, each involving three series of the type ${}_3F_2$, and the cognate relations between well-poised series of the type ${}_6F_5(-1)$. All the relations between the members of a group of 192 ${}_7F_6$'s have been investigated systematically.

In the last section of the paper the relations between members of the allied group of 160 Saalschützian ${}_4F_3$'s are discussed. There is no relation involving less than four of these series.

2. We adopt the notation

$$\psi[a;\ b,\ c,\ d,\ e,\ f]$$

$$\equiv \frac{\Gamma(1+a)}{\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)\,\Gamma(1-s)}$$

$$\times {}_7F_6\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ b,\ c,\ d,\ e,\ f \\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a-f \end{matrix}\right], \quad (2.1)$$

where

$$s = b+c+d+e+f-2a-1. \quad (2.11)$$

---

* *Proc. London Math. Soc.* (2), 36 (1934), 235.

† (Cambridge, 1935), 63.

The series ${}_7F_6$ converges when $1-s>0$. In this notation, Bailey's identities* may be written

$$\psi[a;\ b,\ c,\ d,\ e,\ f]$$

$$=\psi[b+c-s;\ b,\ c,\ 1+a-d-e,\ 1+c-d-f,\ 1+a-e-f]$$

$$=\psi[1+a-s-b;\ 1+a-b-c,\ 1+a-b-d,\ 1+a-b-e,\ 1+a-b-f,\ 1-s]. \tag{2.2}$$

The number of equivalent forms found by permutation of the symbols b to f is 16.

The fundamental relation† from which the identities are derived is equivalent to

$$\frac{\sin\pi(d+e+f-a)}{\pi}\psi[a;\ b,\ c,\ d,\ e,\ f]$$

$$=\frac{1}{D_1N_2}\,{}_4F_3\left[\begin{matrix}d,\ e,\ f,\ 1+a-b-c\\ 1+a-b,\ 1+a-c,\ d+e+f-a\end{matrix}\right]$$

$$-\frac{1}{D_2N_1}\,{}_4F_3\left[\begin{matrix}1+a-e-f,\ 1+a-d-f,\ 1+a-d-e,\ 1-s\\ 1-s+b,\ 1-s+c,\ 2+a-d-e-f\end{matrix}\right], \tag{2.3}$$

where

$$D_1=\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(d+e+f-a), \tag{2.31}$$

$$N_1=\Gamma(d)\,\Gamma(e)\,\Gamma(f)\,\Gamma(1+a-b-c), \tag{2.32}$$

$$D_2=\Gamma(1-s+b)\,\Gamma(1-s+c)\,\Gamma(2+a-d-e-f), \tag{2.33}$$

$$N_2=\Gamma(1+a-e-f)\,\Gamma(1+a-d-f)\,\Gamma(1+a-d-e)\,\Gamma(1-s). \tag{2.34}$$

The relation may also be written

$$\frac{\sin\pi w}{\pi}\psi[x+y+z-u;\ x,\ y,\ z,\ w-t,\ v-t]$$

$$=\frac{1}{D_1N_2}\,{}_4F_3\left[\begin{matrix}x,\ y,\ z,\ t\\ u,\ v,\ w\end{matrix}\right]$$

$$-\frac{1}{D_2N_1}\,{}_4F_3\left[\begin{matrix}1+x-u,\ 1+y-u,\ 1+z-u,\ 1+t-u\\ 2-u,\ 1+v-u,\ 1+w-u\end{matrix}\right]. \tag{2.4}$$

---

* *Cf. Tract*, 62.

† *Cf. Tract*, 29.

This relation holds provided that the Saalschützian condition

$$u+v+w-x-y-z-t=1 \tag{2.41}$$

is satisfied.

With (2.4) as a guide we look for a new relation analogous to (2.3) and involving the first of the ${}_4F_3$'s in that equation. Such a relation is

$$\frac{\sin\pi(1+a-b)}{\pi}\,\psi[e+f-c;\ e,\ f,\ 1+a-b-c,\ 1+a-c-d,\ e+f-a]$$

$$=\frac{1}{D_1N_3}\,{}_4F_3\left[\begin{matrix}d,\ e,\ f,\ 1+a-b-c\\ 1+a-b,\ 1+a-c,\ d+e+f-a\end{matrix}\right]$$

$$-\frac{1}{D_3N_1}\,{}_4F_3\left[\begin{matrix}b+d-a,\ b+e-a,\ b+f-a,\ 1-c\\ 1+b-c,\ 1+s-c,\ 1-a+b\end{matrix}\right]. \tag{2.5}$$

Here D_1 and N_1 have the same values as before, while

$$D_3=\Gamma(1+b-c)\,\Gamma(1+s-c)\,\Gamma(1-a+b) \tag{2.51}$$

and

$$N_3=\Gamma(b+d-a)\,\Gamma(b+e-a)\,\Gamma(b+f-a)\,\Gamma(1-c). \tag{2.52}$$

The sum of the first parameters in the denominators of the ${}_4F_3$'s in (2.4) is 2, and we notice that the sum of the parameters $1+s-c$, $1-s+c$ in the last ${}_4F_3$'s of (2.5) and (2.3) is also 2. With this clue we find that

$$\frac{\sin\pi(1+s-c)}{\pi}\,\psi[2b-a;\ b,\ b+c-a,\ b+d-a,\ b+e-a,\ b+f-a]$$

$$=\frac{1}{D_3N_2}\,{}_4F_3\left[\begin{matrix}b+d-a,\ b+e-a,\ b+f-a,\ 1-c\\ 1+b-c,\ 1+s-c,\ 1-a+b\end{matrix}\right]$$

$$-\frac{1}{D_2N_3}\,{}_4F_3\left[\begin{matrix}1+a-e-f,\ 1+a-d-f,\ 1+a-d-e,\ 1-s\\ 1-s+b,\ 1-s+c,\ 2+a-d-e-f\end{matrix}\right] \tag{2.6}$$

By elimination of the ${}_4F_3$'s from equations (2.3), (2.5), and (2.6), it is found that

$$\frac{\sin\pi(d+e+f-a)\,\psi[a;\ b,\ c,\ d,\ e,\ f]}{\Gamma(b+d-a)\,\Gamma(b+e-a)\,\Gamma(b+f-a)\,\Gamma(1-c)}$$

$$=\frac{\sin\pi(b-a)\,\psi[e+f-c;\ e,\ f,\ 1+a-b-c,\ 1+a-c-d,\ e+f-a]}{\Gamma(1+a-e-f)\,\Gamma(1+a-d-f)\,\Gamma(1+a-d-e)\,\Gamma(1-s)}$$

$$+\frac{\sin\pi(c-s)\,\psi[2b-a;\ b,\ b+c-a,\ b+d-a,\ b+e-a,\ b+f-a]}{\Gamma(d)\,\Gamma(e)\,\Gamma(f)\,\Gamma(1+a-b-c)}. \tag{2.7}$$

It will be seen that this is a relation between three series of the type ${}_7F_6$.

3. The symmetry of this relation and of the earlier ones is brought out by the introduction of the following notation, x_0, x_1, ..., x_5 being six independent numbers:

$$\left.\begin{aligned} a &= \tfrac{1}{2}-3x_0+x_1+x_2+x_3+x_4+x_5 \\ b &= \tfrac{1}{2}-x_0-x_1+x_2+x_3+x_4+x_5 \\ c &= \tfrac{1}{2}-x_0+x_1-x_2+x_3+x_4+x_5 \\ d &= \tfrac{1}{2}-x_0+x_1+x_2-x_3+x_4+x_5 \\ e &= \tfrac{1}{2}-x_0+x_1+x_2+x_3-x_4+x_5 \\ f &= \tfrac{1}{2}-x_0+x_1+x_2+x_3+x_4-x_5 \end{aligned}\right\}. \qquad (3.1)$$

From these six equations it follows that

$$\begin{aligned} s &\equiv b+c+d+e+f-2a-1 \\ &= \tfrac{1}{2}+x_0+x_1+x_2+x_3+x_4+x_5. \end{aligned} \qquad (3.2)$$

Examination of the parameters defining the ψ's in equation (2.2) shows that

(*a*) The leading parameter is equal to $\tfrac{1}{2}-3x_0\pm x_1\pm x_2\pm x_3\pm x_4\pm x_5$.

The signs to be given to the ambiguities are such that the total number of negative signs is odd.

(*b*) The five following parameters are values of

$$\tfrac{1}{2}-x_0\pm x_1\pm x_2\pm x_3\pm x_4\pm x_5.$$

The signs to be given to the ambiguities are such that the number of negative signs is even. In each case all the signs but one are to be the same as in the leading parameter.

The common value of the sixteen expressions derived from (2.2) will be denoted by $Gp(0)$. The functions $Gp(1)$, $Gp(2)$, ... may be defined by similar rules.

Another group of functions is obtained by changing the signs of the x's. Thus, in any of the ψ's defining $Gn(0)$, the leading parameter is

$$\tfrac{1}{2}+3x_0\pm x_1\pm x_2\pm x_3\pm x_4\pm x_5,$$

the other rules being the same as in the definition of $Gp(0)$.

The equivalents of the parameters in terms of a, b, c, d, e, and f are set out in the following table. It is to be understood that, in the expansions of $Gp(0)$ and $Gn(0)$, permutations of the letters b to f are allowed. In the

expansions of $Gp(1)$ and $Gn(1)$ permutations of the letters c to f are allowed. The number of permutations of each form is given in the last column of the table.

PARAMETERS OF 192 ALLIED SERIES ${}_7F_6$.

$s \equiv b+c+d+e+f-2a-1$.

	Numerator parameters.	Convergency Indicator.	No.
$Gp(0)$	$a;\ b, c, d, e, f$	$2(1-s)$	1
	$b+c-s;\ b, c, 1+a-e-f, 1+a-f-d, 1+a-d-e$	$2(1+a-b-c)$	10
	$1+a-b-s;\ 1-s, 1+a-b-c, 1+a-b-d, 1+a-b-e, 1+a-b-f$	$2b$	5
$Gp(1)$	$2b-a;\ b, b+c-a, b+d-a, b+e-a, b+f-a$	$2(1-s)$	1
	$1-a+b-s;\ 1-s, 1-c, 1-d, 1-e, 1-f$	$2b$	1
$Gp(2)$, etc., are permutations	$2b+c-s-a;\ b, b+c-a, 1+a-e-f, 1+a-f-d, 1+a-d-e$	$2(1-c)$	4
	$1+b-c-s;\ 1-s, 1-c, 1+a-c-d, 1+a-c-e, 1+a-c-f$	$2(b+c-a)$	4
	$1+b-c-d;\ 1-c, 1-d, b+e-a, b+f-a, 1+a-c-d$	$2(1+a-e-f)$	6
$Gn(0)$	$1-a;\ 1-b, 1-c, 1-d, 1-e, 1-f$	$2s$	1
	$1-b-c+s;\ 1-b, 1-c, e+f-a, f+d-a, d+e-a$	$2(b+c-a)$	10
	$s+b-a;\ s, b+c-a, b+d-a, b+e-a, b+f-a$	$2(1-b)$	5
$Gn(1)$	$1-2b+a;\ 1-b, 1-b-c+a, 1-b-d+a, 1-b-e+a, 1-b-f+a$	$2s$	1
	$a-b+s;\ s, c, d, e, f$	$2(1-b)$	1
$Gn(2)$, etc., are permutations	$1-2b-c+s+a;\ 1-b, 1-b-c+a, e+f-a, d+f-a, d+e-a$	$2c$	4
	$c+s-b;\ s, c, c+d-a, c+e-a, c+f-a,$	$2(1-b-c+a)$	4
	$c+d-b;\ c, d, 1+a-b-e, 1+a-b-f, c+d-a$	$2(e+f-a)$	6

Examining the coefficients in equation (2.7), we observe that

$$d+e+f-a = 1+2x_1+2x_2, \quad c-s = -2x_0-2x_2,$$

$$b-a = 2x_0-2x_1, \quad b+d-a = \tfrac{1}{2}+x_0-x_1+x_2-x_3+x_4+x_5,$$

and so on.

Accordingly, equation (2.7) may be rewritten as

$$\frac{\sin 2\pi(x_1+x_2)\, Gp(0)}{\Pi'[\Gamma(\frac{1}{2}+x_0-x_1+x_2\pm x_3\pm x_4\pm x_5)]} - \frac{\sin 2\pi(x_0+x_2)\, Gp(1)}{\Pi'[\Gamma(\frac{1}{2}-x_0+x_1+x_2\pm x_3\pm x_4\pm x_5)]}$$

$$= \frac{\sin 2\pi(x_1-x_0)\, Gn(2)}{\Pi'[\Gamma(\frac{1}{2}-x_0-x_1-x_2\pm x_3\pm x_4\pm x_5)]}. \qquad (3.3)$$

The products indicated by the symbol Π' are subject to the convention that the number of negative signs in any parameter is even.

From two equations, analogous to (3.3), connecting $Gp(0)$, $Gp(1)$, $Gn(3)$, and $Gp(0)$, $Gp(2)$, $Gn(3)$ respectively, $Gn(3)$ can be eliminated. The resulting equation is

$$\frac{\sin 2\pi(x_1-x_2)\,Gp(0)}{\Pi'[\Gamma(\frac{1}{2}+x_0-x_1-x_2\pm x_3\pm x_4\pm x_5)]}+\frac{\sin 2\pi(x_2-x_0)\,Gp(1)}{\Pi'[\Gamma(\frac{1}{2}-x_0+x_1-x_2\pm x_3\pm x_4\pm x_5)]}$$

$$+\frac{\sin 2\pi(x_0-x_1)\,Gp(2)}{\Pi'[\Gamma(\frac{1}{2}-x_0-x_1+x_2\pm x_3\pm x_4\pm x_5)]}=0. \quad (3.4)$$

Further, by eliminating $Gp(2)$ from two equations of the types (3.3) and (3.4), it can be proved that

$$\frac{\sin 2\pi(x_0+x_1)\,Gp(0)}{\Pi'[\Gamma(\frac{1}{2}+x_0-x_1\pm x_2\pm x_3\pm x_4\pm x_5)]}+\frac{\sin 2\pi(x_0-x_1)\,Gn(0)}{\Pi'[\Gamma(\frac{1}{2}-x_0-x_1\pm x_2\pm x_3\pm x_4\pm x_5)]}$$

$$=R(0)\,Gp(1), \quad (3.5)$$

where $R(0)$ is defined by any of the equivalent equations

$$16\pi^4 R(0)=-\sin 8\pi x_0+2\sin 4\pi x_0\,\Sigma_1{}^5\cos 4\pi x$$

$$+16\sin 2\pi x_0\,\Pi_1{}^5\cos 2\pi x+16\cos 2\pi x_0\,\Pi_1{}^5\sin 2\pi x, \quad (3.6)$$

$$16\pi^4 R(0)=-\sin 8\pi x_0-\Sigma_1{}^5\sin[2\pi(1-2x_0\pm 2x)]$$

$$+\Sigma'\sin[2\pi(\tfrac{1}{2}-x_0\pm x_1\pm x_2\pm x_3\pm x_4\pm x_5)], \quad (3.61)$$

and

$$16\pi^4 R(0)=\sin 2\pi(1-s+a)-\Sigma\sin 2\pi(1+a-b)-\Sigma\sin 2\pi(1+b-s)$$

$$+\Sigma\sin 2\pi(1+a-b-c)+\Sigma\sin 2\pi b+\sin 2\pi(1-s), \quad (3.62)$$

the Σ' of (3.61) being subject to the same convention as Π'. Analogues of (3.4) and (3.5) connect three Gn's and express one Gn in terms of a complementary pair Gp and Gn. Since the expression on the right of (3.6) is merely changed in sign by changing the signs of all the x's, the same value of $R(0)$ occurs in (3.5) and its analogue.

In (3.3) and (3.4) series can be substituted for the Gp's and Gn's so as to have the same convergency indicator. In (3.3) the indicator may be $2b$, $2(1+a-c-d)$, $2(1+a-c-e)$, or $2(1+a-c-f)$. In (3.4) the indicator may be $2(1-s)$, $2(1+a-d-e)$, $2(1+a-e-f)$, or $2(1+a-d-f)$. On the other hand, in (3.5) no such choice is possible.

4. On comparing the table of allied series with the corresponding table* for the case in which one of the parameters is a negative integer, so that the series may terminate, we notice that the number of series in the general case is 192 while the number in the special case is only 90. When $b = -n$ all the expansions of $Gp(0)$ are to be found in the special table with the exception of

$$\psi[1+a+n-s;\ 1-s,\ 1+a+n-c,\ 1+a+n-d,\ 1+a+n-e,\ 1+a+n-f].$$

This expression is, however, indeterminate, for the ${}_7F_6$ involved is divergent, the convergency indicator being $-2n$, and the ${}_7F_6$ has to be divided by $\Gamma(-n)$ to obtain the ψ.

Similarly, fifteen forms of each of the functions $Gp(1)$, $Gn(2)$, $Gn(3)$, $Gn(4)$, and $Gn(5)$ survive. The other functions $Gn(0)$, etc., form another related set, but there is no connection between the two sets. This is explained by the intrusion of $\Gamma(-n)$ into the equations which in the general case connect the functions in the two sets.

5. When one of the parameters, say b, tends to infinity, the ${}_7F_6$ may degenerate into ${}_6F_5(-1)$ or into ${}_3F_2$. Of the sixteen ${}_7F_6$'s used to expand $Gp(0)$, five become ${}_6F_5$'s and ten others ${}_3F_2$'s, while one, in which all the parameters tend to infinity, is nugatory. The functions $Gp(0)$, $Gp(1)$, $Gp(2)$, $Gp(3)$, $Gp(4)$, and $Gp(5)$, multiplied by the appropriate factorials, tend when $b \to -\infty$ to the functions† $Fp(0)$, $Fn(1)$, $Fp(2)$, $Fp(3)$, $Fp(4)$, and $Fp(5)$, used in the analysis of series of the type ${}_6F_5(-1)$.

6. The relations between well-poised series of the type ${}_7F_6$ could be derived as limiting cases of Bailey's relations between series of the type ${}_9F_8$, but the work would be rather heavy. It is worth noticing that, if $g \to +\infty$, $h \to -\infty$ in Bailey's equation‡, which involves four ${}_9F_8$'s, two of the reduced series are related to our $Gp(0)$, while the other two are related to $Gp(1)$ and $Gn(5)$, so that the resulting equation, like (2.7), involves only three ${}_7F_6$'s.

[*Added 29 April*, 1935. A Referee has kindly pointed out that, if c is put equal to $-n$ in Bailey's equation, then the four-term relation reduces to one containing three ${}_9F_8$'s. By substituting for d and making $n \to \infty$, an equation equivalent to (2.7) is obtained. The work is still heavy, but it is more direct, since the appeal to equation (2.2), which establishes the equivalence of the forms of $Gp(0)$, is avoided.]

* Whipple, *Proc. London Math. Soc.* (2), 25 (1926), 535. *Cf. Tract*, 60.

† Whipple, *Proc. London Math. Soc.* (2), 24 (1926), 247–263.

‡ *Tract*, 51.

7. Bailey's equation (2.3) involves a pair of Saalschützian ${}_4F_3$'s. There are twenty ways in which $Gp(0)$ can be resolved* into such a pair of series. The symmetry of the equation is brought out by writing

$$S(0,\ 1,\ 2) \equiv \frac{{}_4F_3\left[\begin{matrix}\frac{1}{2}-x_0-x_1-x_2\pm x_3\pm x_4\pm x_5 \\ 1-2x_1-2x_2,\ 1-2x_2-2x_0,\ 1-2x_0-2x_1\end{matrix}\right]}{\Gamma(1-2x_1-2x_2)\,\Gamma(1-2x_2-2x_0)\,\Gamma(1-2x_0-2x_1)}. \quad (7.1)$$

Here each of the four parameters in the numerator is subject to the usual convention, that the number of negative signs before the x's is even.

In this notation (2.3) becomes

$$\frac{\sin 2\pi(x_1+x_2)\,Gp(0)}{\pi}$$

$$= \frac{S(0,\ 1,\ 2)}{\Pi'[\Gamma(\frac{1}{2}-x_0+x_1+x_2\pm x_3\pm x_4\pm x_5)]} - \frac{S(0,\ -1,\ -2)}{\Pi'[\Gamma(\frac{1}{2}-x_0-x_1-x_2\pm x_3\pm x_4\pm x_5)]}. \quad (7.2)$$

The number of S's obtained by permutation of the indices and by changing signs is 160. The total number of resolutions of the type (7.2) is 240. Each S occurs in three of these.

The proof of (3.3) would be facilitated by the new notation and (3.4) can be deduced from (7.2) without appealing to (3.3).

A relation involving four Saalschützian series can be deduced at once from (7.2). There are eight allied functions $S(\pm 0,\ \pm 1,\ \pm 2)$ and six relations of this type between them. It looks as though, by elimination, a relation involving only three of the functions could be deduced. This is not the case, however, for the six relations are not independent.

By substitution from (7.2) in (3.4) a relation between the four functions $S(0,\ 1,\ 2)$, $S(0,\ -1,\ -2)$, $S(-0,\ 1,\ -2)$, and $S(-0,\ -1,\ 2)$ can be found. The coefficient of $S(0,\ 1,\ 2)$ in the equation is the sum of three terms. Relations involving the same four functions can be derived, however, from analogues of (3.3), each involving one Gp and two Gn's, and it is seen that the four relations must be identical. Each is, in fact, equivalent to

$$\frac{\sin 2\pi(x_1-x_2)\sin 2\pi(x_2-x_0)\sin 2\pi(x_0-x_1)}{\Pi'[\Gamma(\frac{1}{2}+x_0+x_1+x_2\pm x_3\pm x_4\pm x_5)]}\,S(0,\ 1,\ 2)$$

$$+\Sigma\,\frac{\sin 2\pi(x_1-x_2)\sin 2\pi(x_2+x_0)\sin 2\pi(x_0+x_1)}{\Pi'[\Gamma(\frac{1}{2}+x_0-x_1-x_2\pm x_3\pm x_4\pm x_5)]}\,S(0,\ -1,\ -2)=0. \quad (7.3)$$

* *Cf. Tract*, § 4.4 (4) and § 7.5 (3).

To obtain a direct proof of (7.3) from (3.4) we should have to use the identity

$$\sin 2\pi(x_1-x_2)\sin 2\pi(x_2-x_0)\sin 2\pi(x_0-x_1)\Pi'[\cos\pi(x_0+x_1+x_2\pm x_3\pm x_4\pm x_5)]$$
$$+\Sigma\sin 2\pi(x_1-x_2)\sin 2\pi(x_2+x_0)\sin 2\pi(x_0+x_1)$$
$$\times\Pi'[\cos\pi(x_0-x_1-x_2\pm x_3\pm x_4\pm x_5)]=0.$$

This is equivalent to an equation of the eighth degree in $\tan\pi x_0$ and, since the equation is satisfied by nine values of $\tan\pi x_0$, viz. $\pm\tan\pi x_1$, $\pm\tan\pi x_2$, $\pm\cot\pi x_1$, $\pm\cot\pi x_2$, and 0, the identity is established. The statement that the six primitive relations between the eight S functions are not independent is therefore justified.

The identity (7.3) can be proved by considering the Barnes integral

$$\int\frac{\Gamma(2x_1+2x_2-z)\,\Gamma(2x_2+2x_0-z)\,\Gamma(2x_0+2x_1-z)\,\Gamma(-z)}{\Pi'[\Gamma(\frac{1}{2}+x_0+x_1+x_2\pm x_3\pm x_4\pm x_5-z)]}\,dz.$$

The identity is therefore more elementary than the other formulae of the paper.

6 Addison Road,
Chiswick, W.4.

CESÀRO MEANS OF POWER SERIES (2)

By A. E. GWILLIAM.

[Received 1 November, 1934.—Read 15 November, 1934.]

I. *Introduction.*

1.1. Throughout this paper we shall suppose that $z = re^{i\theta}$, that

$$f(z) = u(r, \theta) + iv(r, \theta) = \sum_0^\infty a_n z^n$$

is a power series regular in $|z| \leqslant 1$, that

$$\lambda > 0, \quad k > -1,$$

and that A is an absolute constant, or a number depending only on λ and k; A may have different values even in successive contexts.

We write

$$M_\lambda(f) = M_\lambda(f, r) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z)|^\lambda \, d\theta\right)^{1/\lambda},$$

and say that $f(z)$ belongs to the (complex) class L^λ if $M_\lambda(f, r)$ is bounded for all $r < 1$. It is well known that if $f(z)$ belongs to L^λ, then, for almost all θ, $f(z)$ has a radial limit, so that

$$\lim_{r\to 1} f(z) = f(e^{i\theta})$$

exists for almost all θ. Furthermore, $f(e^{i\theta})$ belongs to the (real) class L^λ.

The Cesàro sum and mean of order k of $f(z)$, namely $s_n^k(f, z)$ and $\sigma_n^k(f, z)$ respectively, are defined by the formulae

$$A_n^k \sigma_n^k(f, z) = s_n^k(f, z) = \sum_{\nu=0}^{n} A_{n-\nu}^k a_\nu z^\nu,$$

where

$$A_n^k = \binom{n+k}{n} = \frac{\Gamma(n+k+1)}{\Gamma(k+1)\,\Gamma(n+1)} \sim \frac{n^k}{\Gamma(k+1)}.$$

It is well known that, if $w=\rho e^{i\phi}$, $z=re^{i\theta}$, where $0<\rho<1$, $0<r\leqslant 1$, then

$$(1-w)^{-k-1}f(zw)=\sum_{n=0}^{\infty}s_n{}^k(f,z)w^n.$$

Consequently

$$s_n{}^k(f,e^{i\theta})=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(\rho e^{i(\theta+\phi)})}{(1-\rho e^{i\phi})^{k+1}}\rho^{-n}e^{-in\phi}\,d\phi.$$

We shall write

$$\sigma_n{}^k(f,z)=\sigma_n{}^k(u,r,\theta)+i\sigma_n{}^k(v,r,\theta).$$

1.2. Hardy and Littlewood* have shown that, if $f(z)$ belongs to L^λ, and if

(1.2.1) $\quad k>0$ when $\lambda\geqslant 1$, $\quad k>(1/\lambda)-1$ when $0<\lambda\leqslant 1$,

then

$$\int_{-\pi}^{\pi}|\sigma_n{}^k(f,e^{i\theta})-f(e^{i\theta})|^\lambda\,d\theta=o(1). \tag{1.2.2}$$

We shall give an alternative proof of this theorem, showing that it may be derived from the following result. If

$$M_\lambda(f')=o\{(1-r)^{-1}\}, \tag{1.2.3}$$

and if k satisfies (1.2.1), then

$$\int_{-\pi}^{\pi}|\sigma_n{}^k(f,e^{i\theta})-f(\rho e^{i\theta})|^\lambda\,d\theta=o(1), \tag{1.2.4}$$

where

$$\rho=1-(1/n).$$

This result is true also if o is replaced by O. Having proved this result, we are then able to solve a problem left open by Hardy and Littlewood. We show that, if k satisfies (1.2.1), then

$$\int_{-\pi}^{\pi}|\sigma_n{}^k(u,1,\theta)|^\lambda\,d\theta=O(1). \tag{1.2.5}$$

1.3. We need four known theorems, which we give in the form of lemmas.

* Hardy and Littlewood, (3), Theorem 2.

LEMMA 1*. ("Max".) *If* $r<\rho\leqslant 1$, *then*

$$\int_{-\pi}^{\pi}\max_{|\phi|\leqslant(\rho-r)}|f(re^{i(\theta+\phi)})|^{\lambda}\,d\theta\leqslant A\int_{-\pi}^{\pi}|f(\rho e^{i\theta})|^{\lambda}\,d\theta. \tag{1.3.1}$$

LEMMA 2†. *If* $f(z)$ *belongs to* L^{λ}, *then*

$$\int_{-\pi}^{\pi}|f(e^{i\theta})-f(\rho e^{i\theta})|^{\lambda}\,d\theta=o(1), \tag{1.3.2}$$

when $\rho\to 1$.

LEMMA 3‡. *If* $f(z)$ *belongs to* L^{λ}, *then*

$$M_{\lambda}(f')=o\{(1-r)^{-1}\}. \tag{1.3.3}$$

LEMMA 4§. *If* $u(r,\theta)$ *belongs to* L^{λ}, *then*

$$M_{\lambda}(f')=O\{(1-r)^{-1}\}. \tag{1.3.4}$$

2.

2.1. We come now to the essential theorem of the paper.

THEOREM 1. *If*

$$M_{\lambda}(f')=o\{(1-r)^{-1}\}, \tag{2.1.1}$$

and if

$$k>0 \text{ when } \lambda\geqslant 1,\quad k>(1/\lambda)-1 \text{ when } 0<\lambda\leqslant 1, \tag{2.1.2}$$

then

$$\int_{-\pi}^{\pi}|\sigma_n{}^{k}(f,e^{i\theta})-f(\rho e^{i\theta})|^{\lambda}\,d\theta=o(1), \tag{2.1.3}$$

where

$$\rho=1-(1/n).$$

All this holds if o *is replaced throughout by* O.

---

* Hardy and Littlewood (1), Theorem 27.

† See F. Riesz (4), 95.

‡ Hardy and Littlewood, (2), Theorem 4.

§ Hardy and Littlewood, (2), Theorem 2.

We prove this theorem only in the o form. The O form follows in the same way. For simplicity in printing, we shall write

$$\eta = \pi/n = \pi(1-\rho).$$

We need three preliminary lemmas.

LEMMA 5. *If a is either $k+1$ or $k+2$, then*

$$|1-\rho e^{i\phi}|^{-a} \leqslant A(\eta^a + |\phi|^a)^{-1}. \tag{2.1.4}$$

This result is elementary.

LEMMA 6. *We have*

$$\left| \frac{1}{(1-\rho e^{i(\phi+\eta)})^{k+1}} - \frac{1}{(1-\rho e^{i\phi})^{k+1}} \right| \leqslant \frac{A}{n} \frac{1}{\eta^{k+2}+|\phi|^{k+2}}. \tag{2.1.5}$$

For the left-hand side of (2.1.5) is equal to

$$\left| \int_\phi^{\phi+\eta} \frac{i(k+1)\rho e^{i\psi}\, d\psi}{(1-\rho e^{i\psi})^{k+2}} \right| \leqslant \frac{A}{n} \max_{\phi \leqslant \psi \leqslant \phi+\eta} \left| \frac{1}{1-\rho e^{i\psi}} \right|^{k+2}$$

$$\leqslant \frac{A}{n} \max_{\phi \leqslant \psi \leqslant \phi+\eta} \left(\frac{1}{\eta^{k+2}+|\psi|^{k+2}} \right)$$

by Lemma 5. This gives the result.

LEMMA 7. *If $\psi > 0$, then*

$$\int_\psi^\pi \frac{d\phi}{\eta^{k+2}+\phi^{k+2}} \leqslant \frac{A}{\eta^{k+1}+\psi^{k+1}}. \tag{2.1.6}$$

We consider the two possible cases, (1) when $\psi \geqslant \eta$, and (2) when $\psi < \eta$. In case (1), we have

$$\int_\psi^\pi \frac{d\phi}{\eta^{k+2}+\phi^{k+2}} \leqslant \int_\psi^\infty \frac{d\phi}{\phi^{k+2}} = \frac{A}{2\psi^{k+1}} \leqslant \frac{A}{\eta^{k+1}+\psi^{k+1}}.$$

In case (2), we have

$$\int_\psi^\pi \frac{d\phi}{\eta^{k+2}+\phi^{k+2}} \leqslant \int_0^\infty \frac{d\phi}{\eta^{k+2}+\phi^{k+2}} = \frac{A}{2\eta^{k+1}} \leqslant \frac{A}{\eta^{k+1}+\psi^{k+1}}.$$

This completes the proof of the lemma.

We come now to the proof of the theorem. We have

$$A_n^k \sigma_n^k(f, e^{i\theta}) = s_n^k(f, e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\rho e^{i(\theta+\phi)})}{(1-\rho e^{i\phi})^{k+1}} \rho^{-n} e^{-in\phi}\, d\phi.$$

Noticing that $$2e^{-in\phi} = e^{-in\phi} - e^{-in(\phi-\eta)},$$

we see that

$$\begin{aligned} &A_n^k[\sigma_n^k(f, e^{i\theta}) - f(\rho e^{i\theta})] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\rho e^{i(\theta+\phi)}) - f(\rho e^{i\theta})}{(1-\rho e^{i\phi})^{k+1}} \rho^{-n} e^{-in\phi}\, d\phi \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{f(\rho e^{i(\theta+\phi)}) - f(\rho e^{i\theta})}{(1-\rho e^{i\phi})^{k+1}} \rho^{-n} e^{-in\phi}\, d\phi \\ &\qquad - \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{f(\rho e^{i(\theta+\phi+\eta)}) - f(\rho e^{i\theta})}{(1-\rho e^{i(\phi+\eta)})^{k+1}} \rho^{-n} e^{-in\phi}\, d\phi \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 = \frac{1}{4\pi}\int_0^{\pi} [f(\rho e^{i(\theta+\phi)}) - f(\rho e^{i\theta})]\left[\frac{1}{(1-\rho e^{i\phi})^{k+1}} - \frac{1}{(1-\rho e^{i(\phi+\eta)})^{k+1}}\right] \rho^{-n} e^{-in\phi}\, d\phi,$$

$$I_2 = \frac{1}{4\pi}\int_{-\pi}^{0} [f(\rho e^{i(\theta+\phi)}) - f(\rho e^{i\theta})]\left[\frac{1}{(1-\rho e^{i\phi})^{k+1}} - \frac{1}{(1-\rho e^{i(\phi+\eta)})^{k+1}}\right] \rho^{-n} e^{-in\phi}\, d\phi,$$

$$I_3 = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{f(\rho e^{i(\theta+\phi)}) - f(\rho e^{i(\theta+\phi+\eta)})}{(1-\rho e^{i(\phi+\eta)})^{k+1}} \rho^{-n} e^{-in\phi}\, d\phi.$$

Now, using Lemmas 6 and 7, we see that

$$\begin{aligned} |I_1| &\leqslant \frac{A}{n}\int_0^{\pi} \frac{d\phi}{\eta^{k+2} + \phi^{k+2}} \int_0^{\phi} |f'(\rho e^{i(\theta+\psi)})|\, d\psi \\ &= \frac{A}{n}\int_0^{\pi} |f'(\rho e^{i(\theta+\psi)})|\, d\psi \int_{\psi}^{\pi} \frac{d\phi}{\eta^{k+2} + \phi^{k+2}} \\ &\leqslant \frac{A}{n}\int_0^{\pi} \frac{|f'(\rho e^{i(\phi+\psi)})|\, d\psi}{\eta^{k+1} + \psi^{k+1}}. \end{aligned}$$

Similarly,

$$|I_2| \leqslant \frac{A}{n}\int_{-\pi}^{0} \frac{|f'(\rho e^{i(\theta+\psi)})|\, d\psi}{\eta^{k+1} + |\psi|^{k+1}}.$$

Also, by Lemma 5,

$$|I_3| \leqslant A \int_{-\pi}^{\pi} \frac{d\phi}{\eta^{k+1}+|\phi|^{k+1}} \int_{\phi}^{(\phi+\eta)} |f'(\rho e^{i(\theta+\psi)}|\, d\psi$$

$$\leqslant \frac{A}{n} \int_{-\pi}^{\pi} \left[\max_{\phi \leqslant \psi \leqslant \phi+\eta} |f'(\rho e^{i(\theta+\psi)})| \right] \frac{d\phi}{\eta^{k+1}+|\phi|^{k+1}}.$$

Thus

$$|\sigma_n{}^k(f, e^{i\theta}) - f(\rho e^{i\theta})| \leqslant \frac{A}{n^{k+1}} \int_{-\pi}^{\pi} \left[\max_{\phi \leqslant \psi \leqslant \phi+\eta} |f'(\rho e^{i(\theta+\psi)})| \right] \frac{d\phi}{\eta^{k+1}+|\phi|^{k+1}}.$$

We now consider separately the two possible cases; (1) when $\lambda \geqslant 1$, and (2) when $0 < \lambda \leqslant 1$.

(1) $\lambda \geqslant 1$, $k > 0$. Then, by Minkowski's inequality,

$$\left(\int_{-\pi}^{\pi} |\sigma_n{}^k(f, e^{i\theta}) - f(\rho e^{i\theta})|^{\lambda} d\theta \right)^{1/\lambda}$$

$$\leqslant \frac{A}{n^{k+1}} \left[\int_{-\pi}^{\pi} d\theta \left(\int_{-\pi}^{\pi} \left[\max_{\phi \leqslant \psi \leqslant \phi+\eta} |f'(\rho e^{i(\theta+\psi)})| \right] \frac{d\phi}{\eta^{k+1}+|\phi|^{k+1}} \right)^{\lambda} \right]^{1/\lambda}$$

$$\leqslant \frac{A}{n^{k+1}} \int_{-\pi}^{\pi} \frac{d\phi}{\eta^{k+1}+|\phi|^{k+1}} \left(\int_{-\pi}^{\pi} \max_{\phi \leqslant \psi \leqslant \phi+\eta} |f'(\rho e^{i(\theta+\psi)})|^{\lambda} d\theta \right)^{1/\lambda}$$

$$\leqslant \frac{A}{n^{k+1}} \int_{-\pi}^{\pi} \frac{d\phi}{\eta^{k+1}+|\phi|^{k+1}} \left(\int_{-\pi}^{\pi} |f'\{\tfrac{1}{2}(1+\rho)\, e^{i\theta}\}|^{\lambda} d\theta \right)^{1/\lambda}$$

by "Max". Now

$$\left(\int_{-\pi}^{\pi} |f'\{\tfrac{1}{2}(1+\rho)\, e^{i\theta}\}^{\lambda} d\theta \right)^{1/\lambda} = o(n),$$

and

$$\int_{-\pi}^{\pi} \frac{d\phi}{\eta^{k+1}+|\phi|^{k+1}} \leqslant \frac{1}{\eta^k} \int_{-\infty}^{\infty} \frac{d\phi}{1+|\phi|^{k+1}} = An^k.$$

Consequently

$$\left(\int_{-\pi}^{\pi} |\sigma_n{}^k(f, e^{i\theta}) - f(\rho e^{i\theta})|^{\lambda} d\theta \right)^{1/\lambda} = o(1).$$

This proves the theorem in the case $\lambda \geqslant 1$.

(2) $0 < \lambda \leqslant 1$, $k > (1/\lambda) - 1$. Let

$$\theta_m = \pi m(1-\rho), \quad \theta_{M-1} < \pi \leqslant \theta_M.$$

Then

$$\int_{-\pi}^{\pi} |\sigma_n^k(f, e^{i\theta}) - f(\rho e^{i\theta})|^\lambda \, d\theta$$

$$\leqslant \frac{A}{n^{\lambda(k+1)}} \int_{-\pi}^{\pi} d\theta \left(\int_{-\pi}^{\pi} \left[\max_{\phi \leqslant \psi \leqslant \phi + \eta} |f'(\rho e^{i(\theta+\psi)})| \right] \frac{d\phi}{\eta^{k+1} + |\phi|^{k+1}} \right)^\lambda$$

$$\leqslant \frac{A}{n^{\lambda(k+1)}} \sum_{m=-M}^{M-1} \int_{-\pi}^{\pi} d\theta \left(\int_{\theta_m}^{\theta_{m+1}} \left[\max_{\phi \leqslant \psi \leqslant \phi + \eta} |f'(\rho e^{i(\theta+\psi)})| \right] \frac{d\phi}{\eta^{k+1} + |\phi|^{k+1}} \right)^\lambda$$

$$\leqslant \frac{A}{n^{\lambda(k+1)}} \sum_{-M}^{M-1} \int_{-\pi}^{\pi} \max_{\theta_m \leqslant \psi \leqslant \theta_{m+2}} |f'(\rho e^{i(\theta+\psi)})|^\lambda \, d\theta \left(\int_{\theta_m}^{\theta_{m+1}} \frac{d\phi}{\eta^{k+1} + |\phi|^{k+1}} \right)^\lambda.$$

Now, by "Max",

$$\int_{-\pi}^{\pi} \left[\max_{\theta_m \leqslant \psi \leqslant \theta_{m+2}} |f'(\rho e^{i(\theta+\psi)})|^\lambda \right] d\theta \leqslant A \int_{-\pi}^{\pi} |f'\{\tfrac{1}{2}(1+\rho) e^{i\theta}\}|^\lambda \, d\theta = o(n^\lambda).$$

Also

$$\sum_{-M}^{M-1} \left(\int_{\theta_m}^{\theta_{m+1}} \frac{d\phi}{\eta^{k+1} + |\phi|^{k+1}} \right)^\lambda \leqslant 2 \left(\int_{\theta_0}^{\theta_1} \frac{d\phi}{\eta^{k+1}} \right)^\lambda + 2 \sum_{1}^{\infty} \left(\int_{\theta_m}^{\theta_{m+1}} \frac{d\phi}{\phi^{k+1}} \right)^\lambda$$

$$\leqslant An^{\lambda k} + An^{\lambda k} \sum_{1}^{\infty} m^{-\lambda(k+1)} \leqslant An^{\lambda k}$$

since $\lambda(k+1) > 1$. Thus

$$\int_{-\pi}^{\pi} |\sigma_n^k(f, e^{i\theta}) - f(\rho e^{i\theta})|^\lambda \, d\theta = o(1),$$

which completes the proof of the theorem.

2.2. We are now able to prove the following theorems.

THEOREM 2. *If $f(z)$ belongs to L^λ, and if k satisfies* (2.1.2), *then*

$$\int_{-\pi}^{\pi} |\sigma_n^k(f, e^{i\theta}) - f(e^{i\theta})|^\lambda \, d\theta = o(1). \tag{2.2.1}$$

For, if $f(z)$ belongs to L^λ, then $M_\lambda(f') = o\{(1-r)\}^{-1}$, and Theorem 1 and Lemma 2 give us (2.2.1).

THEOREM 3. *If $u(r, \theta)$ belongs to L^λ, and if k satisfies* (2.1.2), *then*

$$\int_{-\pi}^{\pi} |\sigma_n^k(u, 1, \theta)|^\lambda \, d\theta = O(1). \tag{2.2.2}$$

For then $M_\lambda(f') = O\{(1-r)^{-1}\}$, and so

$$\int_{-\pi}^{\pi} |\sigma_n{}^k(f, e^{i\theta}) - f(\rho e^{i\theta})|^\lambda d\theta = O(1),$$

where $\rho = 1-(1/n)$. Thus

$$\int_{-\pi}^{\pi} |\sigma_n{}^k(u, 1, \theta) - u(\rho, \theta)|^\lambda d\theta = O(1),$$

which gives (2.2.2).

References.

1. G. H. Hardy and J. E. Littlewood, "A maximal theorem with function-theoretic applications", *Acta Math.*, 54 (1930), 81–116.
2. ————, "Some properties of conjugate functions", *Journal für Math.*, 167 (1931), 405–423.
3. ————, "Cesàro means of power series", *Proc. London Math. Soc.* (2), 36 (1932), 516–530.
4. F. Riesz, "Über die Randwerte einer analytischen Funktion", *Math. Zeitschrift*, 18 (1923), 87–95.

Trinity College,
Cambridge.

ON LIPSCHITZ CONDITIONS

By A. E. GWILLIAM.

[Received 1 November, 1934.—Read 15 November, 1934.]

1. *Introduction.*

1.1. We suppose throughout this paper that $f(z)$ is regular in $|z|<1$, that $z=re^{i\theta}$, and that

$$k>0, \quad p\geqslant 1, \quad 0<\lambda\leqslant 1, \quad 0<\alpha\leqslant 1.$$

For the sake of convenience we frequently write

$$f(z)=f(r,\theta)=u(r,\theta)+iv(r,\theta), \quad f'(z)=f'(r,\theta),$$

and so on.

We suppose that A is an absolute constant or a number depending only on k, p, λ, and α, that B is a number depending on other specified variables, and that C is a number given in the enunciation. Throughout each separate proof, each B or C will be assumed constant, although A, unless qualified by an index, may denote different numbers even in successive contexts.

The "γ-th" integral $f_\gamma(z)$ of $f(z)$ is defined, for all $\gamma>0$, by the formula

$$f_\gamma(z)=\frac{1}{\Gamma(\gamma)}\int_0^z (z-u)^{\gamma-1}f(u)\,du,$$

where the path of integration is rectilinear, and $(z-u)^{\gamma-1}$ takes its principal value at the origin.

If $\gamma=-N$, where N is a positive integer, then we define $f_\gamma(z)$ as the "N-th" derivative of $f(z)$.

If $-N<\gamma=N+\alpha<-N+1$, then we define $f_\gamma(z)$ by

$$f_\gamma(z)=\{f_{-N}(z)\}_\alpha.$$

Then we have

$$(f_\alpha)_\beta=f_{\alpha+\beta} \tag{1.1.1}$$

provided that* either $\alpha \geqslant 0$, or, otherwise, that $f(z)$ has a zero of order $[|\alpha|]+1$ at the origin.

We write

$$M_k(f) = M_k(f, r) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z)|^k \, d\theta\right)^{1/k},$$

and say that $f(z)$ belongs to the (complex) class L^k if $M_k(f, r)$ is bounded for all $r < 1$.

The function $f(z)$ is said to belong to the Lipschitz class $\mathrm{Lip}(\alpha, k)$ if

$$\left(\int_{-\pi}^{\pi} |f(r, \theta+h)-f(r, \theta)|^k \, d\theta\right)^{1/k} \leqslant C\,|h|^{\alpha}.$$

1.2. Hardy and Littlewood† have proved that the three conditions

$$f(z) \in \mathrm{Lip}\,(\alpha, p), \tag{1.2.1}$$

$$M_p(f') = O\{(1-r)^{\alpha-1}\}, \tag{1.2.2}$$

and

$$f_\gamma(z) \in \mathrm{Lip}(\alpha+\gamma, p), \tag{1.2.3}$$

where

$$p \geqslant 1, \quad 0 < \alpha+\gamma < 1,$$

are equivalent. Furthermore, if $\alpha = 1$, then (1.2.1) and (1.2.2) are equivalent.

Hardy and Littlewood‡ have also shown that, if $0 < \lambda \leqslant 1$, $\alpha \geqslant 0$, and if

$$M_\lambda(u) = O\{(1-r)^{-\alpha}\}, \tag{1.2.4}$$

then

$$M_\lambda(f') = O\{(1-r)^{-\alpha-1}\}. \tag{1.2.5}$$

By means of this theorem we are able to show that, if u belongs to $\mathrm{Lip}\,(\alpha, \lambda)$, where $0 < \lambda \leqslant 1$, $0 < \alpha < 1$, then so does f. This is not true when $\alpha = 1$. We also show that, by means of this last result, certain theorems of Hardy

* These conditions are necessary only because of the vanishing of the first coefficients in the expansion of the terms in (1.1.1).

† See Hardy and Littlewood, 1, 2, and 3, for these and similar theorems.

‡ Hardy and Littlewood, 4, Theorem 2. The constant implied in (1.2.5) plays a fundamental rôle in one of my theorems. Since Hardy and Littlewood only state this constant without proof, I shall prove this result by a method slightly different from theirs.

and Littlewood on fractional integrals can be extended from the case of the index $p \geqslant 1$ to the case of the index $k > 0$.

1.3. We need four theorems of Hardy and Littlewood, and one lemma.

THEOREM 1. ("Max".)* *If* $\rho > r$, *then*

$$\int_{-\pi}^{\pi} \max_{0 \leqslant \phi \leqslant \rho - r} |f(r, \theta+\phi)|^k d\theta \leqslant A \int_{-\pi}^{\pi} |f(\rho, \theta)|^k d\theta, \tag{1.3.1}$$

and

$$\int_{-\pi}^{\pi} \max_{0 \leqslant t \leqslant \rho} |f(t, \theta)|^k d\theta \leqslant A \int_{-\pi}^{\pi} |f(\rho, \theta)|^k d\theta. \tag{1.3.2}$$

THEOREM 2.† *If* $\rho > r$, *then*

$$M_k(f', r) \leqslant A(\rho - r)^{-1} M_k(f, \rho). \tag{1.3.3}$$

THEOREM 3.‡ *If* $a > 0$, $f(0) = 0$, *and if*

$$M_k(f') \leqslant C(1-r)^{-a-1}, \tag{1.3.4}$$

then

$$M_k(f) \leqslant BC(1-r)^{-a}, \tag{1.3.5}$$

where $B = B(k, a)$. *If, further,* $0 < A \leqslant a$, *then* B *may be replaced by an* A.

THEOREM 4§. *If* $N(r) \geqslant 0$, $a \geqslant 0$, $b \geqslant 0$, $0 < \mu \leqslant 1$, *and if, for all* $\frac{1}{2} \leqslant r < \rho \leqslant \frac{1}{2}(1+r) < 1$, *we have*

$$N(r) \leqslant C(1-r)^{-a}(\rho - r)^{-b}[N(\rho)]^{1-\mu}, \tag{1.3.6}$$

then $N(r)$ *is of finite order: there are constants* B_1 *and* B_2 *such that*

$$N(r) \leqslant B_1(1-r)^{-B_2}.$$

---

* See Hardy and Littlewood, **5**, Theorem 27. Actually, "Max" contains slightly more than we quote. We shall always refer to this theorem as "Max".

† Hardy and Littlewood, **3**, Theorem 46. Their theorem is more general than this, and they do not actually state it in this form. It will be seen, however, that only a slight modification of their theorem is necessary.

‡ Hardy and Littlewood, **4**, Theorem 3.

§ Hardy and Littlewood, **4**, 409. Their equation (2.2.2) is equivalent to our equation (1.3.6) with a slight difference of constants. Their proof, however, does suffice for our theorem.

LEMMA 1. *If* $N(r) \geqslant 1$, $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$, $d \geqslant 0$, $e \geqslant 0$, $f > 0$, *and if, for all* $0 \leqslant r < \rho \leqslant \frac{1}{2}(1+r) < 1$, $0 \leqslant h \leqslant \frac{1}{2}(\rho - r)$, *we have*

$$(1.3.7) \quad N(r) \leqslant C_1(1-r)^{-a}(\rho-r)^{-b}h^{-c} + C_2(1-r)^{-d}(\rho-r)^{-e}h^f N(\rho),$$

then $N(r)$ *is of finite order.*

For each particular pair of values of ρ and r, we have two possible cases: (1), when

$$C_1(1-r)^{-a}(\rho-r)^{-b-c} > C_2(1-r)^{-d}(\rho-r)^{-e+f}N(\rho),$$

and (2), when

$$C_1(1-r)^{-a}(\rho-r)^{-b-c} \leqslant C_2(1-r)^{-d}(\rho-r)^{-e+f}N(\rho).$$

In case (1), let $h = \frac{1}{2}(\rho-r)$, so that (1.3.7) gives us

$$N(r) \leqslant B_1(1-r)^{-a}(\rho-r)^{-b-c}.$$

In case (2), let

$$h^{f+c} = 2^{-f-c}\, C_1\, C_2^{-1}(1-r)^{d-a}(\rho-r)^{e-b}[N(\rho)]^{-1},$$

so that $h \leqslant \frac{1}{2}(\rho-r)$, and both terms on the right-hand side of (1.3.7) are of the same order. Then (1.3.7) gives us

$$N(r) \leqslant B_2(1-r)^{-a_1}(\rho-r)^{-b_1}[N(\rho)]^{1-\mu},$$

where

$$a_1 = (fa+cd)/(f+c), \quad b_1 = (fb+ce)/(f+c), \quad \mu = f/(f+c).$$

Thus, in either case,

$$N(r) \leqslant B_3(1-r)^{-a_2}(\rho-r)^{-b_2}[N(\rho)]^{1-\mu},$$

where $$a_2 = \max(a, a_1), \quad b_2 = \max(b+c, b_1).$$

Consequently, $N(r)$ satisfies the conditions of Theorem 4, and so $N(r)$ is of finite order.

Throughout this paper we shall make use of some of the many forms and extensions of Minkowski's inequality. The particular forms we use are dealt with fully in Theorems 25, 27, 199, and 202 of *Inequalities* (1934) by Hardy, Littlewood, and Pólya.

2.

2.1. In this section the theorems are given only for the case of the index λ, where $0 < \lambda \leqslant 1$. They are already known* for the case of the

* As already pointed out, Theorem 6 also is known, and is due to Hardy and Littlewood.

index $p \geqslant 1$. To avoid certain glosses (when applying Theorem 3), we suppose that $f(0) = 0$, $f'(0) = 0$. This is only a trivial assumption.

THEOREM 5. *If* $0 < \lambda \leqslant 1$, $0 < \alpha \leqslant 1$, *and if*

$$M_\lambda(f') \leqslant C(1-r)^{\alpha-1}, \tag{2.1.1}$$

then $f(z)$ *belongs to* Lip (α, λ).

Suppose that $h > 0$. Then

$$f(r, \theta+h)-f(r, \theta) = I_1+I_2+I_3,$$

where

$$I_1 = f(r, \theta+h)-f(r-h, \theta+h), \quad I_2 = f(r-h, \theta+h)-f(r-h, \theta),$$

$$I_3 = f(r-h, \theta)-f(r, \theta),$$

so that

$$\int_{-\pi}^{\pi} |f(r, \theta+h)-f(r, \theta)|^\lambda d\theta \leqslant \int_{-\pi}^{\pi} |I|^\lambda d\theta + \int_{-\pi}^{\pi} |I_2|^\lambda d\theta + \int_{-\pi}^{\pi} |I_3|^\lambda d\theta.$$

Let $r_n = r(1-2^{-n})$, so that $r_N \leqslant r-h < r_{N+1}$. Then

$$\int_{-\pi}^{\pi} |I_1|^\lambda d\theta = \int_{-\pi}^{\pi} |I_3|^\lambda d\theta \leqslant \int_{-\pi}^{\pi} d\theta \left(\int_{r-h}^{r} |f'(t, \theta)|\, dt\right)^\lambda$$

$$\leqslant \sum_N^\infty \int_{-\pi}^{\pi} d\theta \left(\int_{r_n}^{r_{n+1}} |f'(t, \theta)|\, dt\right)^\lambda$$

$$\leqslant \sum_N^\infty (r_{n+1}-r_n)^\lambda \int_{-\pi}^{\pi} \max_{r_n \leqslant t \leqslant r_{n+1}} |f'(t, \theta)|^\lambda d\theta$$

$$\leqslant A \sum_N^\infty (r_{n+1}-r_n)^\lambda \int_{-\pi}^{\pi} |f'(r_{n+1}, \theta)|^\lambda d\theta$$

(by "Max")

$$\leqslant AC^\lambda \sum_N^\infty (r_{n+1}-r_n)^\lambda (1-r_{n+1})^{\lambda(\alpha-1)}$$

$$\leqslant AC^\lambda \sum_N^\infty (r_{n+1}-r_n)^\lambda (r-r_{n+1})^{\lambda(\alpha-1)}$$

$$= AC^\lambda \sum_N^\infty (r-r_{n+1})^{\lambda\alpha} = AC^\lambda \sum_N^\infty 2^{-n\lambda\alpha} r^{\lambda\alpha}$$

$$\leqslant AC^\lambda 2^{-N\lambda\alpha} r^{\lambda\alpha} \leqslant AC^\lambda h^{\lambda\alpha}.$$

Also,

$$\int_{-\pi}^{\pi} |I_2|^\lambda d\theta \leqslant \int_{-\pi}^{\pi} d\theta \left(\int_0^h |f'(r-h, \theta+\phi)| d\phi \right)^\lambda$$

$$\leqslant h^\lambda \int_{-\pi}^{\pi} \max_{0 \leqslant \phi \leqslant h} |f'(r-h, \theta+\phi)|^\lambda d\theta$$

$$\leqslant A h^\lambda \int_{-\pi}^{\pi} |f'(1-\tfrac{1}{2}h, \theta)|^\lambda d\theta \leqslant AC^\lambda h^{\lambda a},$$

by "Max". This completes the proof of the theorem.

2.2. THEOREM 6. *If* $0 < \lambda \leqslant 1$, $a \geqslant 0$, *and if*

$$M_\lambda(u) \leqslant C(1-r)^{-a}, \tag{2.2.1}$$

then

$$M_\lambda(f') \leqslant AC(1-r)^{-a-1}. \tag{2.2.2}$$

We show first that f' is of finite order. We take $r < \rho \leqslant \frac{1}{2}(1+r)$, $0 < h \leqslant \frac{1}{2}(\rho-r)$. Then, by the second mean value theorem, we have

$$\frac{\partial}{\partial\theta} u(r, \theta) = \frac{1}{h}[u(r, \theta+h) - u(r, \theta)] - \frac{h}{2} \frac{\partial^2}{\partial\theta^2} u(r, \theta+\xi_1 h),$$

$$\frac{\partial}{\partial r} u(r, \theta) = \frac{1}{h}[u(r+h, \theta) - u(r, \theta)] - \frac{h}{2} \frac{\partial^2}{\partial r^2} u(r+\xi_2 h, \theta),$$

where $0 \leqslant \xi_1 \leqslant 1$, $0 \leqslant \xi_2 \leqslant 1$. Thus, if

$$N(r) = \int_{-\pi}^{\pi} |f'(r, \theta)|^\lambda d\theta,$$

since

$$izf' = \frac{\partial u}{\partial\theta} + ir \frac{\partial u}{\partial r},$$

we have, for $r \geqslant \frac{1}{2}$,

$$N(r) \leqslant \frac{A}{h^\lambda}\left[3\int_{-\pi}^{\pi} |u(r, \theta)|^\lambda d\theta + \int_{-\pi}^{\pi} |u(r+h, \theta)|^\lambda d\theta\right] \tag{2.2.3}$$

$$+ Ah^\lambda\left[\int_{-\pi}^{\pi} \left|\frac{\partial^2}{\partial\theta^2} u(r, \theta+\xi_1 h)\right|^\lambda d\theta + \int_{-\pi}^{\pi} \left|\frac{\partial^2}{\partial r^2} u(r+\xi_2 h, \theta)\right|^\lambda d\theta\right].$$

Now, since $1-(r+h) \geqslant \frac{1}{2}(1-r)$, we have

$$\int_{-\pi}^{\pi} |u(r+h, \theta)|^\lambda d\theta \leqslant AC^\lambda (1-r)^{-\lambda a}.$$

Also,
$$zf'+z^2f''=-\frac{\partial^2 u}{\partial\theta^2}-i\frac{\partial^2 v}{\partial\theta^2},$$

so that, by "Max" and Theorem 2,

$$(2.2.4)\quad \int_{-\pi}^{\pi}\left|\frac{\partial^2}{\partial\theta^2}u(r,\ \theta+\xi_1 h)\right|^\lambda d\theta$$
$$\leqslant\int_{-\pi}^{\pi}|f'(r,\ \theta+\xi_1 h)|^\lambda d\theta+\int_{-\pi}^{\pi}|f''(r,\ \theta+\xi_1 h)|^\lambda d\theta$$
$$\leqslant A\int_{-\pi}^{\pi}|f'(\rho,\ \theta)|^\lambda d\theta+A\int_{-\pi}^{\pi}|f''\{\tfrac{1}{2}(\rho+r),\ \theta\}|^\lambda d\theta$$
$$\leqslant AN(\rho)+A(\rho-r)^{-\lambda}N(\rho)\leqslant A(\rho-r)^{-\lambda}N(\rho).$$

Similarly,
$$\int_{-\pi}^{\pi}\left|\frac{\partial^2}{\partial r^2}u(r+\xi_2 h,\ \theta)\right|^\lambda d\theta\leqslant A(\rho-r)^{-\lambda}N(\rho),$$

and so

$$(2.2.5)\qquad N(r)\leqslant AC^\lambda(1-r)^{-\lambda a}h^{-\lambda}+A(\rho-r)^{-\lambda}h^\lambda N(\rho).$$

Thus, by Lemma 1, we see that $N(r)$ is of finite order: there are constants $b_1>(a+1)$, $B_1>4^{2\lambda}$, such that

$$(2.2.6)\qquad N(r)\leqslant B_1C^\lambda(1-r)^{-\lambda b_1}.$$

Now suppose that there are constants $b_n>(a+1)$, $B_n>4^{2\lambda}$, such that

$$(2.2.7)\qquad N(r)\leqslant B_nC^\lambda(1-r)^{-\lambda b_n}.$$

Let $\rho=\frac{1}{2}(1+r)$, and

$$h^{2\lambda}=B_n^{-1}(1-r)^{-\lambda(b_n+1-a)},$$

so that $h\leqslant\frac{1}{2}(\rho-r)$. If we replace $N(\rho)$ in (2.2.5) by the bound given by (2.2.7), we have

$$(2.2.8)\qquad N(r)\leqslant A_1B_n^{\frac{1}{2}}C^\lambda(1-r)^{-\frac{1}{2}\lambda(b_n+a-1)},$$

where A_1 is a function of λ only: we may assume that $A_1\geqslant 4^\lambda$.

Let
$$b_{n+1}=\tfrac{1}{2}(b_n+a+1),\quad B_{n+1}=A_1B_n^{\frac{1}{2}},$$

so that $B_{n+1}\geqslant 4^{2\lambda}$ for all n. Also,

$$b_n\to(a+1),\quad B_n=A_1^{(1+2^{-1}+\ldots 2^{-n+2})}B_1^{2^{-n+1}}\to A_1^2,$$

as $n\to\infty$.

Then, by induction, (2.2.8) shows that, if (2.2.7) is satisfied (as it is for $n=1$) for any particular value of n, then it is satisfied for all succeeding values of n. Consequently,

$$N(r) \leqslant \lim_{n\to\infty} B_n C^\lambda (1-r)^{-\lambda b_n} = A C^\lambda (1-r)^{-\lambda(a+1)},$$

which is the required result.

2.3. THEOREM 7. *If* $0<\lambda\leqslant 1$, $0<a\leqslant 1$, *and if* $u(r,\theta)$ *belongs to* Lip (a,λ), *so that*

$$\int_{-\pi}^{\pi} |u(r,\theta+h)-u(r,\theta)|^\lambda \, d\theta \leqslant C^\lambda h^{\lambda a}, \tag{2.3.1}$$

then

$$\int_{-\pi}^{\pi} \left|\frac{\partial^2 u}{\partial\theta^2}\right|^\lambda d\theta \leqslant A C^\lambda (1-r)^{-\lambda(1-a)}. \tag{2.3.2}$$

We show first that f' is of finite order. We take any $a>0$. Then we have

$$\int_{-\pi}^{\pi} |u(r,\theta+h)-u(r,\theta)|^\lambda \, d\theta \leqslant C^\lambda h^{\lambda a}(1-r)^{-\lambda a}.$$

Thus, if h is fixed, so that $[f(r,\theta+h)-f(r,\theta)]$ is regular, then

$$\int_{-\pi}^{\pi} |f(r,\theta+h)-f(r,\theta)|^\lambda \, d\theta \leqslant B C^\lambda h^{\lambda a}(1-r)^{-\lambda a}, \tag{2.3.3}$$

where $B=B(\lambda,a)$. Since the constants in (2.3.3) are independent of h, clearly (2.3.3) also holds for varying h.

We let

$$\tfrac{1}{2}\leqslant r<\rho\leqslant \tfrac{1}{2}(1+r)<1, \quad 0<h\leqslant \tfrac{1}{2}(\rho-r).$$

The second mean value theorem gives

$$\frac{\partial}{\partial\theta}u(r,\theta) = \frac{1}{h}[u(r,\theta+h)-u(r,\theta)] - \frac{h}{2}\frac{\partial^2}{\partial\theta^2}u(r,\theta+\xi_1 h), \tag{2.3.4}$$

$$\frac{\partial}{\partial\theta}v(r,\theta) = \frac{1}{h}[v(r,\theta+h)-v(r,\theta)] - \frac{h}{2}\frac{\partial^2}{\partial\theta^2}v(r,\theta+\xi_2 h),$$

where $0\leqslant\xi_1\leqslant 1$, $0\leqslant\xi_2\leqslant 1$. Hence

$$izf' = \frac{\partial u}{\partial\theta}+i\frac{\partial v}{\partial\theta}$$

$$= \frac{1}{h}[f(r,\theta+h)-f(r,\theta)] - \frac{h}{2}\left[\frac{\partial^2}{\partial\theta^2}u(r,\theta+\xi_1 h)+i\frac{\partial^2}{\partial\theta^2}v(r,\theta+\xi_2 h)\right].$$

But, from (2.2.4), we have

$$\int_{-\pi}^{\pi} \left| \frac{\partial^2}{\partial\theta^2} u(r, \theta+\xi_1 h) \right|^\lambda d\theta \leqslant A(\rho-r)^{-\lambda} \int_{-\pi}^{\pi} |f'(\rho, \theta)|^\lambda d\theta,$$

and, similarly,

$$\int_{-\pi}^{\pi} \left| \frac{\partial^2}{\partial\theta^2} v(r, \theta+\xi_2 h) \right|^\lambda d\theta \leqslant A(\rho-r)^{-\lambda} \int_{-\pi}^{\pi} |f'(\rho, \theta)|^\lambda d\theta.$$

Thus, since $|z| \geqslant \frac{1}{2}$,

$$\int_{-\pi}^{\pi} |f'(r, \theta)|^\lambda d\theta \leqslant ABC^\lambda (1-r)^{-\lambda\alpha} h^{-\lambda(1-\alpha)} + Ah^\lambda (\rho-r)^{-\lambda} \int_{-\pi}^{\pi} |f'(\rho, \theta)|^\lambda d\theta,$$

and so, by Lemma 1, f' is of finite order.

Now let $\rho = \frac{1}{2}(1+r)$. Since f' is of finite order and

$$izf' = \frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta},$$

there are constants $b_1 > 1-\alpha$, $B_1 > 4^{\lambda(2-\alpha)}$, such that

$$N(r) \leqslant B_1 C^\lambda (1-r)^{-\lambda b_1}, \tag{2.3.5}$$

where now

$$N(r) = \int_{-\pi}^{\pi} \left| \frac{\partial u}{\partial\theta} \right|^\lambda d\theta. \tag{2.3.6}$$

Suppose that $b_n > 1-\alpha$, $B_n > 4^{\lambda(2-\alpha)}$, and that

$$N(r) \leqslant B_n C^\lambda (1-r)^{-\lambda b_n}. \tag{2.3.7}$$

Then, by the preceding theorem, since

$$izf' = \frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta},$$

and $b_n > 1-\alpha > 0$, we have

$$\int_{-\pi}^{\pi} \left| \frac{d}{dz}(izf') \right|^\lambda d\theta \leqslant AB_n C^\lambda (1-r)^{-\lambda(b_n+1)}.$$

Also, since

$$iz\frac{d}{dz}(izf') = \frac{\partial^2 u}{\partial\theta^2} + i\frac{\partial^2 v}{\partial\theta^2},$$

we have

$$\left| \frac{\partial^2 u}{\partial\theta^2} \right| \leqslant \left| \frac{d}{dz}(izf') \right|.$$

Consequently, if we put $w = \rho e^{i\theta}$, then, by "Max", we have

$$\int_{-\pi}^{\pi} \left| \frac{\partial^2}{\partial\theta^2} u(r, \theta+\xi_1 h) \right|^\lambda d\theta \leqslant \int_{-\pi}^{\pi} \left| \frac{d}{dw} \{iwf'(w)\} \right|^\lambda d\theta$$

$$\leqslant A B_n C^\lambda (1-r)^{-\lambda(b_n+1)},$$

since $\rho = \frac{1}{2}(1+r)$, $0 \leqslant h \leqslant \frac{1}{2}(\rho-r)$.

Now (2.3.4) gives

$$(2.3.8) \qquad N(r) \leqslant C^\lambda h^{-\lambda(1-\alpha)} + Ah^\lambda \int_{-\pi}^{\pi} \left| \frac{\partial^2}{\partial\theta^2} u(r, \theta+\xi_1 h) \right|^\lambda d\theta$$

$$\leqslant C^\lambda h^{-\lambda(1-\alpha)} + A B_n C^\lambda h^\lambda (1-r)^{-\lambda(b_n+1)}$$

$$\leqslant A_1 B_n^{(1-\alpha)/(2-\alpha)} C^\lambda (1-r)^{-\lambda(b_n+1)(1-\alpha)/(2-\alpha)},$$

where we have put

$$h^{\lambda(2-\alpha)} = B_n^{-1}(1-r)^{\lambda(b_n+1)},$$

so that $h \leqslant \frac{1}{2}(\rho-r)$.

We may assume that $A_1 \geqslant 4^\lambda$. Hence, if

$$b_{n+1} = (b_n+1)(1-\alpha)/(2-\alpha), \quad B_{n+1} = A_1 B_n^{(1-\alpha)/(2-\alpha)},$$

for all n, then $\qquad b_n > (1-\alpha), \quad B_n \geqslant 4^{\lambda(2-\alpha)},$

and, further, $\qquad b_n \to (1-\alpha), \quad B_n \to A_1^{2-\alpha}.$

As in the last theorem (by induction), we see that

$$N(r) \leqslant \lim_{n\to\infty} B_n C^\lambda (1-r)^{-\lambda b_n} = AC^\lambda (1-r)^{-\lambda(1-\alpha)}.$$

This completes the proof of the theorem.

2.4. We are now able to give the following theorems.

THEOREM 8. *A necessary and sufficient condition that $f(z)$ should belong to* Lip (α, λ), *where* $0 < \lambda \leqslant 1$, $0 < \alpha \leqslant 1$, *is that* $M_\lambda(f') = O\{(1-r)^{\alpha-1}\}$.

Theorem 5 shows that the condition is sufficient. If now $f(z)$ belongs to Lip (α, λ), then so do $u(r, \theta)$ and $v(r, \theta)$. Consequently, Theorem 7 gives, for $|z| \geqslant \frac{1}{2}$,

$$M_\lambda(f') \leqslant AM_\lambda\left(\frac{\partial u}{\partial\theta}\right) + AM_\lambda\left(\frac{\partial v}{\partial\theta}\right) = O\{(1-r)^{\alpha-1}\},$$

which shows that the condition is necessary.

THEOREM 9. *If $f(z)$ belongs to* Lip (α, λ), *where* $0<\lambda\leqslant 1$, $0<\alpha\leqslant 1$, *then $f(z)$ belongs to L^λ.*

For it is well known that, if $M_\lambda(f')=O\{(1-r)^{-a}\}$, where $a<1$, then $f(z)$ belongs to L^λ.

THEOREM 10. *If $0<\lambda\leqslant 1$, $0<\alpha<1$, and if u belongs to* Lip(α, λ), *then so does f.*

For
$$M_\lambda\left(\frac{\partial u}{\partial\theta}\right)=O\{(1-r)^{\alpha-1}\},$$
by Theorem 7, and so, by Theorems 6 and 3, we get
$$M_\lambda(f')=O\{(1-r)^{\alpha-1}\}.$$
Here we must have $\alpha<1$. This gives the required result.

The theorem is false when $\alpha=1$. We show this in the case $\lambda=1$. Similar examples can be constructed for other values of λ.

We let
$$f(z)=-i\log(1-z), \quad f(0)=0.$$
Here f does not belong to Lip (1, 1). For
$$f'=i(1-z)^{-1},$$
and this does not belong to L. But u does belong to Lip (1, 1). For
$$-\pi\leqslant u(r,\theta)=\tan^{-1}[r\sin\theta/\{1-r\cos\theta\}]\leqslant\pi.$$
We notice that $[u(r,\theta+h)-u(r,\theta)]$ is a continuous function of θ, which has only a finite number of zeros. This number is bounded for all r and h. Hence, θ_ρ denoting a typical zero, we have
$$\begin{aligned}\int_{-\pi}^{\pi}|u(r,\theta+h)-u(r,\theta)|\,d\theta&=\sum_l\left|\int_{\theta_l}^{\theta_{l+1}}[u(r,\theta+h)-u(r,\theta)]\,d\theta\right|\\&=\sum_l\left|\int_{(\theta_{l+1}-h)}^{\theta_{l+1}}u(r,\theta)\,d\theta-\int_{(\theta_l-h)}^{\theta_l}u(r,\theta)\,d\theta\right|\\&\leqslant 2\sum_l\int_{\theta_l-h}^{\theta_l+h}|u(r,\theta)|\,d\theta\leqslant Ah,\end{aligned}$$
since $|u(r,\theta)|\leqslant\pi$, and l is bounded.

THEOREM 11. *If $0<\lambda\leqslant 1$, $0<\alpha\leqslant 1$, if $f\in$* Lip (α, λ), *and if*
$$\lambda<\lambda'<\lambda/(1-a\lambda)=\lambda'(1-a'\lambda'),$$
then $f\in$ Lip (α', λ').

THEOREM 12. *If* $0 < \lambda \leqslant 1$, $0 < \alpha \leqslant 1$, $0 < \alpha + \beta < 1$, *and if* $f \in \mathrm{Lip}(\alpha, \lambda)$, *then* $f_\beta \in \mathrm{Lip}(\alpha+\beta, \lambda)$.

As Hardy and Littlewood show in their papers **1** and **3**, these theorems, and other similar ones, follow from the theorems which we have already proved, and theorems already known to be true for $0 < \lambda \leqslant 1$. Thus we need go no further into detail.

Memoirs referred to.

1. G. H. Hardy and J. E. Littlewood, "A convergence criterion for Fourier series", *Math. Zeitschrift*, 28 (1928), 612–634.
2. ————, "Some properties of fractional integrals, (1)", *Math. Zeitschrift*, 27 (1928), 565–606.
3. ————, "Some properties of fractional integrals, (2)", *Math. Zeitschrift*, 34 (1931), 403–439.
4. ————, "Some properties of conjugate functions", *Journal für Math.*, 167 (1931), 405–423.
5. ————, "A maximal theorem with function-theoretic applications", *Acta Math.*, 54 (1930), 81–116.

Trinity College,
Cambridge.

SOME FORMULAE FOR PRIMALS IN FOUR DIMENSIONS (III)

By L. ROTH.

[Received 7 January, 1935.—Read 17 January, 1935.]

In a recent memoir B. Segre* has applied the concept of series of equivalence, developed by Severi†, to the discussion of a non-singular algebraic variety V_3. The numerous results obtained connect the covariants of linear systems of surfaces on V_3 with the characters of the (virtual) canonical system; and throughout his investigation Segre assumes the relation‡

$$2P_a = \Omega_0 - \Omega_1 + \Omega_2 - 4 \tag{1}$$

between the arithmetic genus of V_3 and the (virtual) grade, linear genus and arithmetical genus of its canonical system. This relation has been established§ by elementary methods and by the use of Todd's formula‖ for P_a. The object of the present note is to show how certain of Segre's results can be used to obtain (1) and to calculate Ω_0, Ω_1, Ω_2 more rapidly than is possible by the method previously given.

1. If $\mathcal{C}$ is any curve on V_3, we denote by $\kappa_{\mathcal{C}}$ a set of its canonical series, and by $\chi_{\mathcal{C}}$ the set of points in which it meets a (virtual) canonical surface X: the *numbers* of such points are denoted by the corresponding symbols enclosed in square brackets, so that $[\chi_{\mathcal{C}}]$ is the "canonical number" or immersion character of $\mathcal{C}$.

---

* B. Segre, 1.
† Severi, 1.
‡ Severi, 2.
§ Roth, 1.
‖ Todd, 1.

If $|S|$ is a linear system of surfaces on V_3, $\mathcal{C}$ a characteristic curve of $|S|$, and γ a characteristic set [or, symbolically, $\mathcal{C} \equiv (S^2)$, $\gamma \equiv (S^3)$], then*

$$\kappa_{\mathcal{C}} - \chi_{\mathcal{C}} = 2\gamma. \tag{2}$$

If this formula is applied to the system $|X|$, for which

$$[\kappa_{\mathcal{C}}] = 2\Omega_1 - 2, \quad [\chi_{\mathcal{C}}] = [\gamma] = \Omega_0,$$

we obtain the relation†

$$2\Omega_1 - 2 = 3\Omega_0. \tag{3}$$

An important covariant series of equivalence of $|S|$ is that containing the sets $(S\mathcal{X})$, where $\mathcal{X} \equiv (X^2)$ is a characteristic curve of $|X|$. If ϕ_s denotes the "canonical series" of S, of order $\omega_s - 1$ (where ω_s is the arithmetical linear genus of S), we have‡

$$(S\mathcal{X}) \equiv \phi_s + 3\gamma - 2\kappa_{\mathcal{C}}. \tag{4}$$

Applying this result to the canonical system and using (3), we obtain

$$\omega_X = 4\Omega_0 + 1. \tag{5}$$

2. Suppose now that $|S|$ is a net of surfaces on V_3. If ψ_s denotes a set of the "Severi series" of S, and $\delta_{\mathcal{C}}$ a Jacobian set of a pencil of characteristic curves, it is well known that

$$\psi_s \equiv \delta_{\mathcal{C}} - \gamma - 2\kappa_{\mathcal{C}}. \tag{6}$$

For the canonical system $[\psi_X] = 12(\Omega_2 + 1) - 4\Omega_0$, by Noether's relation.

Another series of equivalence of V_3 is defined by

$$\eta \equiv \psi_X - 2(X^3). \tag{7}$$

The number $[\eta]$ is an absolute invariant of V_3; in fact,

$$[\eta] = 12(\Omega_2 - \Omega_1 + \Omega_0 + 2). \tag{8}$$

If $\mathcal{J}$ denotes the Jacobian curve of $|S|$, we have the important result§

$$\eta \equiv \chi_{\mathcal{J}} + 3\gamma - 3\phi_s. \tag{9}$$

---

* Segre, 1, 492.
† Noether, 1.
‡ Segre, 1, 503.
§ Segre, 1, 513.

From equations (8) and (9) we may establish (1). For if $|S|$ is a net of prime sections of a V_3 of S_4 having ordinary singularities, $\mathfrak{f}$ is the curve of contact of tangent primes passing through the base-line l of the net. In the notation previously employed*, $[\gamma]=n$, $[\phi_s]=j+m-6a+9n$, and $\mathfrak{f}$ is of order m. To calculate $[\chi_{\mathfrak{f}}]$ we observe that $\mathfrak{f}$ meets the double surface B of V_3 in the X points of contact of tangent primes through l which touch V_3 in points of B, and in ζ points of the pinch-curve C_j. If C_j has classes j_1 and j_0 of immersion in V_3 and B respectively, then†

$$\zeta = 5j_1-2j_0-j. \tag{10}$$

The class w of V_3 is given by

$$w = m(n-1)-X-2\zeta. \tag{11}$$

And since $[\chi_{\mathfrak{f}}]=m(n-5)-X-\zeta$, it follows that

$$[\chi_{\mathfrak{f}}]=w+\zeta-4m. \tag{12}$$

Substituting in (9), we have

$$[\eta]=w+18a-24n-7m-4j+5j_1-2j_0. \tag{13}$$

Now Todd‡ has shown that this expression represents $24(P_a-1)$. Hence, using (8), we derive (1).

3. If $|S|$ is a linear system ∞^4 on V_3, without base-points, its image is a primal V_3' of S_4 having ordinary singularities. In particular, the pinch-curve C_j of V_3' is represented by the ∞^1 points, each of which is a double base-point for a pencil of surfaces contained in $|S|$. In the notation of § 2, Segre proves§ that *the pinch-curve of* V_3' *has order* $\omega-i+4a-6n-5$, *genus* $\frac{1}{2}I+\frac{3}{2}\Omega_0-48P_a+10\omega-5i+\frac{25}{2}a-25n+16$, *and canonical number* $\Omega_0-24P_a+5\omega-5n+19$.

In these expressions I, i are the Zeuthen-Segre invariants of V_3' and its prime sections, *i.e.*

$$I = w-2i-a+2n-2, \tag{14}$$

$$i = m-2a+3n-4. \tag{15}$$

---

* Roth, 2.
† Roth, 3.
‡ Todd, 1.
§ Segre, 1, 551.

The first of these results serves merely to verify the values of ω and i, since it reduces to j; but the second or third gives a method for calculating Ω_0. For it is known* that C_j is of genus $2j_1-\frac{1}{2}j_0-\frac{3}{2}j+1$, whence, using (1), (13), (14), and (15), we obtain

$$\Omega_0 = 48a-12m-64n+w-13j+8j_1-3j_0, \tag{16}$$

in agreement with the result already given elsewhere†. Hence, from (1) and (3), we have

$$12(\Omega_2+1) = 306a-79m-408n+7w-82j+53j_1-20j_0. \tag{17}$$

4. As a simple application of the formulae for primals in S_4, we calculate the characters of the primal V_3^n whose double surface B is the complete intersection of two non-singular primals of orders α and β respectively‡. The order, rank, class, and *ceto* of B are at once found by the Severi intersection formulae to be

$$\mu_0 = \alpha\beta, \quad \mu_1 = \alpha\beta(\alpha+\beta-2), \quad \mu_2 = \alpha\beta\{(\alpha-1)^2+(\alpha-1)(\beta-1)+(\beta-1)^2\},$$

$$\nu_2 = \alpha\beta(\alpha-1)(\beta-1).$$

The rank of V_3^n is thus

$$a = n(n-1)-2\alpha\beta. \tag{18}$$

To find its class m and the order j of the pinch-curve C_j, we use the fact that V_3^n has no triple curve: the corresponding condition§,

$$\binom{n}{3}-\tfrac{1}{2}an+\tfrac{1}{3}(2a+2j+m) = 0,$$

with the formula $$\mu_1 = a(n-1)-\tfrac{5}{2}j-m,$$

gives the values $$j = 2\alpha\beta(n-\alpha-\beta), \tag{19}$$

$$m = n(n-1)^2-7n\alpha\beta+4\alpha\beta(\alpha+\beta+1). \tag{20}$$

The class w of V_3^n is given by‖

$$w = n(n-1)^3-\mu_0(17n^2-63n+64-\mu_0)+\mu_1(21n-46)-8\mu_2-10\nu_2.$$

---

* Roth, 2.
† Roth, 1.
‡ This case is worked out from first principles, somewhat laboriously, by Segre.
§ Roth, 2.
‖ Roth, 3.

Substituting in this formula, we obtain

$$w = n(n-1)^3 - 17n^2\alpha\beta + 21n\alpha\beta(\alpha+\beta+1)$$
$$-2\alpha\beta\{4(\alpha^2+\beta^2)+7\alpha\beta+6(\alpha+\beta)+3\}. \quad (21)$$

The classes j_1 and j_0 of immersion of C_j are given by*

$$j_1 = \tfrac{1}{2}j(n-2) = \alpha\beta(n-2)(n-\alpha-\beta), \quad (22)$$

$$j_0 = 2\mu_1(n-2) - 2(\mu_2+\nu_2) + 2\alpha\beta(n-\alpha-\beta)(\alpha+\beta-2). \quad (23)$$

Substituting these characters in (13), (16), and (17), we obtain

$$24(P_a-1) = n^4 - 10n^3 + 35n^2 - 50n - 2\alpha\beta\{6n^2 - 6n(\alpha+\beta+5)$$
$$+2(\alpha^2+\beta^2)+3\alpha\beta+15(\alpha+\beta)+35\}, \quad (24)$$

$$\Omega_0 = n^4 - 15n^3 + 75n^2 - 125n - \alpha\beta\{9n^2 - 7n(\alpha+\beta)$$
$$-75n+2(\alpha^2+\beta^2)+2\alpha\beta+30(\alpha+\beta)+150\}, \quad (25)$$

$$12(\Omega_2+1) = 7n^4 - 100n^3 + 485n^2 - 800n - \alpha\beta\{66n^2 - 54n(\alpha+\beta)$$
$$-510n+16(\alpha^2+\beta^2)+18\alpha\beta+210(\alpha+\beta)+970\}. \quad (26)$$

References.

Noether, 1, *Math. Annalen*, 8 (1875), 495.
Roth, 1, *Proc. London Math. Soc.* (2), 40 (1935), 217.
——, 2, *Ibid.* (2), 35 (1933), 540.
——, 3, *Ibid.* (2), 39 (1935), 334.
B. Segre, 1, *Mem. Reale Acc. d'Italia*, 5 (1934), 479.
Severi, 1, *Comm. Math. Helvetici*, 4 (1932), 268.
——, 2, *Rend. di Palermo*, 28 (1909), 33.
Todd, 1, *Journal London Math. Soc.*, 9 (1934), 206.

Imperial College of Science,
London.

* Roth, 2.

ON INDUCED AND COMPOUND MATRICES

By D. E. LITTLEWOOD.

[Received 16 January, 1935.—Read 17 January, 1935.]

1. In a recent paper Aitken* has described methods for finding the classical canonical form of the compound and induced matrices of a given matrix of which the canonical form is known, for the case of multiple roots of the characteristic equation. He also considers the case of the direct product of two matrices.

It is the purpose of this note to simplify Aitken's results, and to extend them to a more general type of invariant matrix. A generating function is obtained, from the coefficients of which the classical canonical form of any such matrix may readily be deduced.

2. *Schur's invariant matrices.*

Let $A = [a_{st}]$ be the matrix of transformation on n variables $x_1, x_2, \ldots, x_n$; *i.e.*,

$$Y = AX,$$

where $X = [x_s]$ and $Y = [y_s]$ are column vectors. The matrix of transformation on the $\binom{n+m-1}{m}$ homogeneous products of the x's of degree m is the *m-th induced (Schäflian) matrix* $A^{[m]}$ of the matrix A.

Again, if m vectors $X_1, X_2, \ldots, X_m$, where $X_r = [x_{sr}]$, are transformed by the same matrix A, so that

$$Y_r = [y_{sr}] = AX_r,$$

there are $\binom{n}{m}$ determinants of the form $|x_{e_s t}|$, where t runs from 1 to m, and $e_1, \ldots, e_m$ represent any m of the numbers 1, 2, ..., n in natural order,

* A. C. Aitken, *Proc. London Math. Soc.* (2), 38 (1934), 354–376.

which are transformed amongst themselves, the matrix of transformation being the *m-th compound matrix* $A^{(m)}$

Induced and compound matrices satisfy the equations*

$$[BA]^{[m]} = [B]^{[m]}[A]^{[m]},$$

$$[BA]^{(m)} = [B]^{(m)}[A]^{(m)},$$

and are special cases of invariant matrix forms which have been studied by Schur†.

Let $T(A)$ be a matrix whose elements are polynomials in the elements of a given matrix A. Then, if

$$T(AB) = T(A)\,T(B), \tag{1}$$

$T(A)$ is called an invariant matrix form, or simply invariant matrix of the matrix A.

Schur shows that an irreducible invariant matrix, *i.e.* an invariant matrix which is not expressible by transformation as the direct sum of two invariant matrices, is homogeneous in the elements of A. He shows that an irreducible invariant matrix of degree m is equivalent to, *i.e.* may be transformed into, one of a finite number of invariant matrices which correspond to the characters of the symmetric group of order $m!$. As for representations of a group, these distinct irreducible invariant matrices are specified by their spurs, which are invariant for transformations.

The spurs of the irreducible invariant matrices of a given matrix A are the S-functions‡ of the roots of the characteristic equation of A.

The spur of the m-th induced matrix $A^{[m]}$ is $\{m\}$ or h_m, and the spur of the m-th compound matrix $A^{(m)}$ is $\{1^m\}$ or a_m§. We write $A^{[\lambda_1, \ldots, \lambda_p]}$ for the invariant matrix with spur $\{\lambda_1, \ldots, \lambda_p\}$, and the m-th compound matrix will henceforth be written $A^{[1^m]}$.

As an example, if $X = [x_s]$ and $X' = [x_s']$ are both transformed by a matrix A, then the $\frac{1}{3}m(m^2-1)$ linearly independent expressions of the form

$$2x_1x_2x_3' - x_2x_3x_1' - x_3x_1x_2'$$

or

$$x_1x_2x_1' - x_1^2x_2'$$

are transformed by the matrix $A^{[21]}$.

---

* A. C. Aitken, *loc. cit.*

† Schur, "Ueber einer Klasse von Matrizen", *Diss. Berlin* (1901).

‡ See D. E. Littlewood and A. R. Richardson, *Phil. Trans. Royal Soc.* (A), 233 (1934), 107–115; for definition, and Schur, *loc. cit.*

§ Schur, *loc. cit.*

Given a specific matrix A, of which the roots of the characteristic equation are distinct, the classical canonical form of $A^{[\lambda_1, \ldots, \lambda_p]}$ may be found as follows*.

Let $u_1, u_2, \ldots, u_n$ be the roots of the characteristic equation of A. Express $\{\lambda_1, \ldots, \lambda_p\}$ in terms of the roots

$$\{\lambda_1, \ldots, \lambda_p\} = \Sigma b_{\alpha\beta\gamma\ldots} u_1^{\alpha} u_2^{\beta} u_3^{\gamma} \ldots .$$

Then, corresponding to the term $b_{\alpha\beta\gamma\ldots} u_1^{\alpha} u_2^{\beta} u_3^{\gamma} \ldots$, there is an element $u_1^{\alpha} u_2^{\beta} u_3^{\gamma} \ldots$ repeated $b_{\alpha\beta\gamma\ldots}$ times in the leading diagonal*.

The number of rows or columns in the matrix $A^{[\lambda_1, \ldots, \lambda_p]}$ is the spur of the unit matrix, and hence the S-function $\{\lambda\}$ of the roots of $(1-x)^n = 0$. This is equal to $\chi_0^{(\lambda)}$ times the product of the first λ_i terms from each i-th row of the set of numbers†

$$\begin{array}{lllll} n, & n+1, & n+2, & n+3, & \ldots \\ n-1, & n, & n+1, & n+2, & \ldots \\ n-2, & n-1, & \ldots & & \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

The case of multiple roots of the characteristic equation of A, with non-diagonal terms in the classical canonical form, presents further difficulties, and this is the problem for which Aitken gives the method of solution.

3. *Compound matrices.*

We shall, for the rest of the paper, use the following nomenclature. Since we are dealing strictly with the classical canonical form of a matrix, we shall assume every matrix to be reduced to this form.

We shall use $\dot{+}$ to denote the direct sum of two matrices, *i.e.*,

$$A \dot{+} B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

$\dot{-}$ for a corresponding difference, and $\mathbf{X}$ to denote the direct product of two matrices. Thus, if $A = [a_{st}]$ and $B = [b_{st}]$ are matrices of order n^2 and m^2 respectively, $A \mathbf{X} B$ denotes the matrix of order $n^2 m^2$ with elements

* Schur, *loc. cit.*

† D. E. Littlewood and A. R. Richardson, *Quart. J. of Math.* (Oxford), 5 (1934), 269.

$a_{st}b_{s't'}$, the two suffixes s and s' defining the rows, and t and t' the columns. Scalar multiplication by n denotes the direct sum of the same matrix repeated n times.

The matrix sometimes denoted by $C_n(\mu)$, namely with n terms in the leading diagonal each equal to μ, and $n-1$ terms each equal to unity in the first diagonal above, will be denoted by $[n]_\mu$.

Hence, *e.g.*, $[3]_\mu \dot{+} [2]_0 \dot{+} [1]_\nu$ will denote the matrix

$$\begin{bmatrix} \mu, & 1, & & & & \\ & \mu, & 1, & & & \\ & & \mu, & & & \\ & & & 0, & 1, & \\ & & & & 0, & \\ & & & & & \nu \end{bmatrix},$$

Taking account of canonical form only, addition and multiplication are clearly commutative, and multiplication is distributive with respect to addition.

Aitken* gives a rule for the canonical form of the direct product of two matrices which may be expresed as follows.

THEOREM I.

$$[n]_\mu \mathbf{X} [m]_\nu = [n+m-1]_{\mu\nu} \dot{+} [n+m-3]_{\mu\nu} \dot{+} [n+m-5]_{\mu\nu} \dot{+} \ldots \dot{+} [n-m+1]_{\mu\nu},$$

for $\mu \neq 0$, $\nu \neq 0$, $n \geqslant m$.

For the case of zero roots of the characteristic equation, Aitken's rule may be expressed by the following theorems.

THEOREM II.

$$[n]_0 \mathbf{X} [m]_0 = 2[1]_0 \dot{+} 2[2]_0 \dot{+} \ldots \dot{+} 2[m-1]_0 \dot{+} (n-m+1)[m]_0,$$

for $n \geqslant m$.

THEOREM III.

$$[n]_\mu \mathbf{X} [m]_0 = n[m]_0,$$

for $\mu \neq 0$.

* See also W. E. Roth, *Bull. American Math. Soc.*, 40 (1934), 461–468.

It should be noted that square brackets [ ] denote a matrix; round brackets ( ) denote a scalar multiplier.

Our next step is to simplify Aitken's result for the m-th compound matrix $A^{[1^m]}$. We take first $A = [n]_\mu$.

The rows and columns of the compound matrix are defined by the sets of m numbers $(p_1, p_2, \ldots, p_m)$ for which $1 \leqslant p_1 < \ldots < p_m \leqslant n$. Aitken proves that the degree r_1 of the greatest submatrix $[r_1]_{\mu^m}$ is equal to the length of the longest chain that can be obtained as follows. Each set $(p_1, \ldots, p_m)$ is linked to another set in which any one of the p's is increased by unity. The first set in the chain is clearly $(1, 2, \ldots, m)$, and the last $(n-m+1, n-m+2, \ldots, n)$. When the sets of the first chain are removed, the length of the greatest chain that can be made from the remaining sets gives the degree of the second largest submatrix, and so on. The greatest length of chain for all possible choices of earlier chains is taken in each case.

Each set $(p_1, \ldots, p_m)$ we now associate with the partition

$$(p_m - m, p_{m-1} - m + 1, \ldots, p_1 - 1).$$

The sets are thus in one-one correspondence with all partitions $(\lambda_1, \ldots, \lambda_p)$ such that the number of parts is less than or equal to m and the magnitude of each part less than or equal to $n-m$, including the partition (0). The chains are obtained by adding unity to any of the parts.

The least partition is (0) and the greatest $[(n-m)^m]$. Hence the length of the greatest chain is $m(n-m)+1$. This chain uses up one partition of every number from 0 to $m(n-m)$.

The next chain begins with a partition of the least number of which there are at least two partitions. If the beginning of the chain is known, the end of the chain may be determined, since for each partition $(\lambda_1, \ldots, \lambda_m)$ of r into not more than m parts each less than or equal to $n-m$, there is a corresponding partition $(n-m-\lambda_m, \ldots, n-m-\lambda_1)$ of $m(n-m)-r$. Hence, if r_2 is the least number of which there are two partitions into not more than m parts each less than or equal to $n-m$, the maximum length of the second chain is $m(n-m)+1-2r_2$.

Similarly the maximum length of the i-th chain is $m(n-m)+1-2r_i$, where r_i is the least number of which there are at least i partitions into not more than m parts each less than or equal to $n-m$.

This gives the maximum possible length for the chains. There is no guarantee that these lengths are attained, since, although there may remain partitions of two consecutive numbers, these may not link according to the rule, *e.g.*, (3) and (2^2) do not link. We shall assume for the present that these lengths may be attained by a suitable choice of chains. This assumption will be justified in the last section of the paper.

We thus arrive at

THEOREM IV.

$$[n]_{\mu}^{[1^m]} = [m(n-m)+1]_{\mu^m} \dot{+} c_1[m(n-m)-1]_{\mu^m} \dot{+} \ldots \dot{+} c_r[m(n-m)-2r+1]_{\mu^m} \dot{+} \ldots,$$

where c_r is the number of partitions of r minus the number of partitions of $(r-1)$ into not more than m parts each less than or equal to $n-m$, for $\mu \neq 0$.

The case of a singular matrix, $\mu = 0$, is simpler. Aitken's criterion for the formation of chains is as follows: the set $(p_1, \ldots, p_m)$ is linked to $(p_1+1, \ldots, p_m+1)$. Hence any partition is linked to the partition in which each of the m parts is increased by unity. It is convenient to deal with the conjugate partition. The conjugate partition $(\lambda_1, \ldots, \lambda_p)$ is linked to $(m, \lambda_1, \ldots, \lambda_p)$. The first partition of a chain must clearly have $\lambda_1 < m$, and the last have $n-m$ parts. The length of the chain is $n-m+1$ minus the number of parts in the first partition. Hence we have

THEOREM V.

$$[n]_0^{[1^m]} = [n-m+1]_0 \dot{+} c_1[n-m]_0 \dot{+} c_2[n-m-1]_0 \dot{+} \ldots \dot{+} c_{n-m}[1]_0,$$

where c_1 is the number of partitions into i parts each less than or equal to $m-1$.

To complete the solution of the problem for all compound matrices we need only

THEOREM VI.

$$[A \dot{+} B]^{[1^m]} = [A]^{[1^m]} \dot{+} [B]^{[1^m]} \dot{+} \sum_{r=1}^{m-1} [A]^{[1^r]} \times [B]^{[1^{m-r}]}.$$

Aitken proves this theorem by considering matrices in canonical form. The proof is almost intuitive for any matrices. If A is the matrix of transformation on $x_{1r}, x_{2r}, \ldots, x_{nr}$, and B on $x_{n+1,r}, \ldots, x_{n+n',r}$, for $1 \leqslant r \leqslant m$, then the matrix of transformation on the determinants $|x_{e_s t}|$ splits up into the direct sum of the matrices of transformation on determinants with assigned degrees in the first n and the last n' x's respectively. The theorem follows readily.

Example. Find the fourth compound matrix of $[4]_{\mu} \dot{+} [3]_0 \dot{+} [1]_{\nu}$.

From Theorems IV and V,

$$[4]_{\mu}^{[1^2]} = [5]_{\mu^2} \dot{+} [1]_{\mu^2}, \quad [4]_{\mu}^{[1^3]} = [4]_{\mu^3}, \quad [4]_{\mu}^{[1^4]} = [1]_{\mu^4},$$

$$[3]_0^{[1^2]} = [2]_0 \dot{+} [1]_0, \quad [3]_0^{[1^3]} = [1]_0.$$

Hence

$$\{[4]_\mu \dot{+} [3]_0 \dot{+} [1]_\nu\}^{[14]}$$

$$= [1]_{\mu^4} \dot{+} [4]_{\mu^3} \{[3]_0 \dot{+} [1]_\nu\}$$

$$+ \{[5]_{\mu^2} \dot{+} [1]_{\mu^3}\} \times \{[2]_0 \dot{+} [1]_0\} \dot{+} \{[5]_{\mu^2} \dot{+} [1]_{\mu^2}\} \times [3]_0 \times [1]_\nu$$

$$+ [4]_\mu \times [1]_0 \dot{+} [4]_\mu \times \{[2]_0 \dot{+} [1]_0\} \times [1] \dot{+} [1]_0 \times [1]_\nu$$

$$= [1]_{\mu^4} \dot{+} [4]_{\mu^3 \nu} \dot{+} 10[3]_0 \dot{+} 10[2]_0 \dot{+} 15[1]_0.$$

4. *Induced and other invariant matrices.*

The case of all other invariant matrices now presents no new difficulty. Since the spur of the direct product of two matrices is the product of the spurs, and the spur of a direct sum is the sum of the spurs, there is an isomorphism between invariant matrices and the corresponding S-functions. Thus, since

$$h_1^2 = h_2 + a_2,$$

the direct product of a matrix with itself is reducible by transformation into the direct sum of the second induced and the second compound matrices. Theorems I to VI, together with the rule for expressing any S-function as a determinant in the functions a_r, suffice for all cases.

For example, since

$$\{3, 1\} = \begin{vmatrix} a_2, & a_3, & a_4 \\ 1, & a_1, & a_2 \\ & 1, & a_1 \end{vmatrix} = a_2 a_1^2 + a_4 - a_2^2 - a_1 a_3,$$

we have

$$[A]^{[31]} = A^{[12]} \times A \times A \dot{+} A^{[14]} \dot{-} A^{[12]} \times A^{[12]} \dot{-} A \times A^{[13]}.$$

Putting $A = [5]$, substituting and evaluating by the above theorems, we obtain

$$[5]_\mu^{[31]} = [15]_{\mu^4} \dot{+} [13]_{\mu^4} \dot{+} 2[11]_{\mu^4} \dot{+} 2[9]_{\mu^4} \dot{+} 3[7]_{\mu^4} \dot{+} 2[5]_{\mu^4} \dot{+} 2[3]_{\mu^4},$$

for $\mu \neq 0$.

The case of induced matrices could be solved in this manner, but it is simpler to start afresh from Aitken's rules.

The rows and columns of the induced matrix are defined by sets $(p_1, p_2, \ldots, p_m)$ in which repetitions are allowed. Chains are obtained in the same way by increasing one of the p's by unity. Assuming that $p_1 \geqslant p_2 \geqslant \ldots \geqslant p_m$, the corresponding partition is

$$(p_1-1,\ p_2-1,\ \ldots,\ p_m-1),$$

and the greatest magnitude for each part is $n-1$. Hence

THEOREM VII.

$$[n]_\mu^{[1^m]} = [n-m+1]_\mu^{[m]}.$$

This theorem also holds for $\mu = 0$. It is true independently of the assumption, made for Theorem IV, that the maximum lengths for the chains are attained. With the assumption we obtain

THEOREM VIII.

$$[n]_\mu^{[m]} = [m(n-1)+1]_{\mu^m} \dot{+} c_1[m(n-1)-1]_{\mu^m} \dot{+} \ldots \dot{+} c_r[m(n-1)-2r+1]_{\mu^m} \dot{+} \ldots,$$

where c_r is the number of partitions of r minus the number of partitions of $r-1$ into not more than m parts each less than or equal to $n-1$, for $\mu \neq 0$.

THEOREM IX.

$$[n]_0^{[m]} = [n]_0 \dot{+} c_1[n-1]_0 \dot{+} c_2[n-2]_0 \dot{+} \ldots \dot{+} c_{n-1}[1]_0,$$

where c_r is the number of partitions into r parts each less than or equal to $m-1$.

5. *A generating function.*

The above results may be combined in the form of a generating function whose coefficients exhibit the canonical form of any invariant matrix.

The coefficient of $\rho^n x^m$ in

$$f(x) = 1/\{(1-x)(1-\rho x)\ldots(1-\rho^p x)\}$$

is clearly the number of partitions of n into not more than m parts each less than or equal to p. Denote the product $(1-\rho)(1-\rho^2)\ldots(1-\rho^r)$ by $[r]!$. The coefficient of x^m in $f(x)$ is $[p+m]!/[p]!\,[m]!$. Hence the number of partitions of n into not more than m parts each less than or equal to p is equal to the coefficient of ρ^n in $[p+m]!/[p]!\,[m]!$.

First correspondence rule.

A polynomial in ρ,

$$\phi(\rho) = c_0 + c_1\rho + c_2\rho^2 + \ldots + c_k\rho^k,$$

of reciprocal form, *i.e.*, such that $c_i = c_{k-i}$, in which the positive integral coefficients c_i are non-decreasing up to the middle term or terms, is associated with a matrix

$$c_0[k+1] \dot{+} (c_1 - c_0)[k-1] \dot{+} (c_2 - c_1)[k-3] \dot{+} (c_3 - c_2)[k-5] \dot{+} \ldots.$$

If $\phi(\rho)$ is the coefficient of a monomial expression in the roots of the characteristic equation of a matrix A, this monomial expression is in each case the leading diagonal term in the matrix.

THEOREM X. *If* $A = \Sigma[n_r]_{\mu_r}$, $\mu_r \neq 0$, *let* $f(x) = \Pi f_r(x)$, *where*

$$f_r(x) = 1/\{(1-\mu_r x)(1-\mu_r\rho x)\ldots(1-\mu_r\rho^{n_r-1}x)\},$$

and let $\{\lambda_1, \ldots, \lambda_p\}$ *be any S-function associated with the series* $f(x) = 1 + \Sigma h_r x^r$. *Then, if* $\{\lambda_1, \ldots, \lambda_p\} = \Sigma\phi_{\alpha\beta\gamma\ldots}\mu_1^{\alpha}\mu_2^{\beta}\ldots$, *the classical canonical form of the matrix* $A^{[\lambda_1,\ldots,\lambda_p]}$ *is the direct sum of the matrices associated with the coefficient* $\phi_{\alpha\beta\gamma\ldots}(\rho)$ *according to the first correspondence rule.*

We first prove the theorem for the induced and compound matrices of $[n]_\mu$. The corresponding function is

$$f(x) = 1/\{(1-\mu x)(1-\mu\rho x)\ldots(1-\mu\rho^{n-1}x)\}.$$

For this series

$$h_m = \frac{\mu^m[n+m-1]!}{[m]!\,[n-1]!}, \quad a_m = \frac{\mu^m[n]!}{[m]![n-m]!},$$

in which the coefficient of $\mu^m\rho^p$ is equal to the number of partitions of p into not more than m parts each less than or equal to $n-1$ in the case of h_m, and less than or equal to $n-m$ in the case of a_m. Theorems VIII and IV complete the proof for these cases.

To complete the proof in the general case it is only necessary to establish an isomorphism between the functions $\phi(\rho)$ and the corresponding matrices in regard to addition and multiplication.

The isomorphism is obvious in the case of addition provided that the two functions of ρ that are added have the same mean degree. The mean degree of each symmetric function of the roots which goes to form $\phi_{\alpha\beta\gamma\ldots}(\rho)$ is clearly $\frac{1}{2}\alpha(n_1+1)+\frac{1}{2}\beta(n_2+1)+\frac{1}{2}\gamma(n_3+1)+\ldots$, and the condition is

satisfied. It should be noted, however, that roots of the characteristic equation of A which correspond to different elementary divisors must be kept symbolically distinct until the calculation is complete, or terms of different mean degree may be added and an incorrect result obtained.

For multiplication, it is clear that the product of $1+\rho+\rho^2+\ldots+\rho^{n-1}$ by $1+\rho+\ldots+\rho^{m-1}$ is in correspondence with the product $[n]\mathbf{X}[m]$. Hence the isomorphism and the theorem are established.

Singular submatrices require different treatment. We introduce a new series of symbols $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$, which are commutative and satisfy

$$\epsilon_i \epsilon_j = \epsilon_i, \quad (i \leqslant j), \quad \rho\epsilon_i = \epsilon_i, \quad \mu_r \epsilon_i = \epsilon_i. \tag{2}$$

Second correspondence rule.

The polynomial $\psi(\epsilon) = \Sigma c_i \epsilon_i$, in which the positive integral coefficients c_i satisfy $c_i \geqslant c_{i+1}$, is associated with the matrix

$$\Sigma(c_i - c_{i+1})[i]_0.$$

THEOREM XI. *If* $A = \Sigma[n_r]_{\mu_r} + \Sigma[m_r]_0$, $\mu_r \neq 0$, *and* $\{\lambda_1, \ldots, \lambda_p\}$ *is an* S*-function associated with* $f(x) = \Pi f_r(x) \Pi g_r(x)$, *where* $f_r(x)$ *is defined as in Theorem X, and* $g_r(x) = 1/\{(1-\epsilon_1 x)(1-\epsilon_2 x)\ldots(1-\epsilon_{m_r} x)\}$, *then, if* $\{\lambda_1, \ldots, \lambda_p\} = \Sigma \phi_{\alpha\beta\gamma\ldots}(\rho)\mu_1^{\alpha}\mu_2^{\beta}\ldots + \psi(\epsilon)$, *the classical canonical form of* $A^{[\lambda_1, \ldots, \lambda_p]}$ *is the direct sum of the matrices associated with* $\phi_{\alpha\beta\gamma\ldots}(\rho)$ *and* $\psi(\epsilon)$ *according to the first and second correspondence rules.*

The generating function which expresses the S-function as a coefficient is*

$$\Pi f(x_r)\Delta(x_1, \ldots, x_n) = \Sigma \pm \{\lambda_1, \ldots, \lambda_p\} x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \ldots x_n^{\lambda_n}.$$

If we put $\rho = 1$, and $\epsilon_1 = \epsilon_2 = \ldots = 0$, the S-function becomes the S-function of the roots of the characteristic equation.

To simplify calculation two rules are useful. For the S-function of a product $f(x) = f'(x)f''(x)$, denoting the S-functions respectively by $\{\lambda\}$, $\{\lambda\}'$, and $\{\lambda\}''$, we have

$$\{\lambda\} = \{\lambda\}' + \{\lambda\}'' + \Sigma g_{\nu\mu\lambda}\{\nu\}'\{\mu\}'',$$

where $g_{\nu\mu\lambda}$ is the coefficient of $\{\lambda\}$ in the product $\{\nu\}\{\mu\}$*.

The S-functions of $1/\{(1-x)(1-\rho x)\ldots(1-\rho^{n-1}x)\}$ are given by†

$$\{\lambda_1, \ldots, \lambda_p\} = \frac{\Pi(1-\rho^{\lambda_r - \lambda_s - r + s})}{\Pi[\lambda_r + p - r]!}$$

* D. E. Littlewood, *Proc. London Math. Soc.* (2), 40 (1935), 54, 56.

† D. E. Littlewood and A. R. Richardson, *Quart. J. of Math.* (Oxford), 6 (1935), 186.

times the product of the first λ_i terms from each i-th row of the set of numbers for which the j-th term in the i-th row is $\rho^{i-1}-\rho^{n+j-1}$.

As an example we evaluate $\left[[4]_\mu \dot{+} [2]_0\right]^{[31]}$.

We have

$$\left[[4]_\mu \dot{+} [2]_0\right]^{[31]} = [4]_\mu^{[31]} \dot{+} [2]_0^{[31]} \dot{+} \{[4]_\mu^{[3]} \dot{+} [4]_\mu^{[21]}\} \mathbf{X} [2]_0 \dot{+} \{[4]_\mu^{[2]} \dot{+} [4]_\mu^{[1^2]}\} \mathbf{X} [2]_0^{[2]} \dot{+} [4]_\mu^{[2]} \mathbf{X} [2]_0^{[1^2]} \dot{+} [4]_\mu \mathbf{X} \{[2]_0^{[21]} \dot{+} [2]_0^{[3]}\}.$$

$[4]_\mu^{[31]}$ corresponds to the S-function $\{31\}$ of

$$1/\{(1-\mu x)(1-\rho\mu x)(1-\rho^2\mu x)(1-\rho^3\mu x)\},$$

which is given by

$$\{31\} = \frac{\rho(1-\rho^3)(1-\rho^3)(1-\rho^4)(1-\rho^5)(1-\rho^6)}{(1-\rho)^2(1-\rho^2)(1-\rho^3)(1-\rho^4)}$$

$$= \rho+2\rho^2+4\rho^3+5\rho^4+7\rho^5+7\rho^6+7\rho^7+5\rho^8+4\rho^9+2\rho^{10}+\rho^{11}.$$

Hence $$[4]_\mu^{[31]} = [11]_{\mu^4} \dot{+} [9]_{\mu^4} \dot{+} 2[7]_{\mu^4} \dot{+} [5]_{\mu^4} \dot{+} 2[3]_{\mu^4}.$$

All other submatrices are singular, and we may put $\rho = 1$, $\mu = 1$, so that the S-functions of $(1-x)^{-4}$ are taken.

We obtain

$$\left[[4]_\mu \dot{+} [2]_0\right]^{[31]} \dot{-} [4]_\mu^{[31]} = [2]_0^{[31]} \dot{+} (20+20)[2]_0^{[1]} \dot{+} (10+6)[2]_0^{[2]} \dot{+} 10[2]_0^{[1^2]} \dot{+} 4[2]_0^{[21]} \dot{+} 4[2]_0^{[3]}.$$

The corresponding function is

$$\epsilon_1\epsilon_2(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2)+40(\epsilon_1+\epsilon_2)+16(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2) + 10\epsilon_1\epsilon_2+4\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2)+4(\epsilon_1^3+\epsilon_1^2\epsilon_2+\epsilon_1\epsilon_2^2+\epsilon_2^3) = 105\epsilon_1+60\epsilon_2.$$

Hence

$$\left[[4]_\mu \dot{+} [2]_0\right]^{[31]} = [11]_{\mu^4} \dot{+} [9]_{\mu^4} \dot{+} 2[7]_{\mu^4} \dot{+} [5]_{\mu^4} \dot{+} 2[3]_{\mu^4} \dot{+} 60[2]_0 \dot{+} 45[1]_0.$$

6. We now complete the proof of Theorems IV and VIII.

For any given $\mu \neq 0$, order the induced matrices $[n]_\mu^{[m]}$ in increasing order of $m+n$, and for fixed $m+n$, in increasing order of n. If all these matrices do not satisfy Theorem VIII, let $[n]_\mu^{[m]}$ be the first which does not.

For $r \leqslant m$, by Theorem VII, $[n]_{\mu}^{[1^r]} = [n-r+1]_{\mu}^{[r]}$, and, since $[n-r+1]_{\mu}^{[r]}$ satisfies Theorem VIII, $[n]^{[1^r]}$ will satisfy Theorem IV.

Now consider the equation

$$a_m - a_{m-1}h_1 + a_{m-2}h_2 - \ldots + (-1)^m h_m = 0, \tag{3}$$

and the corresponding equation

$$[n]^{[1^m]} \dot{-} [n]^{[1^{m-1}]}[n] \dot{+} [n]^{[1^{m-2}]}[n]^{[2]} \dot{-} \ldots \dot{+} (-1)^m [n]^{[m]} = 0. \tag{4}$$

The theoretical results given by Theorems IV and VIII must satisfy (4), since the corresponding S-functions of

$$f(x) = 1/\{(1-x)(1-\rho x)\ldots(1-\rho^{n-1}x)\}$$

satisfy (3). Again the correct results must satisfy (4), since this equation is satisfied by all matrices. By our assumption, all matrices in (4) except the last satisfy Theorems IV and VIII. Hence $[n]^{[m]}$ satisfies Theorem VIII, and we arrive at a contradiction.

Thus Theorems IV and VIII must hold in all cases.

University College,
Swansea.

THE DISTRIBUTION OF MOLECULAR VELOCITIES AND THE MEAN MOTION IN A NON-UNIFORM GAS

By D. BURNETT.

[Received and read 15 November, 1934.]

1. In a recent paper* it has been shown how the classical problem of determining the second approximation to the distribution function for a slightly non-uniform gas (*i.e.* one in which there exist small gradients of density, temperature, and mean velocity) can be treated with peculiar appropriateness in terms of Sonine polynomials. In the present paper it is shown that the result can be found to any degree of accuracy by successive approximation.

Several incomplete third approximations to the stresses in the gas have already been given. Maxwell† and Jones‡ include terms involving second derivatives of the temperature and mean velocity, but ignore squares and products of the first derivatives as well as the third and subsequent derivatives; and this neglect of the products of first derivatives avoids the most serious mathematical difficulty of the problem, but at the same time appears to limit somewhat the range of applicability of their results. Enskog§ gives a solution containing products of the temperature gradients.

In the first part of the present paper a method of analysis is developed by means of which the distribution function can be calculated to any degree of approximation for a gas whose molecules attract one another with a force inversely proportional to the n-th power of their distance. Since the

---

* Burnett, *Proc. London Math. Soc.* (2), 39 (1935), 385–430.

† Maxwell, *Phil. Trans. Royal Soc.*, 157 (1867), 49–88; *Coll. Scientific Papers*, 2 (1890), 26–78.

‡ Jones, *Phil. Trans. Royal Soc.* (A), 223 (1923), 1.

§ Enskog, *Inaug. Diss. Uppsala* (1917), ch. 6.

density, temperature, and mean velocity are all supposed to vary from point to point in an arbitrary manner, the coefficients in the distribution function naturally become somewhat complicated when we include products of first differential coefficients or differential coefficients of higher order than the first; but since the importance of the investigation is mainly in connection with the discussion of the stresses in the gas (from which, of course, follow the equations of mean motion), it has been considered sufficient to give the arithmetical work only for the single coefficient in the distribution function on which these stresses depend. This has been done in the two extreme cases of Maxwellian molecules, for which the law of force is the inverse fifth power, and molecules which are elastic spheres, the approximations being carried so far as to include second differential coefficients of density, temperature, and mean velocity, and products of their first derivatives. The determination of the complete distribution function to this order of accuracy from the general formulae given in this paper involves very little additional labour, but seems unnecessary unless the stresses are required to the next approximation.

It is also possible to investigate the circumstances in which the ordinary Navier-Stokes equations of viscous fluid theory are not valid even as a first approximation to the mean motion; and it appears that this does not happen until the variation in the condition of the gas is appreciable in a distance comparable with the molecular mean free path. Some applications of this occur, *e.g.* Rayleigh's discussion* of the effect of viscosity on an aerial wave of finite amplitude or "shock" wave. The extreme case when the effect of collisions is negligible is important in cosmogony†, but the mathematical difficulties in this case are of quite a different character.

2. The gas is supposed to consist of molecules of one type only, of mass m, spherically symmetrical and attracting one another with a force inversely proportional to the n-th power of the distance. If the molecular density is ν then the number of molecules whose velocities relative to the mean motion of the gas have components between (U, V, W) and $(U+dU, V+dV, W+dW)$ is $\nu f(U, V, W)\, dU\, dV\, dW$, where the function $f(U, V, W)$ is to be determined. To a first approximation it is the classical Maxwell distribution function

$$\left(\frac{hm}{\pi}\right)^{\frac{3}{2}} \exp\{-hm(U^2+V^2+W^2)\} = \left(\frac{hm}{\pi}\right)^{\frac{3}{2}} \exp(-hmC^2),$$

* Rayleigh, *Proc. Royal Soc.* (A), 84 (1910), 247.

† *Cf.* Jeans, *Problems of cosmogony* (1919), ch. 10.

in which the slight lack of uniformity in the gas is neglected; the second approximation may be written in the form*

$$f=\left(\frac{hm}{\pi}\right)^{\frac{3}{2}}\exp(-hmC^2)\left\{1+(hm)^{\frac{1}{2}}\sum_n Y_{1n}S_{\frac{3}{2}}^{\,n}(hmC^2)+hm\sum_n Y_{2n}S_{\frac{5}{2}}^{\,n}(hmC^2)\right\},$$

where $\Gamma(m+n+1)\,S_m^{\,n}(x)$ is the coefficient of s^n in the expansion of $(1-s)^{-m-1}e^{-xs/(1-s)}$, and Y_{1n}, Y_{2n} are solid harmonics, with U, V, W as arguments, of degrees 1 and 2 respectively. The actual values are

$$Y_{1n}=B_n\frac{1}{h}\left(U\frac{\partial h}{\partial x}+V\frac{\partial h}{\partial y}+W\frac{\partial h}{\partial z}\right),$$

$$Y_{2n}=C_n\left\{(3U^2-C^2)\frac{\partial u}{\partial x}+(3V^2-C^2)\frac{\partial v}{\partial y}+(3W^2-C^2)\frac{\partial w}{\partial z}\right.$$
$$\left.+3VW\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+3WU\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+3UV\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\},$$

where u, v, w are the components of the mean velocity of the gas and B_n, C_n are coefficients depending on the law of force between pairs of molecules. In the special case when this force is inversely proportional to the fifth power of the distance (Maxwellian molecules), the results are peculiarly simple since B_n is zero unless $n=1$ and C_n is zero unless $n=0$.

3. *The general form of the distribution function.*

Assume that, in general,

$$f=\left(\frac{hm}{\pi}\right)^{\frac{3}{2}}\exp(-hmC^2)\left\{\sum_n n!\,\Gamma(n+\tfrac{3}{2})\,A_n\,S_{\frac{1}{2}}^{\,n}(hmC^2)\right.$$
$$\left.+\sum_\kappa(hm)^{\frac{1}{2}\kappa}\sum_n n!\,\Gamma(\kappa+n+\tfrac{3}{2})\,Y_{\kappa n}\,S_{\kappa+\frac{1}{2}}^{n}(hmC^2)\right\} \quad (3.1)$$

with the same notation as above, $Y_{\kappa n}$ representing a solid harmonic function of U, V, W of degree κ. Then, since (U, V, W) is the velocity of a molecule relative to the mean velocity of the gas as a whole, the mean values of U, V, and W must vanish: now the mean value of U is†

$$\int Uf\,dU\,dV\,dW$$
$$=\frac{(hm)^2}{\pi^{\frac{3}{2}}}\sum n!\,\Gamma(n+\tfrac{5}{2})\int U\,Y_{1n}\,S_{\frac{3}{2}}^{\,n}(hmC^2)\exp(-hmC^2)\,dU\,dV\,dW,$$

---

* *Cf.* Burnett, *loc. cit.*

† The *one* sign of integration is used to imply as many as the context requires.

by the well-known orthogonal property of spherical harmonics, and if

$$Y_{1n} = B_{xn} U + B_{yn} V + B_{zn} W,$$

the integral is equal to

$$\tfrac{1}{3} B_{xn} \int C^2 S_{\frac{3}{2}}^{n}(hmC^2) \exp(-hmC^2)\, dU\, dV\, dW,$$

which vanishes* unless $n = 0$.

Hence B_{x0} and, similarly, B_{y0} and B_{z0} must vanish, *i.e.*

$$Y_{10} = 0. \tag{3.21}$$

In order to make the solution definite, two further conditions must be specified and these may be taken to be†:

(*a*) that A_0 has the same value as for a uniform gas

i.e. $$A_0 = 1 \tag{3.22}$$

since $$\Gamma(\tfrac{3}{2})\, S_{\frac{3}{2}}^{0}(x) = 1;$$

(*b*) that h is chosen so that the mean square of the velocity relative to the mean has the same value as for a uniform gas, *i.e.* that

$$\overline{C^2} = 3/2hm.$$

Now

$$\overline{C^2} = \left(\frac{hm}{\pi}\right)^{\frac{3}{2}} \Sigma n!\, \Gamma(n+\tfrac{3}{2})\, A_n \int C^2 S_{\frac{1}{2}}^{n}(hmC^2) \exp(-hmC^2)\, dU\, dV\, dW$$

$$= \left(\frac{hm}{\pi}\right)^{\frac{3}{2}} \Sigma n!\, \Gamma(n+\tfrac{3}{2})\, A_n\, 4\pi \int_0^\infty C^4 S_{\frac{1}{2}}^{n}(hmC^2) \exp(-hmC^2)\, dC,$$

and this last integral is the coefficient of s^n in the expansion of

$$\frac{(1-s)^{-\frac{3}{2}}}{\Gamma(n+\frac{3}{2})} \int_0^\infty C^4 \exp\left(\frac{-hmC^2}{1-s}\right) dC = \frac{\Gamma(\frac{5}{2})}{2(hm)^{\frac{5}{2}}} \frac{1-s}{\Gamma(n+\frac{3}{2})},$$

so that it is zero if n is greater than unity and

$$\overline{C^2} = \frac{3}{2hm}(A_0 - A_1).$$

* This follows easily from the definition of $S_m^n(x)$, *cf.* Burnett, *loc.cit.*, 387.

† *Cf.* Jeans, *Dynamical theory of gases* (1921), 3rd ed., ch. 8, 276.

Hence, on taking account of the value of A_0, this condition requires that A_1 should be zero. (3.23)

The solid harmonic of degree κ can be expressed linearly in terms of a set of $2\kappa+1$ independent harmonics of degree κ; the particular set employed will be those which are of the form*

$$(-1)^p C^\kappa P_\kappa^{(p)}(\cos\theta)\cos p\phi, \quad (-1)^p C^\kappa P_\kappa^{(p)}(\cos\theta)\sin p\phi,$$

when U, V, W are put equal to $C\sin\theta\cos\phi$, $C\sin\theta\sin\phi$, $C\cos\theta$, respectively, and the notation will be

$$Y_\kappa = C^\kappa P_\kappa(\cos\theta), \tag{3.31}$$

$$Y_\kappa^{(p)} = (-1)^p C^\kappa P_\kappa^{(p)}(\cos\theta)\cos p\phi, \tag{3.32}$$

$$Z_\kappa^{\,p} = (-1)^p C^\kappa P_\kappa^{(p)}(\cos\theta)\sin p\phi, \tag{3.33}$$

where $1 \leqslant p \leqslant \kappa$. The general spherical harmonic will be written in the form

$$Y_{\kappa n} = B_{\kappa n} Y_\kappa + 2\sum_{p=1}^{\kappa} \frac{(\kappa-p)!}{(\kappa+p)!}\left(B_{\kappa n}^{(p)} Y_\kappa^{(p)} + C_{\kappa n}^{(p)} Z_\kappa^{(p)}\right), \tag{3.34}$$

$B_{\kappa n}$, $B_{\kappa n}^{(p)}$, and $C_{\kappa n}^{(p)}$ being constants.

Actually it will be sufficient to determine the coefficients $B_{\kappa n}$ for all values of κ and n; $B_{\kappa n}^{(p)}$ and $C_{\kappa n}^{(p)}$, as will be seen later, can always be deduced from $B_{\kappa n}$ by a transformation of axes.

The solid harmonics are well known; in the above notation those of degrees 1, 2, and 3 are

$\kappa = 1$: $$Y_1 = W, \quad Y_1^{(1)} = U, \quad Z_1^{(1)} = V;$$

$\kappa = 2$: $$Y_2 = \tfrac{1}{2}(3W^2 - C^2), \quad Y_2^{(1)} = 3WU, \quad Y_2^{(2)} = 3(U^2 - V^2),$$

$$Z_2^{(1)} = 3VW, \quad Z_2^{(2)} = 6UV;$$

$\kappa = 3$: $$Y_3 = \tfrac{1}{2}W(5W^2 - 3C^2),$$

$$Y_3^{(1)} = \tfrac{3}{2}U(5W^2 - C^2), \quad Y_3^{(2)} = 15W(U^2 - V^2), \quad Y_3^{(3)} = 15U(U^2 - 3V^2),$$

$$Z_3^{(1)} = \tfrac{3}{2}V(5W^2 - C^2), \quad Z_3^{(2)} = 30UVW, \quad Z_3^{(3)} = 15V(3U^2 - V^2).$$

---

* The definition of $P_\kappa^{(p)}(\cos\theta)$ is that given by Hobson, *Spherical and ellipsoidal harmonics* (1931), § 55, *i.e.* $P_\kappa^{(p)}(x) = (-1)^p(1-x^2)^{\frac{1}{2}p}(d/dx)^p P_\kappa(x)$.

The following properties are noted for reference:

$$\left.\begin{aligned}
(2\kappa+1)\,U\,Y_\kappa &= Y^{(1)}_{\kappa+1} - C^2\,Y^{(1)}_{\kappa-1},\\
(4\kappa+2)\,U\,Y^{(p)}_\kappa &= Y^{(p+1)}_{\kappa+1} - (\kappa-p+1)(\kappa-p+2)\,Y^{(p-1)}_{\kappa+1}\\
&\qquad - C^2\{Y^{(p+1)}_{\kappa-1} - (\kappa+p)(\kappa+p-1)\,Y^{(p-1)}_{\kappa-1}\},\\
(4\kappa+2)\,U\,Z^{(p)}_\kappa &= Z^{(p+1)}_{\kappa+1} - (\kappa-p+1)(\kappa-p+2)\,Z^{(p-1)}_{\kappa+1}\\
&\qquad - C^2\{Z^{(p+1)}_{\kappa-1} - (\kappa+p)(\kappa+p-1)\,Z^{(p-1)}_{\kappa-1}\}.
\end{aligned}\right\} \quad (3.41)$$

$$\left.\begin{aligned}
(2\kappa+1)\,V\,Y_\kappa &= Z^{(1)}_{\kappa+1} - C^2\,Z^{(1)}_{\kappa-1},\\
(4\kappa+2)\,V\,Y^{(p)}_\kappa &= Z^{(p+1)}_{\kappa+1} + (\kappa-p+1)(\kappa-p+2)\,Z^{(p-1)}_{\kappa+1}\\
&\qquad - C^2\{Z^{(p+1)}_{\kappa-1} + (\kappa+p)(\kappa+p-1)\,Z^{(p-1)}_{\kappa-1}\},\\
(4\kappa+2)\,V\,Z^{(p)}_\kappa &= -[Y^{(p+1)}_{\kappa+1} + (\kappa-p+1)(\kappa-p+2)\,Y^{(p-1)}_{\kappa+1}\\
&\qquad - C^2\{Y^{(p+1)}_{\kappa-1} + (\kappa+p)(\kappa+p-1)\,Y^{(p-1)}_{\kappa-1}\}].
\end{aligned}\right\} \quad (3.42)$$

$$\left.\begin{aligned}
(2\kappa+1)\,W\,Y_\kappa &= (\kappa+1)\,Y_{\kappa+1} + \kappa C^2\,Y_{\kappa-1},\\
(2\kappa+1)\,W\,Y^{(p)}_\kappa &= (\kappa-p+1)\,Y^{(p)}_{\kappa+1} + (\kappa+p)\,C^2\,Y^{(p)}_{\kappa-1},\\
(2\kappa+1)\,W\,Z^{(p)}_\kappa &= (\kappa-p+1)\,Z^{(p)}_{\kappa+1} + (\kappa+p)\,C^2\,Z^{(p)}_{\kappa-1}.
\end{aligned}\right\} \quad (3.43)$$

$$\left.\begin{aligned}
\frac{\partial}{\partial U}\,Y_\kappa &= -\,Y^{(1)}_{\kappa-1},\\
\frac{\partial}{\partial U}\,Y^{(p)}_\kappa &= \tfrac{1}{2}[(\kappa+p)(\kappa+p-1)\,Y^{(p-1)}_{\kappa-1} - Y^{(p+1)}_{\kappa-1}],\\
\frac{\partial}{\partial U}\,Z^{(p)}_\kappa &= \tfrac{1}{2}[(\kappa+p)(\kappa+p-1)\,Z^{(p-1)}_{\kappa-1} - Z^{(p+1)}_{\kappa-1}].
\end{aligned}\right\} \quad (3.44)$$

$$\left.\begin{aligned}
\frac{\partial}{\partial V}\,Y_\kappa &= -\,Z^{(1)}_{\kappa-1},\\
\frac{\partial}{\partial V}\,Y^{(p)}_\kappa &= -\tfrac{1}{2}[(\kappa+p)(\kappa+p-1)\,Z^{(p-1)}_{\kappa-1} + Z^{(p+1)}_{\kappa-1}],\\
\frac{\partial}{\partial V}\,Z^{(p)}_\kappa &= \tfrac{1}{2}[(\kappa+p)(\kappa+p-1)\,Y^{(p-1)}_{\kappa-1} + Y^{(p+1)}_{\kappa-1}].
\end{aligned}\right\} \quad (3.45)$$

$$\frac{\partial}{\partial W}\,Y_\kappa = \kappa\,Y_{\kappa-1}, \quad \frac{\partial}{\partial W}\,Y^{(p)}_\kappa = (\kappa+p)\,Y^{(p)}_{\kappa-1}, \quad \frac{\partial}{\partial W}\,Z^{(p)}_\kappa = (\kappa+p)\,Z^{(p)}_{\kappa-1}. \quad (3.46)$$

In all these formulae $Z_\kappa^{(p)}$ is to be taken as zero if $p = 0$ or $p > \kappa$; while $Y_\kappa^{(0)} = Y_\kappa$, and $Y_\kappa^{(p)} = 0$ if $p > \kappa$.

It is also convenient to note here, as an extension of the known result

$$\int_0^\infty C^{2\kappa+2} S^r_{\kappa+\frac{1}{2}}(hmC^2)\, S^s_{\kappa+\frac{1}{2}}(hmC^2) \exp(-hmC^2)\, dC = \frac{1}{2(hm)^{\kappa+\frac{3}{2}}\Gamma(\kappa+r+\frac{3}{2})\, r!} \quad \text{if} \quad r = s,$$

and is zero if r and s are unequal, that

$$\int_0^\infty C^{2\kappa+4} S^r_{\kappa+\frac{1}{2}}(hmC^2)\, S^s_{\kappa+\frac{3}{2}}(hmC^2) \exp(-hmC^2)\, dC$$
$$= \{\Gamma(\kappa+r+\tfrac{3}{2})\,\Gamma(\kappa+s+\tfrac{5}{2})\}^{-1} \times \text{coefficient of } x^r y^s \text{ in}$$
$$(1-x)^{-\kappa-\frac{3}{2}}(1-y)^{-\kappa-\frac{5}{2}} \int_0^\infty C^{2\kappa+4} \exp\left\{-hmC^2\left(1+\frac{x}{1-x}+\frac{y}{1-y}\right)\right\} dC;$$

the value of this last integral is

$$\frac{\Gamma(\kappa+\frac{5}{2})}{2}\left\{\frac{(1-x)(1-y)}{hm(1-xy)}\right\}^{\kappa+\frac{3}{2}},$$

so that we require the coefficient of $x^r y^s$ in

$$\frac{\Gamma(\kappa+\frac{5}{2})}{2} \frac{1-x}{(hm)^{\kappa+\frac{5}{2}}(1-xy)^{\kappa+\frac{5}{2}}}.$$

Hence $$\int_0^\infty C^{2\kappa+4} S^r_{\kappa+\frac{1}{2}}(hmC^2)\, S^s_{\kappa+\frac{3}{2}}(hmC^2) \exp(-hmC^2)\, dC$$

is zero unless $r = s$ or $r = s+1$, in which cases it is

$$\frac{(-1)^{r-s}}{2(hm)^{\kappa+\frac{5}{2}}\Gamma(\kappa+r+\frac{3}{2})\, s!}. \tag{3.51}$$

It can also be shown directly from the definition of $S_m{}^n(x)$ that

$$\frac{d}{dx} S_m{}^n(x) = -S_{m+1}^{n-1}(x). \tag{3.52}$$

$$xS^{r-1}_{\kappa+\frac{5}{2}}(x) = S^{r-1}_{\kappa+\frac{1}{2}}(x) - rS^r_{\kappa+\frac{1}{2}}(x)$$
$$= (\kappa+\tfrac{1}{2})\, S^r_{\kappa+\frac{1}{2}}(x) - S^r_{\kappa-\frac{1}{2}}(x), \tag{3.53}$$

results which will be required in the course of the paper.

4. *The mean value equation.*

Suppose that two molecules of the gas, whose velocities are (u_1, v_1, w_1) and (u_2, v_2, w_2) when they are outside the field of force of one another, approach, interact, and then separate again, their final velocities being (u_1', v_1', w_1'), (u_2', v_2', w_2'); the mean velocity of the gas has components u, v, w, so that the velocities of the molecules relative to the mean are (U_1, V_1, W_1), etc., where $U_1 = u_1 - u$, etc. The relative velocity of the two molecules before and after the encounter will be the same in magnitude and will be represented* by C_R; the angle between the initial and final directions of this relative velocity will be called χ. The relative position of the two molecules at the instant of closest approach is defined† by the parameters α, ϵ, which have the property that, so long as the force between two molecules is proportional to a power of their distance, χ can be expressed as a function of α which does not explicitly contain ϵ or the velocity components. Then, if f_1, f_2, f_1', f_2' are written for $f(U_1, V_1, W_1)$, $f(U_2, V_2, W_2)$, $f(U_1', V_1', W_1')$, $f(U_2', V_2', W_2')$ respectively, Boltzmann's integral equation for the distribution function is

$$\left(\frac{\partial}{\partial t}+u_1\frac{\partial}{\partial x}+v_1\frac{\partial}{\partial y}+w_1\frac{\partial}{\partial z}+X\frac{\partial}{\partial u_1}+Y\frac{\partial}{\partial v_1}+Z\frac{\partial}{\partial w_1}\right)\nu f_1$$

$$= M\nu^2\int (f_1'f_2'-f_1f_2)\, C_R{}^{\mu}\, du_2\, dv_2\, dw_2\, \alpha\, d\alpha\, d\epsilon, \quad (4.1)$$

where X, Y, Z are the components of external force acting on the gas per unit mass and M, μ are constants depending on the inter-molecular force: if two molecules at a distance r apart repel one another with a force $Km^2 r^{-\mu}$, then‡

$$M = (2Km)^{2/(n-1)} \quad (4.11)$$

and

$$\mu = (n-5)/(n-1). \quad (4.12)$$

The extreme values $\mu = 0$ and $\mu = 1$ correspond respectively to Maxwellian molecules and to molecules which are elastic spheres. The integral in (4.1) is taken over all values of u_2, v_2, w_2 from $-\infty$ to $+\infty$, of α from 0 to ∞, and of ϵ from 0 to 2π.

* The notation here differs slightly from that of Chapman, *Phil. Trans. Royal Soc.* (A), 216 (1916), 291, where the relative velocity would be represented by $2C_R$.

† *Cf.* Jeans, *Dynamical theory of gases*, ch. 8.

‡ *Cf.* Jeans, *loc. cit.*, 213, 218.

Multiply this equation throughout by Q_1, an arbitrary* function of U_1, V_1, W_1, and integrate over all values of u_1, v_1, w_1. If $\bar{F}$ is written for $\int F f_1 du_1 dv_1 dw_1$ and some integrations by parts are performed, it is easily found that†

$$\frac{\partial}{\partial t}\nu\overline{Q_1}+\frac{\partial}{\partial x}\nu\overline{u_1 Q_1}+\frac{\partial}{\partial y}\nu\overline{v_1 Q_1}+\frac{\partial}{\partial z}\nu\overline{w_1 Q_1}$$

$$-\nu\left[\overline{\frac{\partial Q_1}{\partial t}}+\overline{u_1\frac{\partial Q_1}{\partial x}}+\overline{v_1\frac{\partial Q_1}{\partial y}}+\overline{w_1\frac{\partial Q_1}{\partial z}}+X\overline{\frac{\partial Q_1}{\partial U_1}}+Y\overline{\frac{\partial Q_1}{\partial V_1}}+Z\overline{\frac{\partial Q_1}{\partial W_1}}\right]$$

$$=M\nu^2\int Q_1(f_1'f_2'-f_1f_2)\,C_R{}^\mu\,du_1dv_1dw_1du_2dv_2dw_2\,\alpha\,d\alpha\,d\epsilon$$

$$=M\nu^2\int(Q_1'-Q_1)f_1f_2\,C_R{}^\mu\,du_1dv_1dw_1du_2dv_2dw_2\,\alpha\,d\alpha\,d\epsilon$$

$$=M\nu^2\int(Q_1'-Q_1)f_1f_2\,C_R{}^\mu dU_1dV_1dW_1dU_2dV_2dW_2\,\alpha\,d\alpha\,d\epsilon \quad (4.2)$$

by obvious changes of variable, Q_1' being the same function of U_1', V_1', W_1' as Q_1 is of U_1, V_1, W_1: it will be noticed that

$$du_1dv_1dw_1du_2dv_2dw_2=du_1'\ldots dw_2'=dU_1\ldots dW_2.$$

The particular values of Q which will be considered are‡

$$Q=(\kappa+\tfrac{1}{2})\pi^{\frac{1}{2}}(hm)^{\frac{1}{2}\kappa}\,Y_\kappa\,S^r_{\kappa+\frac{1}{2}}(hmC^2) \quad (4.3)$$

for different (integral) values of κ and r. Hence

$$\bar{Q}=(\kappa+\tfrac{1}{2})\pi^{\frac{1}{2}}(hm)^{\frac{1}{2}\kappa}\int Y_\kappa\,S^r_{\kappa+\frac{1}{2}}(hmC^2)\,f\,dU\,dV\,dW,$$

and, on substituting for f from (3.1) and (3.34), it follows from the orthogonal properties of the harmonic functions and Sonine polynomials that the only term in the integral which does not vanish is that having $B_{\kappa r}$ as its coefficient; and on performing the integration it is found that

$$\bar{Q}=B_{\kappa r}. \quad (4.31)$$

---

* With the obvious restrictions that it should be continuous and that $\overline{Q}_1$ as defined below should exist.

† *Cf.* Enskog, *loc. cit.*, 10–12.

‡ The suffixes 1 will be dropped in future except when it is desirable to retain them in order to distinguish between the two molecules.

Also, from (3.41), (3.42), (3.43),

$$UQ = \tfrac{1}{2}\pi^{\frac{1}{2}}(hm)^{\frac{1}{2}\kappa} S^r_{\kappa+\frac{1}{2}}(Y^{(1)}_{\kappa+1} - C^2 Y^{(1)}_{\kappa-1}),$$

$$VQ = \tfrac{1}{2}\pi^{\frac{1}{2}}(hm)^{\frac{1}{2}\kappa} S^r_{\kappa+\frac{1}{2}}(Z^{(1)}_{\kappa+1} - C^2 Z^{(1)}_{\kappa-1}),$$

$$WQ = \tfrac{1}{2}\pi^{\frac{1}{2}}(hm)^{\frac{1}{2}\kappa} S^r_{\kappa+\frac{1}{2}}\{(\kappa+1) Y_{\kappa+1} + \kappa C^2 Y_{\kappa-1}\},$$

the argument of the Sonine polynomial being hmC^2 in each case: this argument will in future be omitted unless for some reason it is specially necessary to emphasise it. It follows by integration as above that

$$(hm)^{\frac{1}{2}}\overline{UQ} = \frac{1}{2\kappa+3}\{(\kappa+r+\tfrac{3}{2}) B^{(1)}_{\kappa+1,r} - B^{(1)}_{\kappa+1,r-1}\} - \frac{1}{2\kappa-1}\{B^{(1)}_{\kappa-1,r} - (r+1) B^{(1)}_{\kappa-1,r+1}\}$$

$$= D_{\kappa r}, \quad \text{say,} \tag{4.321}$$

$(hm)^{\frac{1}{2}}\overline{VQ}$ = a similar expression with the B's replaced by C's

$$= E_{\kappa r}, \quad \text{say,} \tag{4.322}$$

$$(hm)^{\frac{1}{2}}\overline{WQ} = \frac{\kappa+1}{2\kappa+3}\{(\kappa+r+\tfrac{3}{2}) B_{\kappa+1,r} - B_{\kappa+1,r-1}\} + \frac{\kappa}{2\kappa-1}\{B_{\kappa-1,r} - (r+1) B_{\kappa-1,r+1}\}$$

$$= F_{\kappa r}. \tag{4.323}$$

If the solid harmonic Y_κ in Q is replaced by $Y^{(p)}_\kappa$, the values of $\overline{UQ}$, $\overline{VQ}$, $\overline{WQ}$ are found similarly from (3.41), (3.42), (3.43); they are

$$(hm)^{\frac{1}{2}}\overline{UQ} = D^{(p)}_{\kappa r}$$

$$= \frac{1}{2(2\kappa+3)}[(\kappa+r+\tfrac{3}{2}) B^{(p+1)}_{\kappa+1,r} - B^{(p+1)}_{\kappa+1,r-1} - (\kappa-p+1)(\kappa-p+2)\{(\kappa+r+\tfrac{3}{2}) B^{(p-1)}_{\kappa+1,r} - B^{(p-1)}_{\kappa+1,r-1}\}]$$

$$- \frac{1}{2(2\kappa-1)}[B^{(p+1)}_{\kappa-1,r} - (r+1) B^{(p+1)}_{\kappa-1,r+1} - (\kappa+p)(\kappa+p-1)\{B^{(p-1)}_{\kappa-1,r} - (r+1) B^{(p-1)}_{\kappa-1,r+1}\}];$$

$$(hm)^{\frac{1}{2}}\overline{VQ} = E^{(p)}_{\kappa r}$$

is the same expression with the B's replaced by C's and the signs of the terms of rank $p-1$ changed.

$$(hm)^{\frac{1}{2}}\,\overline{WQ} = F^{(p)}_{\kappa r} = \frac{\kappa-p+1}{2\kappa+3}\{(\kappa+r+\tfrac{3}{2})\,B^{(p)}_{\kappa+1,r} - B^{(p)}_{\kappa+1,r-1}\}$$
$$+\frac{\kappa+p}{2\kappa-1}\{B^{(p)}_{\kappa-1,r} - (r+1)\,B^{(p)}_{\kappa-1,r+1}\}. \quad (4.331)$$

If Y_κ in Q is replaced by $Z^{(p)}_\kappa$, the corresponding values are

$$(hm)^{\frac{1}{2}}\,\overline{UQ} = G^{(p)}_{\kappa r},$$

the same as $D^{(p)}_{\kappa r}$ except that the B's are replaced by C's.

$$(hm)^{\frac{1}{2}}\,\overline{VQ} = H^{(p)}_{\kappa r}$$

is got from $D^{(p)}_{\kappa r}$ by changing the signs of the terms of rank $p+1$.

$$(hm)^{\frac{1}{2}}\,\overline{WQ} = J^{(p)}_{\kappa r} \quad (4.332)$$

is got from $F^{(p)}_{\kappa r}$ by replacing the B's by C's.

In all these expressions, $B^{(p)}_{\kappa r}$ and $C^{(p)}_{\kappa r}$ are to be taken as zero when p is negative or greater than κ: also, when p is zero, $B^{(0)}_{\kappa r} = B_{\kappa r}$, $C^{(0)}_{\kappa r} = 0$, $B_{0r} = A_r$.

On returning to the expression (4.3) for Q and using (3.44) and (3.52), we get

$$\frac{\partial Q}{\partial U} = -(\kappa+\tfrac{1}{2})\,\pi^{\frac{1}{2}}(hm)^{\frac{1}{2}\kappa}\{Y^{(1)}_{\kappa-1}\,S^r_{\kappa+\frac{1}{2}} + 2hmU\,Y_\kappa\,S^{r-1}_{\kappa+\frac{3}{2}}\},$$

and the terms in the bracket reduce to

$$Y^{(1)}_{\kappa-1}\,S^r_{\kappa+\frac{1}{2}} + \frac{2}{2\kappa+1}\,hm\,S^{r-1}_{\kappa+\frac{3}{2}}(Y^{(1)}_{\kappa+1} - C^2\,Y^{(1)}_{\kappa-1})$$
$$= \frac{2}{2\kappa+1}\{hm\,Y^{(1)}_{\kappa+1}\,S^{r-1}_{\kappa+\frac{3}{2}} + Y^{(1)}_{\kappa-1}\,S^r_{\kappa-\frac{1}{2}}\}, \quad (4.341)$$

by (3.53). Hence

$$\frac{1}{(hm)^{\frac{1}{2}}}\,\frac{\overline{\partial Q}}{\partial U} = -\frac{2}{2\kappa+3}\,B^{(1)}_{\kappa+1,\,r-1} - \frac{2}{2\kappa-1}\,B^{(1)}_{\kappa-1,\,r}, \quad (4.342)$$

and, similarly,

$$\frac{1}{(hm)^{\frac{1}{2}}}\,\frac{\overline{\partial Q}}{\partial V} = -\frac{2}{2\kappa+3}\,C^{(1)}_{\kappa+1,\,r-1} - \frac{2}{2\kappa-1}\,C^{(1)}_{\kappa-1,\,r}, \quad (4.343)$$

$$\frac{1}{(hm)^{\frac{1}{2}}}\,\frac{\overline{\partial Q}}{\partial W} = -\frac{2(\kappa+1)}{2\kappa+3}\,B_{\kappa+1,\,r-1} + \frac{2\kappa}{2\kappa-1}\,B_{\kappa-1,\,r}. \quad (4.344)$$

Also, since $U_1 = u_1 - u$, ...,

$$\frac{\partial Q}{\partial t} = -\frac{\partial Q}{\partial U}\frac{\partial u}{\partial t} - \frac{\partial Q}{\partial V}\frac{\partial v}{\partial t} - \frac{\partial Q}{\partial W}\frac{\partial w}{\partial t} + \frac{\partial Q}{\partial h}\frac{\partial h}{\partial t},$$

so that

$$\frac{\overline{\partial Q}}{\partial t} = -\frac{\partial u}{\partial t}\frac{\overline{\partial Q}}{\partial U} - \frac{\partial v}{\partial t}\frac{\overline{\partial Q}}{\partial V} - \frac{\partial w}{\partial t}\frac{\overline{\partial Q}}{\partial W} + \frac{\partial h}{\partial t}\frac{\overline{\partial Q}}{\partial h}. \tag{4.351}$$

Now

$$\frac{\partial Q}{\partial h} = (\kappa+\tfrac{1}{2})\,\pi^{\frac{1}{2}}\frac{(hm)^{\frac{1}{2}\kappa}}{h}\,Y_\kappa\{\tfrac{1}{2}\kappa\, S^r_{\kappa+\frac{1}{2}} - hmC^2\, S^{r-1}_{\kappa+\frac{3}{2}}\}$$

$$= (\kappa+\tfrac{1}{2})\,\pi^{\frac{1}{2}}\frac{(hm)^{\frac{1}{2}\kappa}}{h}\,Y_\kappa\{(r+\tfrac{1}{2}\kappa)\, S^r_{\kappa+\frac{1}{2}} - S^{r-1}_{\kappa+\frac{1}{2}}\},$$

so that

$$\frac{\overline{\partial Q}}{\partial h} = \frac{1}{h}\,\{(r+\tfrac{1}{2}\kappa)\, B_{\kappa r} - B_{\kappa,\, r-1}\} \tag{4.352}$$

and

$$\frac{\overline{\partial Q}}{\partial t} = \frac{2(hm)^{\frac{1}{2}}}{2\kappa+3}\left\{\frac{\partial u}{\partial t}\, B^{(1)}_{\kappa+1,\, r-1} + \frac{\partial v}{\partial t}\, C^{(1)}_{\kappa+1,\, r-1} + (\kappa+1)\,\frac{\partial w}{\partial t}\, B_{\kappa+1, r-1}\right\}$$

$$+ \frac{2(hm)^{\frac{1}{2}}}{2\kappa-1}\left\{\frac{\partial u}{\partial t}\, B^{(1)}_{\kappa-1,\, r} + \frac{\partial v}{\partial t}\, C^{(1)}_{\kappa-1,\, r} - \kappa\,\frac{\partial w}{\partial t}\, B_{\kappa-1, r}\right\}$$

$$+ \frac{1}{h}\,\frac{\partial h}{\partial t}\,\{(r+\tfrac{1}{2}\kappa)\, B_{\kappa r} - B_{\kappa,\, r-1}\}. \tag{4.353}$$

It follows from (4.341) and (4.331) that $\overline{U\dfrac{\partial Q}{\partial U}}$ is got from (4.342) by replacing $B^{(1)}_{\kappa+1,\, r-1}$ by $D^{(1)}_{\kappa+1,\, r-1}$, $B^{(1)}_{\kappa+1,\, r}$ by $D^{(1)}_{\kappa-1,\, r}$, ...; while $\overline{U\dfrac{\partial Q}{\partial V}}$ is got from (4.343) by replacing the C's by the corresponding G's, and $\overline{U\dfrac{\partial Q}{\partial W}}$ is got from (4.344) by replacing the B's by the corresponding D's. Hence $\overline{U\dfrac{\partial Q}{\partial x}}$ can be written down from (4.353): it is

$$\overline{U\frac{\partial Q}{\partial x}} = \frac{2}{2\kappa+3}\left\{\frac{\partial u}{\partial x}\, D^{(1)}_{\kappa+1,\, r-1} + \frac{\partial v}{\partial x}\, G^{(1)}_{\kappa+1,\, r-1} + (\kappa+1)\,\frac{\partial w}{\partial x}\, D_{\kappa+1, r-1}\right\}$$

$$+ \frac{2}{2\kappa-1}\left\{\frac{\partial u}{\partial x}\, D^{(1)}_{\kappa-1,\, r} + \frac{\partial v}{\partial x}\, G^{(1)}_{\kappa-1,\, r} - \kappa\,\frac{\partial w}{\partial x}\, D_{\kappa-1, r}\right\}$$

$$+ \frac{1}{h(hm)^{\frac{1}{2}}}\,\frac{\partial h}{\partial x}\,\{(r+\tfrac{1}{2}\kappa)\, D_{\kappa r} - D_{\kappa,\, r-1}\}. \tag{4.36}$$

Next, $\overline{V\frac{\partial Q}{\partial y}}$ is got by replacing the differential coefficients with respect to x in this expression by the corresponding differential coefficients with respect to y, the D's by the corresponding E's, and the G's by H's; and $\overline{W\frac{\partial Q}{\partial z}}$ is got by replacing the differential coefficients with respect to x by those with respect to z, the D's by F's and the G's by J's. Finally,

$$\overline{\frac{\partial Q}{\partial x}}, \quad \overline{\frac{\partial Q}{\partial y}}, \quad \overline{\frac{\partial Q}{\partial z}}$$

are got from (4.353) by replacing the differential coefficients with respect to t by the corresponding differential coefficients with respect to x, y, z respectively.

5. The left-hand side of equation (4.2) can now be written in terms of the coefficients $B^{(p)}_{\kappa r}$, $C^{(p)}_{\kappa r}$ for any of the values of Q given by (4.3); and before we evaluate the integral on the right the special cases may be considered in which this integral is known, for dynamical reasons, to be zero. It evidently represents the mean value of the change in Q due to collisions between the molecules, and therefore vanishes when

(i) $Q =$ constant, *i.e.* $\kappa = r = 0$.

(ii) $Q = W$, *i.e.* $\kappa = 1$, $r = 0$. (Conservation of linear momentum.)

(iii) $Q = C^2$, which, taken in conjunction with (i), corresponds to $\kappa = 0$, $r = 1$. (Conservation of energy.)

In these cases the equation reduces as follows:

Case (i). $\kappa = r = 0$ $\qquad \overline{Q} = A_0 = 1,$

$$(hm)^{\frac{1}{2}}\overline{UQ} = \tfrac{1}{2}B^{(1)}_{10} = 0,$$

from (3.21), and, similarly,

$$\overline{VQ} = \overline{WQ} = 0.$$

Also

$$\overline{\frac{\partial Q}{\partial t}} = \overline{\frac{\partial Q}{\partial x}} = \overline{\frac{\partial Q}{\partial y}} = \overline{\frac{\partial Q}{\partial z}} = \overline{U\frac{\partial Q}{\partial x}} = \overline{V\frac{\partial Q}{\partial y}} = \overline{W\frac{\partial Q}{\partial z}} = 0,$$

and, on substituting these values, it is found that

$$\frac{\partial \nu}{\partial t}+\frac{\partial}{\partial x}\nu u+\frac{\partial}{\partial y}\nu v+\frac{\partial}{\partial z}\nu w=0, \tag{5.11}$$

which is the usual hydrodynamical equation of continuity; it may be used to reduce the left-hand side of (4.2) to

$$\nu\left[\frac{\partial \overline{Q}}{\partial t}+u\frac{\partial \overline{Q}}{\partial x}+v\frac{\partial \overline{Q}}{\partial y}+w\frac{\partial \overline{Q}}{\partial z}-\frac{\partial \overline{Q}}{\partial t}-u\frac{\partial \overline{Q}}{\partial x}-v\frac{\partial \overline{Q}}{\partial y}-w\frac{\partial \overline{Q}}{\partial z}\right.$$

$$\left.-\overline{U\frac{\partial Q}{\partial x}}-\overline{V\frac{\partial Q}{\partial y}}-\overline{W\frac{\partial Q}{\partial z}}-X\frac{\partial \overline{Q}}{\partial U}-Y\frac{\partial \overline{Q}}{\partial V}-Z\frac{\partial \overline{Q}}{\partial W}\right]$$

$$+\frac{\partial}{\partial x}\nu\overline{UQ}+\frac{\partial}{\partial y}\nu\overline{VQ}+\frac{\partial}{\partial z}\nu\overline{WQ}. \tag{5.12}$$

Case (ii). $\kappa=1, \quad r=0. \qquad \overline{Q}=B_{10}=0.$

$$\overline{UQ}=\tfrac{1}{2}(hm)^{-\frac{1}{2}}B_{20}^{(1)}, \quad \overline{VQ}=\tfrac{1}{2}(hm)^{-\frac{1}{2}}C_{20}^{(1)},$$

$$\overline{WQ}=(hm)^{-\frac{1}{2}}(1+B_{20}).$$

$$\frac{\overline{\partial Q}}{\partial U}=\frac{\overline{\partial Q}}{\partial V}=0, \quad \frac{\overline{\partial Q}}{\partial W}=2(hm)^{\frac{1}{2}}.$$

$$\frac{\overline{\partial Q}}{\partial x}=-2(hm)^{\frac{1}{2}}\frac{\partial w}{\partial x}, \quad \frac{\overline{\partial Q}}{\partial y}=-2(hm)^{\frac{1}{2}}\frac{\partial w}{\partial y}, \quad \frac{\overline{\partial Q}}{\partial z}=-2(hm)^{\frac{1}{2}}\frac{\partial w}{\partial z},$$

$$\frac{\overline{\partial Q}}{\partial t}=-2(hm)^{\frac{1}{2}}\frac{\partial w}{\partial t}.$$

$$\overline{U\frac{\partial Q}{\partial x}}=2\left\{\frac{\partial u}{\partial x}D_{00}^{(1)}+\frac{\partial v}{\partial x}G_{00}^{(1)}\right\}+\frac{1}{2h}\frac{\partial h}{\partial x}D_{10}$$

$$=\frac{1}{2h}\frac{\partial h}{\partial x}\frac{1}{2(hm)^{\frac{1}{2}}}B_{20}^{(1)},$$

from (4.36), (4.331), (4.332), remembering that A_1, B_{10}, $B_{10}^{(1)}$, and $C_{10}^{(1)}$ are all zero.

Similarly,

$$\overline{V\frac{\partial Q}{\partial y}}=\frac{1}{4h(hm)^{\frac{1}{2}}}C_{20}^{(1)}\frac{\partial h}{\partial y}, \qquad \overline{W\frac{\partial Q}{\partial z}}=\frac{1}{2h(hm)^{\frac{1}{2}}}\frac{\partial h}{\partial z}(1+B_{20}).$$

Hence Boltzmann's equation gives

$$\tfrac{1}{2}\left[\frac{\partial}{\partial x}\frac{\nu B_{20}^{(1)}}{(hm)^{\frac{1}{2}}}+\frac{\partial}{\partial y}\frac{\nu C_{20}^{(1)}}{(hm)^{\frac{1}{2}}}+2\frac{\partial}{\partial z}\frac{\nu B_{20}}{(hm)^{\frac{1}{2}}}+2\frac{\partial}{\partial z}\frac{\nu}{(hm)^{\frac{1}{2}}}\right]$$

$$=-2\nu(hm)^{\frac{1}{2}}\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}-Z-\frac{1}{4h^2m}\frac{\partial h}{\partial z}\right]$$

$$+\frac{\nu}{4h(hm)^{\frac{1}{2}}}\left[B_{20}^{(1)}\frac{\partial h}{\partial x}+C_{20}^{(1)}\frac{\partial h}{\partial y}+2B_{20}\frac{\partial h}{\partial z}\right],$$

i.e.

$$Z-\frac{dw}{dt}-\frac{1}{\nu m}\frac{\partial}{\partial z}\frac{\nu}{2h}=\frac{1}{4\nu m}\left[\frac{\partial}{\partial x}\frac{\nu B_{20}^{(1)}}{h}+\frac{\partial}{\partial y}\frac{\nu C_{20}^{(1)}}{h}+2\frac{\partial}{\partial z}\frac{\nu B_{20}}{h}\right], \qquad (5.2)$$

where

$$\frac{dw}{dt}=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}.$$

This and two symmetrical equations are the *exact* equations of mean motion of the gas; the terms on the right represent the effect of the viscous stresses in the gas and it is noteworthy that this effect depends only on the coefficients of the harmonics of the second degree in the distribution function, so that, so far as the mean motion is concerned, only the value of B_{20} need be calculated. The apparent lack of symmetry on the right-hand side of (5.2) is due to the choice of unsymmetrical solid harmonics in the specification of the distribution function (3.1).

Case (iii). $\kappa=0, \quad r=1. \qquad \bar{Q}=A_1=0. \qquad (3.23)$

$$\overline{UQ}=\frac{5}{6(hm)^{\frac{1}{2}}}B_{11}^{(1)}, \quad \overline{VQ}=\frac{5}{6(hm)^{\frac{1}{2}}}C_{11}^{(1)}, \quad \overline{WQ}=\frac{5}{6(hm)^{\frac{1}{2}}}B_{11}.$$

$$\frac{\overline{\partial Q}}{\partial U}=\frac{\overline{\partial Q}}{\partial V}=\frac{\overline{\partial Q}}{\partial W}=0.$$

$$\frac{\overline{\partial Q}}{\partial x}=-\frac{1}{h}\frac{\partial h}{\partial x}, \quad \frac{\overline{\partial Q}}{\partial y}=-\frac{1}{h}\frac{\partial h}{\partial y}, \quad \frac{\overline{\partial Q}}{\partial z}=-\frac{1}{h}\frac{\partial h}{\partial z},$$

$$\frac{\partial Q}{\partial t}=-\frac{1}{h}\frac{\partial h}{\partial t}.$$

$$\overline{U\frac{\partial Q}{\partial x}}=\tfrac{2}{3}\frac{\partial u}{\partial x}+\frac{5}{6h(hm)^{\frac{1}{2}}}\frac{\partial h}{\partial x}B_{11}^{(1)},$$

$$\overline{V\frac{\partial Q}{\partial y}}=\tfrac{2}{3}\frac{\partial v}{\partial y}+\frac{5}{6h(hm)^{\frac{1}{2}}}\frac{\partial h}{\partial y}C_{11}^{(1)},$$

$$\overline{W\frac{\partial Q}{\partial z}}=\tfrac{2}{3}\frac{\partial w}{\partial z}+\frac{5}{6h(hm)^{\frac{1}{2}}}\frac{\partial h}{\partial z}B_{11}.$$

When these values are substituted in (5.12), it is found that

$$\frac{1}{h}\frac{dh}{dt}-\tfrac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$=-\tfrac{5}{6}\frac{h}{\nu m^{\frac{1}{2}}}\left\{\frac{\partial}{\partial x}\frac{\nu B_{11}^{(1)}}{h^{\frac{3}{2}}}+\frac{\partial}{\partial y}\frac{\nu C_{11}^{(1)}}{h^{\frac{3}{2}}}+\frac{\partial}{\partial z}\frac{\nu B_{11}}{h^{\frac{3}{2}}}\right\}$$

$$+\tfrac{1}{6}\left\{2B_{20}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+2B_{20}^{(1)}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+2C_{20}^{(1)}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right.$$

$$\left.+B_{20}^{(2)}\left(\frac{\partial w}{\partial x}-\frac{\partial v}{\partial y}\right)+C_{20}^{(2)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}, \quad (5.3)$$

which is the *exact* equation of conservation of energy. The different terms in this expression are capable of physical interpretation; the left-hand side of the equation is proportional to the change of energy due to change in temperature and to adiabatic expansion, while on the right the first bracket represents the effect of heat conduction and the second the dissipation due to viscosity*.

For a first approximation the right-hand side may be neglected and the equation becomes

$$\frac{1}{h}\frac{dh}{dt}=\tfrac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \quad (5.31)$$

$$=-\tfrac{2}{3}\frac{1}{\nu}\frac{\partial \nu}{\partial t},$$

from (5.11), reducing to $h\nu^{\frac{3}{2}}=$ constant or $p\nu^{\frac{5}{3}}=$ constant.

6. For the calculation of the integral on the right of equation (4.2) it will be sufficient to evaluate

$$\int Q' f_1 f_2 C_R{}^{\mu} dU_1 dV_1 dW_1 dU_2 dV_2 dW_2 \alpha d\alpha d\epsilon, \quad (6.11)$$

since the corresponding integral with Q in place of Q' can be deduced by putting $\chi=0$. As has already been noticed, χ does not contain explicitly any of the variables of integration except α, so that the integrations with respect to ϵ and the velocity components can be performed regarding χ as a constant. Substituting for Q', f_1 and f_2, and leaving the integration with

* Compare the discussion of Maxwell's theory in Jeans, *Dynamical theory of gases* (3rd edition), ch. 9, 332.

respect to α to be performed later, we find that integrals have to be evaluated of the general type*

$$\int Y_\kappa(U_1', V_1', W_1')\, Y_{\kappa_1}^{(p_1)}(U_1, V_1, W_1)\, Y_{\kappa_2}^{(p_2)}(U_2, V_2, W_2) \exp\{-hm(C_1^2+C_2^2)\}$$
$$\times S_{\kappa+\frac{1}{2}}^{r}(hmC_1'^2)\, S_{\kappa_1+\frac{1}{2}}^{n_1}(hmC_1^2)\, S_{\kappa_2+\frac{1}{2}}^{n_2}(hmC_2^2)\, C_R^{\mu}\, dU_1 \ldots dW_2\, d\epsilon, \quad (6.12)$$

or, on using the definition of $S_m^n(x)$ (§ 2),

$$\int Y_\kappa(U_1', V_1', W_1')\, Y_{\kappa_1}^{(p_1)}(U_1, V_1, W_1)\, Y_{\kappa_2}^{(p_2)}(U_2, V_2, W_2)$$
$$\times \exp\{-hm(aC_1^2+bC_2^2+cC_1'^2)\}\, C_R^{\mu}\, dU_1 \ldots dW_2\, d\epsilon, \quad (6.13)$$

where
$$a = 1+\frac{s}{1-s} = \frac{1}{1-s}, \quad b = \frac{1}{1-t}, \quad c = \frac{u}{1-u}. \quad (6.14)$$

The integral (6.12) is got from (6.13) by multiplying it by

$$(1-s)^{-\kappa_1-\frac{3}{2}}(1-t)^{-\kappa_2-\frac{3}{2}}(1-u)^{-\kappa-\frac{3}{2}}/\Gamma(\kappa_1+n_1+\tfrac{3}{2})\,\Gamma(\kappa_2+n_2+\tfrac{3}{2})\,\Gamma(\kappa+r+\tfrac{3}{2})$$

and picking out the coefficient of $s^{n_1} t^{n_2} u^r$.

Since $0 \leqslant \mu \leqslant 1$,

$$C_R^{\mu} = \frac{\Gamma(1+\frac{1}{2}\mu)}{(hm)^{\frac{1}{2}\mu}}\, \frac{i}{2\pi} \int_{\infty}^{(0+)} \frac{d\beta}{(-\beta)^{1+\frac{1}{2}\mu}} \exp\{-hm\beta C_R^2\}, \quad (6.15)$$

and the factor C_R^{μ} can be omitted from the integrand in (6.13) if the exponential factor is replaced by

$$\exp\{-hm(aC_1^2+bC_2^2+cC_1'^2+\beta C_R^2)\};$$

there is no difficulty in justifying a change in the order of integration so that the integration with respect to β is performed last; this integration will in the meantime be omitted and β regarded as a constant. The results for Maxwellian molecules for which $\mu = 0$ can evidently be derived by putting $\beta = 0$.

---

* It is not necessary to consider separately the corresponding integrals with

$$Y_{\kappa_1}^{(p_1)}(U_1, V_1, W_1) \quad \text{or} \quad Y_{\kappa_2}^{(p_2)}(U_2, V_2, W_2)$$

—or both—replaced by the corresponding Z functions, since it is evident from the discussion of (6.12) that the integrals containing products like $Y_{\kappa_1}^{(p_1)}(U_1, V_1, W_1)\, Z_{\kappa_2}^{(p_2)}(U_2, V_2, W_2)$ are zero, while those containing $Z_{\kappa_1}^{(p_1)}(U_1, V_1, W_1)\, Z_{\kappa_2}^{(p_2)}(U_2, V_2, W_2)$ are equal to the corresponding integrals (6.12).

It will now be shown that if the value of the integral (6.13) (modified by the above substitution for C_R^{κ}) is known for all (integral) values of κ_1 and κ_2, and for all values of κ up to and including a specified value κ, then the integral with $\kappa+1$ written instead of κ can be deduced. Replace W_1 and W_2 by W_1-w_0 and W_2-w_0: then, since this is equivalent to superposing a velocity $-w_0$ on both molecules, W_1' will be changed to $W_1'-w_0$ and the relative velocity C_R will be unaltered. Evidently the value of the integral is unchanged, so that it is independent of w_0 and differentiation with respect to w_0 will give zero. Hence, on performing this differentiation [*cf.* (3.46)] and then putting w_0 equal to zero, we have

$$\int \exp\{-hm(aC_1^2+bC_2^2+cC_1'^2+\beta C_R^2)\}$$
$$\times[2hm(aW_1+bW_2+cW_1')\,Y_\kappa\,Y_{\kappa_1}^{(p_1)}\,Y_{\kappa_2}^{(p_2)}-\kappa Y_{\kappa-1}\,Y_{\kappa_1}^{(p_1)}\,Y_{\kappa_2}^{(p_2)}$$
$$-(\kappa_1+p_1)\,Y_\kappa\,Y_{\kappa_1-1}^{(p_1)}\,Y_{\kappa_2}^{(p_2)}-(\kappa_2+p_2)\,Y_\kappa\,Y_{\kappa_1}^{(p_1)}\,Y_{\kappa_2-1}^{(p_2)}]\,dU_1\ldots dW_2\,d\epsilon=0,$$

so that, on using (3.43),

$$\int \exp\{-hm(aC_1^2+bC_2^2+cC_1'^2+\beta C_R^2)\}\times$$
$$Y_\kappa\,Y_{\kappa_2}^{(p_2)}\left[\frac{2hma}{2\kappa_1+1}\{(\kappa_1-p_1+1)\,Y_{\kappa_1+1}^{(p_1)}+(\kappa_1+p_1)\,Y_{\kappa_1-1}^{(p_1)}\,C_1^2\}-(\kappa_1+p_1)\,Y_{\kappa_1-1}^{(p_1)}\right]$$
$$+Y_\kappa\,Y_{\kappa_1}^{(p_1)}\left[\frac{2hmb}{2\kappa_2+1}\{(\kappa_2-p_2+1)\,Y_{\kappa_2+1}^{(p_2)}+(\kappa_2+p_2)\,Y_{\kappa_2-1}^{(p_2)}\,C_2^2\}-(\kappa_2+p_2)\,Y_{\kappa_2-1}^{(p_2)}\right]$$
$$+Y_{\kappa_1}^{(p_1)}\,Y_{\kappa_2}^{(p_2)}\left[\frac{2hmc}{2\kappa+1}\{(\kappa+1)\,Y_{\kappa+1}+\kappa Y_{\kappa-1}\,C_1'^2\}-\kappa Y_{\kappa-1}\right]$$
$$\times dU_1\,dV_1\,dW_1\,dU_2\,dV_2\,dW_2\,d\epsilon=0. \quad (6.2)$$

In these formulae $Y_{\kappa_1}^{(p_1)}$, $Y_{\kappa_2}^{(p_2)}$, Y_κ are written for $Y_{\kappa_1}^{(p_1)}(U_1, V_1, W_1)$, $Y_{\kappa_2}^{(p_2)}(U_2, V_2, W_2)$, and $Y_\kappa(U_1', V_1', W_1')$ respectively. The integrals with the factors $Y_{\kappa_1-1}^{(p_1)}C_1^2$, $Y_{\kappa_2-1}^{(p_2)}C_2^2$, $Y_{\kappa-1}C_1'^2$, are got by differentiating known integrals with respect to a, b, or c; and so

$$\int Y_{\kappa+1}\,Y_{\kappa_1}^{(p_1)}\,Y_{\kappa_2}^{(p_2)}\exp\{-hm(aC_1^2+bC_2^2+cC_1'^2+\beta C_R^2)\}\,dU_1\ldots dW_2\,d\epsilon$$

is expressed in terms of known integrals.

It follows that it is sufficient to evaluate the integral in the case $\kappa=0$; to do this we consider first the integration with respect to ϵ. If θ_{12} is the

angle between the directions of C_1 and C_2, it is easily shown* that

$$C_1'^2 = C_1^2 \cos^2 \tfrac{1}{2}\chi + C_2^2 \sin^2 \tfrac{1}{2}\chi + C_1 C_2 \sin\chi \sin\theta_{12} \cos\epsilon, \quad (6.311)$$

$$C_R^2 = C_1^2 + C_2^2 - 2C_1 C_2 \cos\theta_{12}, \quad (6.312)$$

from which it appears that, when κ is zero, ϵ occurs in the integrand only through the term $-hmc\, C_1 C_2 \sin\chi \sin\theta_{12} \cos\epsilon$ in the exponential. The integral with respect to ϵ can be transformed by means of the relation

$$\int_0^{2\pi} \exp(-hmcC_1 C_2 \sin\chi \sin\theta_{12} \cos\epsilon)\, d\epsilon$$

$$= \int_0^{2\pi} \exp\{-hmcC_1 C_2 \sin\chi\, (\sin\phi - i \cos\theta_{12} \cos\phi)\}\, d\phi,$$

both these expressions being equal to a Bessel function of zero order, as can readily be verified by expanding the integrands and integrating term by term. The terms involving $\cos\theta_{12}$ in the integrand so modified can now be expanded by means of a known result in the form

$$\exp\{hm\, C_1 C_2 \cos\theta_{12}(2\beta + ic \sin\chi \cos\phi)\}$$

$$= (2\pi)^{\frac{1}{2}} \sum_n (n+\tfrac{1}{2}) \frac{I_{n+\frac{1}{2}}(x)}{x^{\frac{1}{2}}} P_n(\cos\theta_{12}), \quad (6.331)$$

where $x = hm\, C_1 C_2 (2\beta + ic \sin\chi \cos\phi)$.

Let $U_1 = C_1 \sin\theta_1 \cos\phi_1$, $U_2 = C_2 \sin\theta_2 \cos\phi_2$, etc., and use the result

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, Y_m(\theta, \phi)\, P_n(\cos\theta_1) = 0$$

unless $m = n$, in which case it is $4\pi\, Y_n(\theta', \phi')/(2n+1)$, where (θ', ϕ') is the pole of the harmonic $P_n(\cos\theta_1)$. Hence in the present case the integral with respect to θ_2 and ϕ_2 is

$$C_2^{\kappa_2}\, Y_{\kappa_2}^{(p_2)}(\theta_1, \phi_1)(2\pi)^{\frac{3}{2}} \frac{I_{\kappa_2+\frac{1}{2}}(x)}{x^{\frac{1}{2}}}. \quad (6.332)$$

We now multiply this by $Y_{\kappa_1}^{(p_1)}$ and integrate with respect to θ_1 and ϕ_1; by the orthogonal property of harmonic functions the result is zero unless $p_2 = p_1$ and $\kappa_2 = \kappa_1$, in which case it is

$$C_1^{\kappa_1} C_2^{\kappa_1} \frac{(2\pi)^{\frac{3}{2}}}{2\kappa_1 + 1} \frac{(\kappa_1 + p_1)!}{(\kappa_1 - p_1)!} \frac{I_{\kappa_1+\frac{1}{2}}(x)}{x^{\frac{1}{2}}},$$

* From the expressions for the velocities given by Jeans or Chapman. It will be noted that the θ' of Jeans' analysis is $\pi - \chi$.

if p_1 is not zero and twice this value if $p_1 = 0$: the two results may be combined by inserting a factor η which is defined to be 2 when p_1 is zero and 1 otherwise. Now

$$\frac{I_{\kappa_1+\frac{1}{2}}(x)}{x^{\frac{1}{2}}} = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\pi \exp(-x\cos\vartheta)\, P_{\kappa_1}(\cos\vartheta)\sin\vartheta\, d\vartheta,$$

so that the original integral may be written as

$$\frac{4\pi^2\eta}{2\kappa_1+1}\frac{(\kappa_1+p_1)!}{(\kappa_1-p_1)!}\int_0^\infty\int_0^\infty C_1^{\kappa_1+2}\, C_2^{\kappa_1+2}$$

$$\times\exp\left[-hm\{(a+\beta+c\cos^2\tfrac{1}{2}\chi)\, C_1^{\,2}+(b+\beta+c\sin^2\tfrac{1}{2}\chi)\, C_2^{\,2}\}\right] dC_1\, dC_2$$

$$\times\int_0^\pi P_{\kappa_1}(\cos\vartheta)\sin\vartheta\, d\vartheta\int_0^{2\pi} d\phi$$

$$\times\exp\{-hm\, C_1\, C_2(2\beta\cos\vartheta+ic\sin\chi\cos\phi\cos\vartheta+c\sin\chi\sin\phi)\}.$$

The integral with respect to ϕ can be transformed as in (6.32) and becomes

$$\int_0^{2\pi}\exp(-hmc\, C_1\, C_2\sin\chi\sin\vartheta\cos\epsilon)\, d\epsilon,$$

so that the double integral with respect to ϑ and ϕ is equal to

$$\int_0^\pi d\vartheta\int_0^{2\pi} d\epsilon\, P_{\kappa_1}(\cos\vartheta)\sin\vartheta\exp\{-hm\, C_1\, C_2(2\beta\cos\vartheta+c\sin\vartheta\sin\chi\cos\epsilon)\}.$$

Now let $$\beta = r\cos\psi, \quad c\sin\chi = 2r\sin\psi,$$

$$\cos\vartheta\cos\psi+\sin\vartheta\sin\psi\cos\epsilon = \cos\omega,$$

$$\cos\eta = \frac{\cos\theta-\cos\omega\cos\psi}{\sin\omega\sin\psi}, \tag{6.34}$$

and take ω, η as new angular coordinates* in place of ϑ, ϵ; the integral is

$$\int_0^\pi\sin\omega\, d\omega\int_0^{2\pi} d\eta\, P_{\kappa_1}(\cos\psi\cos\omega+\sin\psi\sin\omega\cos\eta)\exp(-2hmrC_1\, C_2\cos\omega)$$

$$= (2\pi)^{\frac{3}{2}}\frac{I_{\kappa_1+\frac{1}{2}}(2hmr\, C_1\, C_2)}{(2hmr\, C_1\, C_2)^{\frac{1}{2}}}\, P_{\kappa_1}(\cos\psi),$$

* This angle η, introduced only for the purpose of evaluating the integral, will not be confused with the factor η above.

as in (6.332). The integrations with respect to C_1 and C_2 can now be carried out with the help of known results in the theory of Bessel functions*; the integral can be written as $I(\kappa_1, p_1) P_{\kappa_1}(\cos\psi)$, where

$$I(\kappa_1, p_1) = \frac{(2\pi)^{\frac{7}{2}}\eta}{2\kappa_1+1}\frac{(\kappa_1+p_1)!}{(\kappa_1-p_1)!}\frac{1}{(2hmr)^{\frac{1}{2}}}\int_0^\infty\int_0^\infty C_1^{\kappa_1+\frac{3}{2}} C_2^{\kappa_1+\frac{3}{2}} I_{\kappa_1+\frac{1}{2}}(2hmrC_1C_2)$$

$$\times\exp[-hm\{(a+\beta+c\cos^2\tfrac{1}{2}\chi)C_1^2+(b+\beta+c\sin^2\tfrac{1}{2}\chi)C_2^2\}]\,dC_1\,dC_2$$

$$= \eta\frac{(\kappa_1+p_1)!}{(\kappa_1-p_1)!}\frac{\pi^{\frac{7}{2}}\Gamma(\kappa_1+\frac{1}{2})}{(hm)^{\kappa_1+3}}\frac{r^{\kappa_1}}{\{(a+\beta+c\cos^2\frac{1}{2}\chi)(b+\beta+c\sin^2\frac{1}{2}\chi)-r^2\}^{\kappa_1+\frac{3}{2}}}$$

$$= \eta\frac{(\kappa_1+p_1)!}{(\kappa_1-p_1)!}\frac{\pi^{\frac{7}{2}}\Gamma(\kappa_1+\frac{1}{2})}{(hm)^{\kappa_1+3}}\frac{r^{\kappa_1}}{\{ab+ac\sin^2\frac{1}{2}\chi+bc\cos^2\frac{1}{2}\chi+\beta(a+b+c)\}^{\kappa_1+\frac{3}{2}}} \qquad (6.35)$$

by substituting for r^2 from (6.34).

For Maxwellian molecules β is zero; and, when the values of a, b, c are substituted from (6.14), it is found that

$$\Gamma(\kappa_1+n_1+\tfrac{3}{2})\Gamma(\kappa_2+n_2+\tfrac{3}{2})\int Y_{\kappa_1}^{(p_1)} Y_{\kappa_2}^{(p_2)} S_{\frac{1}{2}}^{r}(hmC_1'^2)\, S_{\kappa_1+\frac{1}{2}}^{n_1}(hmC_1^2)\, S_{\kappa_2+\frac{1}{2}}^{n_2}(hmC_2^2)$$

$$\times\exp\{-hm(C_1^2+C_2^2)\}\,dU_1\ldots dW_2\,d\epsilon$$

is zero unless $\kappa_2=\kappa_1$ and $p_2=p_1$, in which case it is the coefficient of $s^{n_1}t^{n_2}u^r$ in the expansion of

$$\eta\frac{(\kappa_1+p_1)!}{(\kappa_1-p_1)!}\frac{\pi^{\frac{7}{2}}\Gamma(\kappa_1+\frac{1}{2})}{(hm)^{\kappa_1+3}\,\Gamma(r+\frac{3}{2})}\frac{(u\sin\frac{1}{2}\chi\cos\frac{1}{2}\chi)^{\kappa_1}P_{\kappa_1}(0)}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^{\kappa_1+\frac{3}{2}}}, \qquad (6.36)$$

since, when $\beta=0$, $\psi=\frac{1}{2}\pi$ and $r=c\sin\frac{1}{2}\chi\cos\frac{1}{2}\chi$ from (6.34). It is evident that the result is zero unless κ_1 is even and $\kappa_1+n_1+n_2=r$.

In the general case when μ is not zero, we still have to evaluate [*cf.* (6.15)]

$$\frac{\Gamma(1+\frac{1}{2}\mu)}{(hm)^{\frac{1}{2}\mu}}\frac{i}{2\pi}\int_\infty^{(0+)}\frac{d\beta}{(-\beta)^{1+\frac{1}{2}\mu}}\frac{r^{\kappa_1}P_{\kappa_1}(\cos\psi)}{\{ab+ac\sin^2\frac{1}{2}\chi+bc\cos^2\frac{1}{2}\chi+\beta(a+b+c)\}^{\kappa_1+\frac{3}{2}}}$$

$$= \frac{\Gamma(1+\frac{1}{2}\mu)}{(hm)^{\frac{1}{2}\mu}}\frac{i}{2\pi}\int_\infty^{(0+)}\frac{d\beta}{(-\beta)^{1+\frac{1}{2}\mu}}r^{\kappa_1}P_{\kappa_1}(\cos\psi)\int_0^\infty\frac{x^{\kappa_1+\frac{1}{2}}dx}{\Gamma(\kappa_1+\frac{3}{2})}$$

$$\times\exp[-x\{ab+ac\sin^2\tfrac{1}{2}\chi+bc\cos^2\tfrac{1}{2}\chi+\beta(a+b+c)\}]. \qquad (6.41)$$

---

* *Cf.* Watson, *Theory of Bessel functions* (1922), (13.3).

Now $r\cos\psi=\beta$, $2r\sin\psi=c\sin\chi$, and so

$$r^{\kappa_1}P_{\kappa_1}(\cos\psi)=\frac{1}{\pi}\int_0^\pi(\beta+\tfrac{1}{2}ic\sin\chi\cos\phi)^{\kappa_1}d\phi,$$

and this is $\kappa_1!$ times the coefficient of y^{κ_1} in

$$\frac{1}{\pi}\int_0^\pi\exp\{y(\beta+\tfrac{1}{2}ic\sin\chi\cos\phi)\}d\phi.$$

Hence the integral (6.41) is the coefficient of y^{κ_1} in

$$\frac{\Gamma(1+\frac{1}{2}\mu)}{(hm)^{\frac{1}{2}\mu}}\frac{\kappa_1!}{\Gamma(\kappa_1+\frac{3}{2})}\frac{i}{2\pi^2}\int_\infty^{(0+)}\frac{d\beta}{(-\beta)^{1+\frac{1}{2}\mu}}\int_0^\infty x^{\kappa_1+\frac{1}{2}}dx\int_0^\pi d\phi$$

$$\times\exp\{-x(ab+ac\sin^2\tfrac{1}{2}\chi+bc\cos^2\tfrac{1}{2}\chi)-\beta x(a+b+c)+(\beta+\tfrac{1}{2}ic\sin\chi\cos\phi)y\},$$

or, what is the same thing, the coefficient of y^{κ_1} in

$$\frac{\Gamma(1+\frac{1}{2}\mu)}{(hm)^{\frac{1}{2}\mu}}\frac{\kappa_1!}{\Gamma(\kappa_1+\frac{3}{2})}\frac{i}{2\pi^2}\int_\infty^{(0+)}\frac{d\beta}{(-\beta)^{1+\frac{1}{2}\mu}}\int_0^\infty x^{\frac{1}{2}}dx\int_0^\pi d\phi$$

$$\times\exp\{-x(ab+ac\sin^2\tfrac{1}{2}\chi+bc\cos^2\tfrac{1}{2}\chi)-\beta x(a+b+c-y)+\tfrac{1}{2}icxy\sin\chi\cos\phi\}.$$

Change of the order of integration is permissible and, when the integrations with respect to β and x are carried out, the result is found to be

$$\frac{\Gamma(\frac{3}{2}+\frac{1}{2}\mu)}{(hm)^{\frac{1}{2}\mu}}\frac{\kappa_1!}{\Gamma(\kappa_1+\frac{3}{2})}\frac{(a+b+c-y)^{\frac{1}{2}\mu}}{\pi}$$

$$\times\int_0^\pi\frac{d\phi}{\{ab+ac\sin^2\tfrac{1}{2}\chi+bc\cos^2\tfrac{1}{2}\chi-\tfrac{1}{2}icy\sin\chi\cos\phi\}^{\frac{3}{2}+\frac{1}{2}\mu}}.$$

The remaining integration with respect to ϕ can be carried out in terms of a Legendre function but, since the coefficients of the various powers of y are required, it is more convenient to expand the integrand in powers of $\cos\phi$ and integrate term by term; it is thus found that the integral is equal to

$$\frac{\Gamma(\frac{3}{2}+\frac{1}{2}\mu)}{(hm)^{\frac{1}{2}\mu}}\frac{\kappa_1!}{\Gamma(\kappa_1+\frac{3}{2})}\frac{(a+b+c-y)^{\frac{1}{2}\mu}}{(ab+ac\sin^2\tfrac{1}{2}\chi+bc\cos^2\tfrac{1}{2}\chi)^{\frac{3}{2}+\frac{1}{2}\mu}}$$

$$\times\Big[1-\frac{(\mu+3)(\mu+5)}{2^6}\frac{c^2y^2\sin^2\chi}{(ab+ac\sin^2\tfrac{1}{2}\chi+bc\cos^2\tfrac{1}{2}\chi)^2}$$

$$+\frac{(\mu+3)(\mu+5)(\mu+7)(\mu+9)}{2^{13}}\frac{c^4y^4\sin^4\chi}{(ab+ac\sin^2\tfrac{1}{2}\chi+bc\cos^2\tfrac{1}{2}\chi)^4}-\dots\Big],$$

and so, when the values of a, b, c are substituted,

$$\Gamma(\kappa_1+n_1+\tfrac{3}{2})\,\Gamma(\kappa_2+n_2+\tfrac{3}{2})\int Y_{\kappa_1}^{(p_1)}\,Y_{\kappa_2}^{(p_2)}\,S_{\frac{1}{2}}^{r}(hmC_1'^2)\,S_{\kappa_1+\frac{1}{2}}^{n_1}(hmC_1^2)\,S_{\kappa_2+\frac{1}{2}}^{n_2}(hmC_2^2)$$
$$\times C_R{}^{\mu}\exp\{-hm(C_1^2+C_2^2)\}\,dU_1\ldots dW_2\,d\epsilon$$

is zero unless $\kappa_2=\kappa_1$ and $p_2=p_1$, in which case it is the coefficient of $s^{n_1}t^{n_2}w^r y^{\kappa_1}$ in the expansion of

$$2\eta\,\frac{\kappa_1!}{2\kappa_1+1}\,\frac{(\kappa_1+p_1)!}{(\kappa_1-p_1)!}\,\frac{\pi^{\frac{1}{2}}}{\Gamma(r+\frac{3}{2})}\,\frac{\Gamma(\frac{3}{2}+\frac{1}{2}\mu)}{(hm)^{\kappa_1+3+\frac{1}{2}\mu}}$$
$$\times\frac{\{2-u-s-t+ust-y(1-u)(1-s)(1-t)\}^{\frac{1}{2}\mu}}{\{1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi\}^{\frac{3}{2}+\frac{1}{2}\mu}}\,(1-s)^{-\kappa_1}\,(1-t)^{-\kappa_1}$$
$$\times\Big[1-\frac{(\mu+3)(\mu+5)}{2^6}\,\frac{u^2y^2(1-s)^2(1-t)^2\sin^2\chi}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^2}$$
$$+\frac{(\mu+3)(\mu+5)(\mu+7)(\mu+9)}{2^{13}}\,\frac{u^4y^4(1-s)^4(1-t)^4\sin^4\chi}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^4}-\ldots\Big]. \quad (6.42)$$

Integrals of the type (6.13) with $\kappa=1,2,3,\ldots$, may now be calculated in succession by the method of differentiation explained in (6.2); it can easily be established by induction that

$$\int Y_\kappa\,Y_{\kappa_1}^{(p_1)}\,Y_{\kappa_2}^{(p_2)}\exp\{-hm(aC_1^2+bC_2^2+cC_1'^2+\beta C_R^2)\}\,dU_1\ldots dW_2\,d\epsilon$$

is zero unless $p_2=p_1$ and $\kappa=|\kappa_2-\kappa_1|+2n$, n being a positive integer or zero, and that, if $\kappa_2>\kappa_1$, $\kappa=\kappa_2-\kappa_1$, $p_2=p_1$, its value is

$$\frac{1.3.5\ldots(2\kappa-1)}{\kappa!\,c^\kappa}\,\frac{(\kappa_1-p_1+1)\ldots(\kappa_1-p_1+\kappa)}{(2\kappa_1+1)\ldots(2\kappa_1+2\kappa-1)}\,I(\kappa+\kappa_1,p_1)$$
$$\times\Big[x^\kappa P_{\kappa_1}-\kappa x^{\kappa-1}aP_{\kappa_1+1}+\frac{\kappa(\kappa-1)}{2!}\,x^{\kappa-2}a^2P_{\kappa_1+2}-\ldots\pm a\;P_{\kappa_1+\kappa}\Big], \quad (6.51)$$

where
$$x=a\tan\tfrac{1}{2}\chi\sin\psi+(a+c)\cos\psi, \quad (6.52)$$

the argument of each Legendre function is $\cos\psi$, and $I(\kappa,p)$ is the function defined in (6.35). If $\kappa_1>\kappa_2$, $\kappa=\kappa_1-\kappa_2$, $p_2=p_1$, the corresponding result is got by interchanging κ_1 and κ_2, a and b, and writing $\pi-\chi$ for χ. In the special case of Maxwellian molecules,

$$\beta=0,\quad \psi=\tfrac{1}{2}\pi,\quad r=c\sin\tfrac{1}{2}\chi\cos\tfrac{1}{2}\chi,\quad x=a\tan\tfrac{1}{2}\chi, \quad (6.53)$$

and (6.51) takes the form

$$\frac{1.3.5\ldots(2\kappa-1)}{\kappa!}\eta\,\frac{(\kappa+\kappa_1+p_1)!}{(\kappa_1-p_1)!}\,\frac{\pi^{\frac{7}{2}}\Gamma(\kappa_1+\frac{1}{2})}{2^\kappa(hm)^{\kappa_1+\kappa+3}}\,\frac{a^\kappa c^{\kappa_1}(\sin\frac{1}{2}\chi\cos\frac{1}{2}\chi)^{\kappa+\kappa_1}}{\{ab+ac\sin^2\frac{1}{2}\chi+bc\cos^2\frac{1}{2}\chi\}^{\kappa+\kappa_1+\frac{3}{2}}}$$

$$\times\Big[\tan^\kappa\tfrac{1}{2}\chi\,P_{\kappa_1}-\kappa\tan^{\kappa-1}\tfrac{1}{2}\chi\,P_{\kappa_1+1}+\frac{\kappa(\kappa-1)}{2!}\tan^{\kappa-2}\tfrac{1}{2}\chi\,P_{\kappa_1+2}-\ldots\pm P_{\kappa_1+\kappa}\Big],\qquad(6.54)$$

in which the argument of the Legendre function is now zero; and, on substituting for a, b, c, we find that

$$\Gamma(\kappa_1+n_1+\tfrac{3}{2})\,\Gamma(\kappa_2+n_2+\tfrac{3}{2})\int Y_\kappa S^r_{\kappa+\frac{1}{2}}Y^{(p_1)}_{\kappa_1}Y^{(p_2)}_{\kappa_2}S^{n_1}_{\kappa_1+\frac{1}{2}}S^{n_2}_{\kappa_2+\frac{1}{2}}$$

$$\times\exp\{-hm(C_1^2+C_2^2)\}\,dU_1\ldots dW_2\,d\epsilon$$

is equal to the coefficient of $s^{n_1}t^{n_2}u^r$ in the expansion of

$$\frac{\eta\pi^3}{(hm)^{\kappa+\kappa_1+3}}\,\frac{\Gamma(\kappa+\frac{1}{2})\,\Gamma(\kappa_1+\frac{1}{2})}{\kappa!\,\Gamma(\kappa+r+\frac{3}{2})}\,\frac{(\kappa+\kappa_1+p_1)!}{(\kappa_1-p_1)!}\,\frac{u^{\kappa_1}(\sin\frac{1}{2}\chi\cos\frac{1}{2}\chi)^{\kappa+\kappa_1}}{\{1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi\}^{\kappa+\kappa_1+\frac{3}{2}}}$$

$$\times\Big[\tan^\kappa\tfrac{1}{2}\chi\,P_{\kappa_1}(0)-\kappa\tan^{\kappa-1}\tfrac{1}{2}\chi\,P_{\kappa_1+1}(0)+\frac{\kappa(\kappa-1)}{2!}\tan^{\kappa-2}\tfrac{1}{2}\chi\,P_{\kappa_1+2}(0)-\ldots\Big],\qquad(6.55)$$

when $\kappa=\kappa_2-\kappa_1$.

The integration of (6.51) with respect to β follows the same lines as before; in fact,

$$r^{\kappa+\kappa_1}[x^\kappa P_{\kappa_1}(\cos\psi)-\kappa x^{\kappa-1}a\,P_{\kappa_1+1}(\cos\psi)+\ldots]$$

$$=\frac{1}{\pi}\int_0^\pi d\phi(\beta+\tfrac{1}{2}ic\sin\chi\cos\phi)^{\kappa_1}\{xr-a(\beta+\tfrac{1}{2}ic\sin\chi\cos\phi)\}^\kappa$$

$$=\frac{c^\kappa}{\pi}\int_0^\pi d\phi(\beta+\tfrac{1}{2}ic\sin\chi\cos\phi)^{\kappa_1}(\beta+a\sin^2\tfrac{1}{2}\chi-\tfrac{1}{2}ia\sin\chi\cos\phi)^\kappa,$$

on substituting the values of x and r; now this is $\kappa!\kappa_1!c^\kappa\times$coefficient of $y^{\kappa_1}z^\kappa$ in

$$\frac{1}{\pi}\int_0^\pi d\phi\exp\{y(\beta+\tfrac{1}{2}ic\sin\chi\cos\phi)+z(\beta+a\sin^2\tfrac{1}{2}\chi-\tfrac{1}{2}ia\sin\chi\cos\phi)\},$$

and, when the integrations are performed, it is found that, if $p_2=p_1$, $\kappa=\kappa_2-\kappa_1$ $(\kappa_2>\kappa_1)$,

$$\Gamma(\kappa_1+n_1+\tfrac{3}{2})\,\Gamma(\kappa_2+n_2+\tfrac{3}{2})\int Y_\kappa S^r_{\kappa+\frac{1}{2}}Y^{(p_1)}_{\kappa_1}Y^{(p_2)}_{\kappa_2}S^{n_1}_{\kappa_1+\frac{1}{2}}S^{n_2}_{\kappa_2+\frac{1}{2}}$$

$$\times C_R{}^\mu\exp\{-hm(C_1^2+C_2^2)\}\,dU_1\ldots dW_2\,d\epsilon$$

is the coefficient of $s^{n_1} t^{n_2} u^r y^{\kappa_1} z^\kappa$ in the expansion of

$$\eta\kappa_1! \frac{\Gamma(\kappa+\frac{1}{2})\,\Gamma(\kappa_1+\frac{1}{2})}{\Gamma(\kappa+\kappa_1+\frac{3}{2})} \frac{(\kappa+\kappa_1+p_1)!}{(\kappa_1-p_1)!} \frac{\pi^3\,\Gamma(\frac{3}{2}+\frac{1}{2}\mu)}{(hm)^{\kappa+\kappa_1+3+\frac{1}{2}\mu}\,\Gamma(\kappa+r+\frac{3}{2})}$$

$$\times \frac{\{2-u-s-t+ust-(y+z)(1-u)(1-s)(1-t)\}^{\frac{1}{2}\mu}}{\{1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi-z(1-u)(1-t)\sin^2\frac{1}{2}\chi\}^{\frac{3}{2}+\frac{1}{2}\mu}}$$

$$\times (1-u)^{-\kappa}(1-s)^{-\kappa_1}(1-t)^{-\kappa_2}$$

$$\times\left[1-\frac{(\mu+3)(\mu+5)}{2^6}\frac{(1-t)^2\sin^2\chi\{z(1-u)-yu(1-s)\}^2}{\{1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi-z(1-u)(1-t)\sin^2\frac{1}{2}\chi\}^2}+\dots\right]. \tag{6.55}$$

The corresponding formulae when $\kappa > |\kappa_2-\kappa_1|$ can also be found without any special difficulty but, except in the case of Maxwellian molecules, they appear to be somewhat unwieldy. The main interest of the investigation, however, lies in the determination of the stresses, which depend only on B_{20}, $B_{20}^{(1)}$, $C_{20}^{(1)}$; and these require only the results when $\kappa = 2$. Now the general integral vanishes unless $\kappa = |\kappa_2-\kappa_1|+2n$, where n is a positive integer or zero, and so when $\kappa = 2$ we must have either $\kappa = |\kappa_2-\kappa_1|$, the case already considered, corresponding to $n = 0$, or else $\kappa_2 = \kappa_1$, $n = 1$. In this latter case it is found that

$$\int Y_2\, Y_{\kappa_1}^{(p_1)}\, Y_{\kappa_2}^{(p_2)} \exp\{-hm(aC_1^2+bC_2^2+cC_1'^2+\beta C_R^2)\}\,dU_1\dots dW_2\,d\epsilon$$

is zero if p_1 and p_2 are unequal, while, if $p_2 = p_1$, it is equal to

$$\frac{3}{c^2}\frac{\kappa_1-p_1+1}{\kappa_1+p_1+1}\frac{\kappa_1^2+\kappa_1-3p_1^2}{4\kappa_1^2-1}\,I(\kappa_1+1,\,p_1)$$

$$\times[(\tfrac{2}{3}ab+\tfrac{1}{3}xy)\,P_{\kappa_1-1}-\tfrac{1}{3}(ay+bx+4ab\cos\psi)\,P_{\kappa_1}+ab\,P_{\kappa_1+1}], \tag{6.61}$$

where $$x = a\tan\tfrac{1}{2}\chi\sin\psi+(a+c)\cos\psi,$$

as in (6.52),

$$y = b\cot\tfrac{1}{2}\chi\sin\psi+(b+c)\cos\psi,$$

and the argument of the Legendre functions is $\cos\psi$. It will be noted that (6.61) vanishes when $\kappa_1 = 0$, since p_1 is essentially not greater than κ_1 and so the factor $\kappa_1^2+\kappa_1-3p_1^2$ is zero. For Maxwellian molecules $\psi = \frac{1}{2}\pi$, $x = a\tan\frac{1}{2}\chi$, $y = b\cot\frac{1}{2}\chi$, and (6.61) takes the form

$$\frac{3ab}{c^2}\frac{\kappa_1-p_1+1}{\kappa_1+p_1+1}\frac{\kappa_1^2+\kappa_1-3p_1^2}{4\kappa_1^2-1}\,I(\kappa_1+1,\,p_1)\,[P_{\kappa_1-1}-\tfrac{2}{3}P_{\kappa_1}+P_{\kappa_1+1}],$$

in which the argument of the Legendre functions is now zero. The corresponding integral for general values of κ (subject to the condition that $\kappa = \kappa_2 - \kappa_1 + 2$) for Maxwellian molecules has the value

$$\frac{1.3.5\ldots(2\kappa-3)}{(\kappa-1)!}\,\frac{a^{\kappa-1}b}{c^{\kappa}}\,\frac{(\kappa_1-p_1+1)\ldots(\kappa_1-p_1+\kappa-1)}{(2\kappa_1-1)\ldots(2\kappa_1+2\kappa-3)}$$

$$\times\frac{\kappa_1(\kappa_1+\kappa-1)-(2\kappa-1)p_1^{\,2}}{\kappa_1+p_1+\kappa-1}\,I(\kappa_1+\kappa-1,\,p_1)$$

$$\times\Big[(2\kappa-1)\tan^{\kappa-2}\tfrac{1}{2}\chi\,P_{\kappa_1-1}-(\kappa-1)(2\kappa-3)\tan^{\kappa-3}\tfrac{1}{2}\chi\,P_{\kappa_1}$$

$$+\frac{(\kappa-1)(\kappa-2)}{2!}\,(2\kappa-5)\tan^{\kappa-4}\tfrac{1}{2}\chi\,P_{\kappa_1+1}+\ldots$$

$$-\tan^{\kappa-1}\tfrac{1}{2}\chi\,P_{\kappa_1}+(\kappa-1).3.\tan^{\kappa-2}\tfrac{1}{2}\chi\,P_{\kappa_1+1}+\ldots\Big];$$

and similar results hold when $\kappa = \kappa_2-\kappa_1+4$, $\kappa = \kappa_2-\kappa_1+6$, ..., but, as noted above, the corresponding formulae for molecules not of Maxwell's type are not nearly so simple in form; fortunately they are not necessary except for a very complete investigation of the distribution function.

In order to complete the integration in the general case we multiply (6.61) by

$$\frac{\Gamma(1+\frac{1}{2}\mu)}{(hm)^{\frac{1}{2}\mu}}\,\frac{i}{2\pi}\,\frac{1}{(-\beta)^{1+\frac{1}{2}\mu}}$$

and integrate with respect to β as in (6.15); in order to get results correct to the second order of small quantities only the values when $\kappa_1 = 0$, 1, or 2 are required. When $\kappa_1 = 0$, the value of the integral is zero as above: the other two cases are most conveniently worked out separately and it is easily found that, when $\kappa_1 = 1$,

$$\int Y_2(U_1', V_1', W_1')\,Y_1^{(p)}(U_1, V_1, W_1)\,Y_1^{(p)}(U_2, V_2, W_2)$$

$$\times\exp\{-hm(aC_1^{\,2}+bC_2^{\,2}+cC_1'^{2})\}C_R^{\,\mu}\,dU_1\ldots dW_2\,d\epsilon$$

$$=\frac{3\pi^4}{4(hm)^{5+\frac{1}{2}\mu}}\,\frac{2-3p^2}{\Gamma(\frac{7}{2})}\,\Gamma\Big(\frac{\mu+7}{2}\Big)\Big[\frac{ab\sin^2\frac{1}{2}\chi\cos^2\frac{1}{2}\chi(a+b+c)^{\frac{1}{2}\mu}}{(ab+ac\sin^2\frac{1}{2}\chi+bc\cos^2\frac{1}{2}\chi)^{\frac{7}{2}+\frac{1}{2}\mu}}$$

$$-\frac{2\mu}{3(\mu+5)}\,\frac{(a\sin^2\frac{1}{2}\chi+b\cos^2\frac{1}{2}\chi)(a+b+c)^{\frac{1}{2}\mu-1}}{(ab+ac\sin^2\frac{1}{2}\chi+bc\cos^2\frac{1}{2}\chi)^{\frac{3}{2}+\frac{1}{2}\mu}}$$

$$+\tfrac{2}{3}\,\frac{\mu(\mu-2)}{(\mu+3)(\mu+5)}\,\frac{(a+b+c)^{\frac{1}{2}\mu-2}}{(ab+ac\sin^2\frac{1}{2}\chi+bc\cos^2\frac{1}{2}\chi)^{\frac{3}{2}+\frac{1}{2}\mu}}\Big]$$

and that

$$\Gamma(n_1+\tfrac{5}{2})\,\Gamma(n_2+\tfrac{5}{2})\int Y_2(U_1',\,V_1',\,W_1')\,S_{\frac{5}{2}}^{r}(hmC_1'^2)\,Y_1^{(p)}(U_1,\,V_1,\,W_1)$$
$$\times Y_1^{(p)}(U_2,\,V_2,\,W_2)\,S_{\frac{3}{2}}^{n_1}(hmC_1^2)\,S_{\frac{3}{2}}^{n_2}(hmC_2^2)\,C_R{}^{\mu}$$
$$\times\exp\{-hm(C_1^2+C_2^2)\}\,dU_1\ldots dW_2\,d\epsilon$$

is equal to the coefficient of $s^{n_1}t^{n_2}u^r$ in the expansion of

$$\frac{3\pi^4}{4(hm)^{5+\frac{1}{2}\mu}}\,\frac{2-3p^2}{\Gamma(\frac{7}{2})}\,\frac{\Gamma(\frac{7}{2}+\frac{1}{2}\mu)}{\Gamma(r+\frac{7}{2})}\left[\frac{\sin^2\frac{1}{2}\chi\cos^2\frac{1}{2}\chi\,(2-u-s-t+ust)^{\frac{1}{2}\mu}}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^{\frac{7}{2}+\frac{1}{2}\mu}}\right.$$
$$-\frac{2\mu}{3(\mu+5)}\,\frac{(1-s\cos^2\frac{1}{2}\chi-t\sin^2\frac{1}{2}\chi)(2-u-s-t+ust)^{\frac{1}{2}\mu-1}}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^{\frac{5}{2}+\frac{1}{2}\mu}}$$
$$\left.+\frac{2\mu(\mu-2)}{3(\mu+3)(\mu+5)}\,(1-s)(1-t)\,\frac{(2-u-s-t+ust)^{\frac{1}{2}\mu-2}}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^{\frac{3}{2}+\frac{1}{2}\mu}}\right]. \quad (6.62)$$

The corresponding result when $\kappa_1=2$ is that

$$\Gamma(n_1+\tfrac{7}{2})\,\Gamma(n_2+\tfrac{7}{2})\int Y_2(U_1',\,V_1',\,W_1')\,S_{\frac{7}{2}}^{r}(hmC_1'^2)\,Y_2^{(p)}(U_1,\,V_1,\,W_1)$$
$$\times Y_2^{(p)}(U_2,\,V_2,\,W_2)\,S_{\frac{5}{2}}^{n_1}(hmC_1^2)\,S_{\frac{5}{2}}^{n_2}(hmC_2^2)\,C_R{}^{\mu}$$
$$\times\exp\{-hm(C_1^2+C_2^2)\}\,dU_1\ldots dW_2\,d\epsilon$$

is equal to the coefficient of $s^{n_1}t^{n_2}u^r$ in the expansion of

$$\eta\,\frac{3\pi^4}{16(hm)^{6+\frac{1}{2}\mu}}\,\frac{(2+p)!}{(2-p)!}\,\frac{2-p^2}{\Gamma(\frac{9}{2})}\,\frac{\Gamma(\frac{9}{2}+\frac{1}{2}\mu)}{\Gamma(r+\frac{7}{2})}\left[\frac{u\sin^2\frac{1}{2}\chi\cos^2\frac{1}{2}\chi(2-u-s-t+ust)^{\frac{1}{2}\mu}}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^{\frac{9}{2}+\frac{1}{2}\mu}}\right.$$
$$-\frac{\mu}{\mu+7}\,\frac{\sin^2\frac{1}{2}\chi\cos^2\frac{1}{2}\chi(3-u-us-ut)(2-u-s-t+ust)^{\frac{1}{2}\mu-1}}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^{\frac{7}{2}+\frac{1}{2}\mu}}$$
$$+\frac{2\mu(\mu-2)}{(\mu+5)(\mu+7)}\,\frac{(1-u)(1-s\cos^2\frac{1}{2}\chi-t\sin^2\frac{1}{2}\chi)(2-u-s-t+ust)^{\frac{1}{2}\mu-2}}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^{\frac{5}{2}+\frac{1}{2}\mu}}$$
$$\left.-\frac{2\mu(\mu-2)(\mu-4)}{(\mu+3)(\mu+5)(\mu+7)}\,\frac{(1-s)(1-t)(1-u)(2-u-s-t+ust)^{\frac{1}{2}\mu-3}}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^{\frac{3}{2}+\frac{1}{2}\mu}}\right]. \quad (6.63)$$

The value of the integral

$$\Gamma(\kappa_1+n_1+\tfrac{3}{2})\,\Gamma(\kappa_2+n_2+\tfrac{3}{2})\int Y_\kappa\,S_{\kappa+\frac{1}{2}}^{r}(hmC_1'^2)\,Y_{\kappa_1}^{(p)}\,Y_{\kappa_2}^{(p)}\,S_{\kappa_1+\frac{1}{2}}^{n_1}(hmC_1^2)$$
$$\times S_{\kappa_2+\frac{1}{2}}^{n_2}(hmC_2^2)\,C_R{}^{\mu}\exp\{-hm(C_1^2+C_2^2)\}\,dU_1\ldots dW_2\,d\epsilon \quad (6.71)$$

being known, the corresponding term in

$$\int Q' f_1 f_2\,C_R{}^{\mu}\,dU_1\ldots dW_2\,d\epsilon$$

can readily be written down. It comprises first [compare the expressions for f, Q, and $Y_{\kappa n}$, (3.1), (4.3), and (3.34)] a factor

$$(hm/\pi)^3\,(hm)^{\frac{1}{2}(\kappa+\kappa_1+\kappa_2)}\,n_1!\,n_2!(\kappa+\tfrac{1}{2})\,\pi^{\frac{1}{2}}\,C(B^{(p)}_{\kappa_1 n_1}B^{(p)}_{\kappa_2 n_2}+C^{(p)}_{\kappa_1 n_1}C^{(p)}_{\kappa_2 n_2}), \quad (6.72)$$

where

$$C=1 \quad \text{if} \quad p=0,$$

$$C=4\,\frac{(\kappa_1-p)!}{(\kappa_1+p)!}\,\frac{(\kappa_2-p)!}{(\kappa_2+p)!} \quad \text{if} \quad p\neq 0; \quad (6.73)$$

and secondly, since, unless $\kappa_1=\kappa_2$ and $n_1=n_2$, $B^{(p)}_{\kappa_1 n_1}B^{(p)}_{\kappa_2 n_2}$ and $C^{(p)}_{\kappa_1 n_1}C^{(p)}_{\kappa_2 n_2}$ arise twice in the product $f_1 f_2$ ($B^{(p)}$ and $C^{(p)}_{\kappa_1 n_1}$ being taken either from f_1 or from f_2), the value of (6.71) together with the corresponding integral with κ_1 and κ_2, n_1 and n_2 interchanged or, what is the same thing, with χ replaced by $\pi-\chi$; if $\kappa_1=\kappa_2$ and $n_1=n_2$, then (6.71) gives the required factor at once.

Since

$$\int Q f_1 f_2\,C_R{}^{\mu}\,dU_1\ldots dW_2\,d\epsilon$$

is got from the above integral by putting χ equal to zero, it follows that

$$\int (Q'-Q) f_1 f_2\,C_R{}^{\mu}\,dU_1\ldots dW_2\,d\epsilon$$

can be written as a series of terms, each of the form

$$F(\chi)-F(0);$$

and the final integration [*cf.* (6.11)]

$$\int_0^{\infty}\{F(\chi)-F(0)\}\,\alpha\,d\alpha \quad (6.74)$$

depends on the relation between χ and α, *i.e.* on the law of force between the molecules. It is expressible in terms of known functions only in the case of molecules which are elastic spheres; in other cases it must be evaluated by means of one or other of the methods of approximate integration, as is done by Maxwell for the special type of molecule considered by him.

These results being established, the right-hand side of equation (4.2) can be written down at once for all values of κ and r. When $\kappa=0$ only terms of type $B^{(p)}_{\kappa_1 n_1}B^{(p)}_{\kappa_1 n_2}$ and $C^{(p)}_{\kappa_1 n_1}C^{(p)}_{\kappa_1 n_2}$ can occur in the result, which may be written in the form

$$\Delta_{0r}=M_\nu[\Sigma A_{n_1}A_{n_2}L_{n_1 n_2 r}+\Sigma B_{1n_1}B_{1n_2}L_{1n_1 n_2 r}$$
$$+\Sigma(B^{(1)}_{1n_1}B^{(1)}_{1n_2}+C^{(1)}_{1n_1}C^{(1)}_{1n_2})\,L^{(1)}_{1n_1 n_2 r}+\ldots], \quad (7.1)$$

the L's being formed as explained above. Now A_0 is unity and A_1 is zero (3.23); and it will appear below that B_{1n}, $B^{(1)}_{1n}$, $C^{(1)}_{1n}$, B_{2n}, ... are of the first order of small quantities (*i.e.* of the order of first differential coefficients of temperature and mean velocity). Also, when $\kappa = 0$ and $r > 0$, the left-hand side of (4.2) is easily seen to be of the second order of small quantities, so that A_2, A_3, ... are not greater than the second order, and unless the stresses in the gas are required to the third order, these coefficients need not be calculated.

Similarly, if $\kappa = 1$, the right-hand side of (4.2) can be written in the form

$$\Delta_{1r} = M_\nu [\Sigma A_{n_1} B_{1n_2} M_{n_1 n_2 r} + \Sigma B_{1n_1} B_{2n_2} M_{1n_1 n_2 r} + \Sigma B_{2n_1} B_{3n_2} M_{2n_1 n_2 r} + \ldots], \quad (7.2)$$

and, if $\kappa = 2$,

$$\Delta_{2r} = M_\nu [\Sigma A_{n_1} B_{2n_2} N_{n_1 n_2 r} + \Sigma B_{1n_1} B_{3n_2} N_{1n_1 n_2 r} + \ldots + \Sigma A_{n_1} A_{n_2} P_{n_1 n_2 r} + E_1 B_{1n_1} B_{1n_2} P_{1n_1 n_2 r} + \ldots]. \quad (7.3)$$

It will be noted that, when $\kappa = 2$, products occur in which $\kappa_2 = \kappa_1$, corresponding to $\kappa = |\kappa_2 - \kappa_1| + 2$; the first of these, $\Sigma A_{n_1} A_{n_2} P_{n_1 n_2 r}$, can be omitted since $P_{n_1 n_2 r}$ is known to be zero [*cf.* (6.61)]. The calculations for Maxwellian molecules and for molecules which are elastic spheres will now be considered separately.

8. *Maxwellian molecules. Terms of first order.*

As was noted above, no such terms can occur in A_2, A_3, ...; and to this order of approximation the left-hand side of (4.2) [*i.e.* the expression (5.12)] when $\kappa = 1$ reduces to the single term

$$\frac{1}{h(hm)^{\frac{1}{2}}} \frac{\partial h}{\partial z} F_{1,r-1} = \frac{1}{h(hm)^{\frac{1}{2}}} \frac{\partial h}{\partial z} A_{r-1} = \frac{1}{h(hm)^{\frac{1}{2}}} \frac{\partial h}{\partial z}$$

if $r = 1$, and is negligible if r is greater than unity. On the right the only terms which are not negligible are [*cf.* (7.2)]

$$\Sigma B_{1n_2} M_{0n_2 r},$$

and $M_{0n_2 r}$ is [*cf.* (6.55), (6.72), (6.74)]

$$\int_0^\infty \{F(\chi) - F(0)\} a \, da,$$

where $F(\chi)$ is the coefficient of $t^{n_2}u^r$ in

$$\left(\frac{hm}{\pi}\right)^3 hm^{n_2}\,!\;\tfrac{3}{2}\pi^{\frac{1}{2}}\,2\,\frac{\pi^3}{(hm)^4}\,\frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})}{\Gamma(r+\frac{5}{2})}\,\frac{\sin^2\frac{1}{2}\chi}{\{1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi\}^{\frac{3}{2}}}$$

plus the same expression with $\pi-\chi$ instead of χ; hence it is zero unless $n_2=r$. If $n_2=r=1$, it is

$$2\pi(\sin^4\tfrac{1}{2}\chi+\cos^4\tfrac{1}{2}\chi),$$

so that

$$\begin{aligned} M_{011} &= -2\pi\int_0^\infty \{1-\sin^4\tfrac{1}{2}\chi-\cos^4\tfrac{1}{2}\chi\}\,a\,da \\ &= -4\pi\int_0^\infty \sin^2\tfrac{1}{2}\chi\cos^2\tfrac{1}{2}\chi\,a\,da \\ &= -J, \end{aligned} \qquad (8.11)$$

where J is Maxwell's integral A_2; its value is erroneously given by him* as 1·3682, and it has been recalculated by Aichi and Tanakadate and by Chapman†, their values being 1·3704 and 1·3700 respectively.

The values of M_{022}, M_{033}, ... are not required. Hence, taking $\kappa=1$, $r=1$, $r=2$, ... in Boltzmann's equation, we are led to

$$\begin{aligned} \frac{1}{h(hm)^{\frac{1}{2}}}\frac{\partial h}{\partial z} &= -M\nu J B_{11}, \\ 0 &= M\nu B_{12}M_{022}, \\ 0 &= M\nu B_{13}M_{033}, \ \ldots, \end{aligned}$$

from which it follows that

$$B_{11} = -\frac{1}{M\nu Jh(hm)^{\frac{1}{2}}}\frac{\partial h}{\partial z}, \qquad (8.12)$$

$B_{12}=B_{13}=\ldots=0$. The values of $B_{11}^{(1)}$ and $C_{11}^{(1)}$ are, by symmetry,

$$-\frac{1}{M\nu Jh(hm)^{\frac{1}{2}}}\frac{\partial h}{\partial x} \quad\text{and}\quad -\frac{1}{M\nu Jh(hm)^{\frac{1}{2}}}\frac{\partial h}{\partial y}$$

respectively.

When $\kappa=2$, it is found that the only terms on the left of (4.2) which can give rise to terms of the order of magnitude of first differential

* Maxwell, *Collected papers*, 2 (1890), 42.

† *Cf.* Chapman, *Manchester Lit. and Phil. Soc. Mem.*, 66 (1922).

coefficients are

$$-\tfrac{2}{3}\Big\{D^{(1)}_{1r}\frac{\partial u}{\partial x}+G^{(1)}_{1r}\frac{\partial v}{\partial x}-2D_{1r}\frac{\partial w}{\partial x}+E^{(1)}_{1r}\frac{\partial u}{\partial y}+H^{(1)}_{1r}\frac{\partial v}{\partial y}-2E_{1r}\frac{\partial w}{\partial y}$$

$$+F^{(1)}_{1r}\frac{\partial u}{\partial z}+J^{(1)}_{1r}\frac{\partial v}{\partial z}-2F_{1r}\frac{\partial w}{\partial z}\Big\}$$

and these reduce to

$$\tfrac{2}{3}\Big(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\Big)$$

when r is zero, and are negligible if r is not zero. On the right the relevant terms are [*cf.* (7.3)],

$$M\nu\Sigma B_{2n_2}N_{0n_2r},$$

the others being of smaller order.

Also $$N_{0n_2r}=\int_0^\infty\{F(\chi)-F(0)\}\,a\,da,$$

where $F(\chi)$ is the coefficient of $t^{n_2}u^r$ in

$$\Big(\frac{hm}{\pi}\Big)^3(hm)^2n_2!\,\tfrac{5}{2}\pi^{\frac{1}{2}}\frac{2\pi^3}{(hm)^5}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(r+\frac{7}{2})}\frac{\sin^2\frac{1}{2}\chi\cos^2\frac{1}{2}\chi(\tan^2\frac{1}{2}\chi-\frac{1}{2})}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^{\frac{7}{2}}}$$

plus the same expression with $\pi-\chi$ written instead of χ. Evidently $F(\chi)$ is zero unless $n_2=r$; if $n_2=r=0$, it is

$$2\pi(\sin^4\tfrac{1}{2}\chi+\cos^4\tfrac{1}{2}\chi-\sin^2\tfrac{1}{2}\chi\cos^2\tfrac{1}{2}\chi),$$

so that

$$N_{000}=-2\pi\int_0^\infty\{1-\sin^4\tfrac{1}{2}\chi-\cos^4\tfrac{1}{2}\chi+\sin^2\tfrac{1}{2}\chi\cos^2\tfrac{1}{2}\chi\}\,a\,da$$

$$=-\tfrac{3}{2}J. \qquad (8.21)$$

The values of N_{011}, N_{022}, ... are not required.

We therefore have the set of equations

$$\tfrac{2}{3}\Big(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\Big)=-\tfrac{3}{2}M\nu JB_{20},$$

$$0=M\nu B_{21}N_{011},$$

$$0=M\nu B_{22}N_{022},\quad\ldots,$$

and so $$B_{20}=-\frac{4}{9M\nu J}\Big(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\Big), \qquad (8.22)$$

$$B_{21}=B_{22}=\ldots=0.$$

The corresponding part of the distribution function is [*cf.* (3.34)]

$$B_{20}\,Y_2 = -\tfrac{2}{9}(3W^2-C^2)\,\frac{1}{M\nu J}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right).$$

Since the distribution function must be invariant for any change of rectangular axes, the other terms of the second degree can be deduced from this one by changing the axes. Let the z axis be taken in the direction of the resultant velocity C, and transform to a new set of rectangular axes by putting $u = l_1 u + m_1 v + n_1 w$, ..., so that

$$\tfrac{1}{2}(3W^2-C^2)\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)$$

is replaced by

$$\begin{aligned}C^2\Big[&2\left(l_3\frac{\partial}{\partial x}+m_3\frac{\partial}{\partial y}+n_3\frac{\partial}{\partial z}\right)(l_3u+m_3v+n_3w)\\&-\left(l_1\frac{\partial}{\partial x}+m_1\frac{\partial}{\partial y}+n_1\frac{\partial}{\partial z}\right)(l_1u+m_1v+n_1w)\\&-\left(l_2\frac{\partial}{\partial x}+m_2\frac{\partial}{\partial y}+n_2\frac{\partial}{\partial z}\right)(l_2u+m_2v+n_2w)\Big]\\=\tfrac{1}{2}(3W^2-C^2)&\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\tfrac{3}{2}(U^2-V^2)\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\\&+3UV\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+3VW\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+3WU\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right),\end{aligned}$$

so that [*cf.* (3.34)]

$$\left.\begin{aligned}B_{20}^{(1)}&=-\frac{4}{3M\nu J}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right),\quad B_{20}^{(2)}=-\frac{8}{3M\nu J}\left(\frac{\partial w}{\partial x}-\frac{\partial v}{\partial y}\right),\\C_{20}^{(1)}&=-\frac{4}{3M\nu J}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),\quad C_{20}^{(2)}=-\frac{8}{3M\nu J}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right).\end{aligned}\right\}\qquad(8.23)$$

When these values are substituted in the equation of mean motion (5.2), it appears that the gas moves, to a first approximation, as a viscous fluid* whose coefficient of viscosity is

$$\mu = \frac{1}{3MJh}.\qquad(8.24)$$

* *Cf.* Lamb, *Hydrodynamics*, ch. 11, 4th edition (1916); Jeans, *Dynamical theory of gases*, 3rd edition, § 329 (1921); Chapman, *loc. cit.*, § 2, 334. This coefficient of viscosity μ will not be confused with the constant μ of § 4.

Terms of second order.

Since $$B_{11},\ B_{11}^{(1)},\ C_{11}^{(1)},\ B_{20},\ B_{20}^{(1)},\ B_{20}^{(2)},\ C_{20}^{(1)},\ C_{20}^{(2)}$$

are the only coefficients of the distribution function which are of the first order, the second order terms in equation (4.2) can readily be picked out. To the order required, it is found that, for $\kappa = 2$, $r = 0$,

$$\bar{Q} = B_{20} = -\frac{4}{9M\nu J}\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right),$$

$$\frac{\partial \overline{Q}}{\partial \overline{U}} = \frac{\partial \overline{Q}}{\partial \overline{V}} = \frac{\partial \overline{Q}}{\partial \overline{W}} = 0;$$

$$\frac{\partial \overline{Q}}{\partial t} = \frac{B_{20}}{h}\frac{\partial h}{\partial t}, \quad \frac{\partial \overline{Q}}{\partial x} = \frac{B_{20}}{h}\frac{\partial h}{\partial x}, \ \ldots.$$

$$\overline{UQ} = \frac{1}{3(hm)^{\frac{1}{2}}} B_{11}^{(1)}, \quad \overline{VQ} = \frac{1}{3(hm)^{\frac{1}{2}}} C_{11}^{(1)}, \quad \overline{WQ} = -\frac{2}{3(hm)^{\frac{1}{2}}} B_{11}.$$

$$\overline{U\frac{\partial Q}{\partial x}} + \overline{V\frac{\partial Q}{\partial y}} + \overline{W\frac{\partial Q}{\partial z}}$$

$$= -\tfrac{2}{3}\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \frac{1}{3h(hm)^{\frac{1}{2}}}\left\{2B_{11}\frac{\partial h}{\partial z} - B_{11}^{(1)}\frac{\partial h}{\partial x} - C_{11}^{(1)}\frac{\partial h}{\partial y}\right\}$$

$$-\tfrac{1}{3}\left[B_{20}\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \tfrac{1}{2}B_{20}^{(1)}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \tfrac{1}{2}C_{20}^{(1)}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right.$$

$$\left. -\tfrac{1}{2}B_{20}^{(2)}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \tfrac{1}{2}C_{20}^{(2)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]$$

$$-\tfrac{2}{3}B_{20}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \tfrac{1}{2}B_{20}^{(1)}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \tfrac{1}{2}C_{20}^{(1)}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right).$$

On substitution of these values in (5.12) it is found, after a little reduction, that it gives, for the left-hand side of (4.2),

$$\nu\left[\frac{dB_{20}}{dt} - \frac{B_{20}}{h}\frac{dh}{dt} + \tfrac{2}{3}\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right.$$

$$+\tfrac{1}{3}\left\{B_{20}\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \tfrac{1}{2}B_{20}^{(1)}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \tfrac{1}{2}C_{20}^{(1)}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right.$$

$$\left.\left. -\tfrac{1}{2}B_{20}^{(2)}\left(\frac{\partial w}{\partial x} - \frac{\partial v}{\partial y}\right) - \tfrac{1}{2}C_{20}^{(2)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\}\right.$$

$$\left. +\tfrac{2}{3}B_{20}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \tfrac{1}{2}B_{20}^{(1)}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) - \tfrac{1}{2}C_{20}^{(1)}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)\right]$$

$$-\tfrac{1}{3}h\left\{2\frac{\partial}{\partial z}\frac{\nu B_{11}}{h(hm)^{\frac{1}{2}}} - \frac{\partial}{\partial x}\frac{\nu B_{11}^{(1)}}{h(hm)^{\frac{1}{2}}} - \frac{\partial}{\partial y}\frac{\nu C_{11}^{(1)}}{h(hm)^{\frac{1}{2}}}\right\}. \qquad (8.31)$$

Also, to the order of approximation adopted,

$$\frac{B_{20}}{h}\frac{dh}{dt} = \tfrac{2}{3}B_{20}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

from (5.3) and (5.31), so that these terms in the bracket cancel one another.

The right-hand side of the equation is given by (7.3), in which, as before, $N_{n_1n_2r}$ is zero unless $n_1+n_2=r$, so that when $r=0$ only the coefficient N_{000} does not vanish; the terms of type $B_{1n_1}B_{3n_2}N_{1n_1n_2r}$ are of the third order of small quantities at least; $P_{n_1n_2r}$ is zero for all values of n_1, n_2, and r; $P_{1n_1n_2r}$ is zero unless $n_1+n_2=r$ [(6.62) with $\mu=0$] and, since B_{10} is zero (3.21), the terms like $B_{1n_1}B_{1n_2}P_{1n_1n_2r}$ do not occur when $r=0$; and similarly $P_{2n_1n_2r}$ vanishes unless $n_1+n_2=r-1$ [(6.63) with $\mu=0$], so that terms like $B_{2n_1}B_{2n_2}P_{2n_1n_2r}$ do not occur. It follows that the equation (4.2) for $\kappa=2$, $r=0$, reduces to

$$-\tfrac{3}{2}M\nu JB_{20} = \tfrac{2}{3}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)-\frac{d}{dt}\frac{4}{9M\nu J}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\ldots,$$

the other terms being as in (8.31) with the first approximations substituted for B_{11}, etc. The complete expression for B_{20} is

$$-\frac{2\mu}{3p}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\frac{2\mu}{3p}\frac{d}{dt}\frac{\mu}{p}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)$$

$$+\frac{\mu^2}{9p^2}\left\{2\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2+3\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+3\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right.$$

$$\left.-6\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2-6\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right\}$$

$$-\frac{\mu^2}{p^2}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2-\left(\frac{\partial w}{\partial x}\right)^2-\left(\frac{\partial w}{\partial y}\right)^2\right\}$$

$$-\frac{\mu h}{\nu mp}\left\{2\frac{\partial}{\partial z}\frac{\mu}{h^2}\frac{\partial h}{\partial z}-\frac{\partial}{\partial x}\frac{\mu}{h^2}\frac{\partial h}{\partial x}-\frac{\partial}{\partial y}\frac{\mu}{h^2}\frac{\partial h}{\partial y}\right\}, \qquad (8.32)$$

in which μ has been written for $1/(3MJh)$ and p for $\nu/(2h)$, the hydrostatic pressure.

By a change of axes, as before, it follows that

$$B_{20}^{(1)} = -\frac{2\mu}{p}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\frac{2\mu}{p}\frac{d}{dt}\frac{\mu}{p}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$
$$+\frac{2\mu^2}{3p^2}\left\{3\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\left(2\frac{\partial v}{\partial y}-\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}\right)\right\}$$
$$-\frac{2\mu^2}{p^2}\left\{\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial z}-\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right)\right\}$$
$$-\frac{3\mu h}{\nu m p}\left\{\frac{\partial}{\partial z}\frac{\mu}{h^2}\frac{\partial h}{\partial x}+\frac{\partial}{\partial x}\frac{\mu}{h^2}\frac{\partial h}{\partial z}\right\},$$

$$C_{20}^{(1)} = -\frac{2\mu}{p}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+\frac{2\mu}{p}\frac{d}{dt}\frac{\mu}{p}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)$$
$$+\frac{2\mu^2}{3p^2}\left\{3\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-2\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\left(2\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}-\frac{\partial w}{\partial z}\right)\right\}$$
$$-\frac{2\mu^2}{p^2}\left\{\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\frac{\partial w}{\partial z}-\left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\right)\right\}$$
$$-\frac{3\mu h}{\nu m p}\left\{\frac{\partial}{\partial z}\frac{\mu}{h^2}\frac{\partial h}{\partial y}+\frac{\partial}{\partial y}\frac{\mu}{h^2}\frac{\partial h}{\partial z}\right\},$$

$$B_{20}^{(2)} = -\frac{4\mu}{p}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\frac{4\mu}{p}\frac{d}{dt}\frac{\mu}{p}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)$$
$$+\frac{2\mu^2}{3p^2}\left\{3\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2-3\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2-4\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\right\}$$
$$-\frac{2\mu^2}{p^2}\left\{\left(\frac{\partial w}{\partial x}\right)^2-\left(\frac{\partial u}{\partial z}\right)^2-\left(\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2-2\left(\frac{\partial u}{\partial y}\right)^2+2\left(\frac{\partial v}{\partial x}\right)^2\right\}$$
$$-\frac{6\mu h}{\nu m p}\left\{\frac{\partial}{\partial x}\frac{\mu}{h^2}\frac{\partial h}{\partial x}-\frac{\partial}{\partial y}\frac{\mu}{h^2}\frac{\partial h}{\partial y}\right\},$$

$$C_{20}^{(2)} = -\frac{4\mu}{p}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{4\mu}{p}\frac{d}{dt}\frac{\mu}{p}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$
$$+\frac{4\mu^2}{3p^2}\left\{3\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)-2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\right\}$$
$$-\frac{4\mu^2}{p^2}\left\{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right)\right\}$$
$$-\frac{6\mu h}{\nu m p}\left\{\frac{\partial}{\partial y}\frac{\mu}{h^2}\frac{\partial h}{\partial x}+\frac{\partial}{\partial x}\frac{\mu}{h^2}\frac{\partial h}{\partial y}\right\}.$$

The components of stress are, of course,

$$P_{zz}-p=pB_{20},\quad P_{zx}=\tfrac{1}{2}pB^{(1)}_{20},\quad P_{zy}=\tfrac{1}{2}pC^{(1)}_{20},$$

and similar expressions.

9. *Molecules which are elastic spheres. Terms of first order.*

In this case the coefficient M_{0n_2r} of (7.2) is

$$\int_0^\infty \{F(\chi)-F(0)\}\,a\,da,$$

where $F(\chi)$ is the coefficient of $t^{n_2}u^r z$ in

$$\left(\frac{hm}{\pi}\right)^3 hm\,.\,n_2!\,\tfrac{3}{2}\pi^{\frac{1}{2}}\,2\,\frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2})}\,\pi^3\,\frac{(1-t)^{-1}(1-u)^{-1}}{(hm)^3\,\Gamma(r+\frac{5}{2})}$$

$$\times\frac{\{2-u-t-z(1-u)(1-t)\}^{\frac{1}{2}}}{\{1-ut\sin^2\frac{1}{2}\chi-z(1-u)(1-t)\sin^2\frac{1}{2}\chi\}^2}$$

$$\times\left[1-\tfrac{3}{8}\,\frac{[1-t)^2\sin^2\chi\,z^2(1-u)^2}{\{1-ut\sin^2\frac{1}{2}\chi-z(1-u)(1-t)\sin^2\frac{1}{2}\chi\}^2}+\cdots\right]$$

plus the same expression with χ replaced by $\pi-\chi$ [*cf.* (6.72), (6.55), with $n_1=0, \kappa_1=0, \mu=1$]; *i.e.*, it is the coefficient of $t^{n_2}u^r$ in

$$\frac{n_2!}{\Gamma(r+\frac{5}{2})}\,\frac{2\pi}{(hm)^{\frac{1}{2}}}\left[2\,\frac{(2-u-t)^{\frac{1}{2}}}{(1-ut\sin^2\frac{1}{2}\chi)^3}\,\sin^2\tfrac{1}{2}\chi-\tfrac{1}{2}\,\frac{(2-u-t)^{-\frac{1}{2}}}{(1-ut\sin^2\frac{1}{2}\chi)^2}\right]$$

plus the same expression with χ replaced by $\pi-\chi$. (9.11)

In this particular case the integration with respect to a can be effected in terms of elementary functions: for a (*cf.* Jeans, *Dynamical theory of gases*, third edition, § 281) is now the shortest distance between the centres of the molecules during an encounter projected on the direction of relative motion, *i.e.* it is $\sigma\sin\frac{1}{2}\chi$, where σ is the diameter of a molecule*, and χ is zero when $a>\sigma$.

* Chapman (*loc. cit.*, p. 325) uses σ as the radius of a molecule, which accounts for a discrepancy of a factor 4 between his results and those of this paper.

Hence

$$\int_0^\infty \{F(\chi)-F(0)\}\, a\, da = \tfrac{1}{2}\sigma^2 \int_0^\pi \{F(\chi)-F(0)\} \sin\tfrac{1}{2}\chi \cos\tfrac{1}{2}\chi\, d\chi.$$

It will still be convenient to use the notation

$$4\pi \int_0^\infty \sin^2\tfrac{1}{2}\chi \cos^2\tfrac{1}{2}\chi\, a\, da = J,$$

so that in this case $J = \frac{1}{3}\pi\sigma^2$; also

$$4\pi \int_0^\infty \sin^4\tfrac{1}{2}\chi \cos^4\tfrac{1}{2}\chi\, a\, da = \tfrac{1}{5}J,$$

$$4\pi \int_0^\infty \sin^6\tfrac{1}{2}\chi \cos^6\tfrac{1}{2}\chi\, a\, da = \tfrac{3}{70}J, \quad \ldots.$$

The coefficients M_{0n_2r} can now be written down at once from (9.1); they are

$$M_{011} = \frac{1}{\Gamma(\frac{7}{2})}\, 4\pi\, \frac{2}{(hm)^{\frac{1}{2}}}\, 3 \int_0^\infty (\sin^4\tfrac{1}{2}\chi + \cos^4\tfrac{1}{2}\chi - 1)\, a\, da$$

$$= -\frac{16}{5}\left(\frac{2}{\pi hm}\right)^{\frac{1}{2}} J,$$

$$M_{021} = -\frac{1}{2} M_{011}, \quad M_{031} = -\frac{3}{16} M_{011}, \quad M_{041} = -\frac{3}{16} M_{011}, \quad \ldots,$$

$$M_{012} = -\frac{1}{14} M_{011}, \quad M_{022} = \frac{45}{28} M_{011},$$

$$M_{032} = -\frac{309}{14.16} M_{011}, \quad M_{042} = -\frac{177}{14.16} M_{011}, \quad \ldots,$$

$$M_{013} = -\frac{1}{7.8.9} M_{011}, \quad M_{023} = -\frac{103}{16.7.9} M_{011},$$

$$M_{033} = \frac{3}{16}\frac{5657}{7.8.9} M_{011}, \quad M_{043} = -\frac{3}{16}\frac{6783}{7.8.9} M_{011}, \quad \ldots,$$

$$M_{014} = -\frac{1}{16.7.9.11} M_{011}, \quad M_{024} = -\frac{59}{64.7.9.11} M_{011},$$

$$M_{034} = -\frac{3}{16}\frac{6783}{16.7.9.11} M_{011}, \quad M_{044} = -\frac{3}{16}\frac{149749}{16.7.9.11}, \quad \ldots. \quad (9.12)$$

Hence to the first order $B_{11}, B_{12}, B_{13}, \ldots$ are determined by the equations

$$M\nu M_{011}\left\{B_{11}-\frac{1}{2}B_{12}-\frac{3}{16}B_{13}-\frac{3}{16}B_{14}+\ldots\right\}=\frac{1}{h(hm)^{\frac{1}{2}}}\frac{\partial h}{\partial z},$$

$$M\nu M_{011}\left\{-\frac{1}{14}B_{11}+\frac{45}{28}B_{12}-\frac{309}{14.16}B_{13}-\frac{177}{14.16}B_{14}+\ldots\right\}=0,$$

$$M\nu M_{011}\left\{-\frac{1}{7.8.9}B_{11}-\frac{103}{16.7.9}B_{12}\right.$$
$$\left.+\frac{3}{16}\frac{5657}{7.8.9}B_{13}-\frac{3}{16}\frac{6783}{7.8.9}B_{14}+\ldots\right\}=0,$$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9.13)$

which give the coefficients as rapidly converging ratios of infinite determinants as in the previous paper*; their values are

$$\frac{B_{11}}{b_1}=\frac{B_{12}}{b_2}=\frac{B_{13}}{b_3}=\frac{B_{14}}{b_4}=\ldots=-\tfrac{5}{3}\tfrac{2}{2}\frac{(2\pi)^{\frac{1}{2}}}{M\nu J}\frac{1}{h}\frac{\partial h}{\partial z}, \qquad (9.14)$$

where, if three rows and columns of each determinant are taken,

$$b_1=1{\cdot}0248, \quad b_2=0{\cdot}04839, \quad b_3=0{\cdot}00332,$$

while, if four rows and columns are taken,

$$b_1=1{\cdot}0252, \quad b_2=0{\cdot}04881, \quad b_3=0{\cdot}003639, \quad b_4=0{\cdot}00025.$$

Hence we may take

$$b_1=1{\cdot}025, \quad b_2=0{\cdot}049, \quad b_3=0{\cdot}004, \quad b_4=0{\cdot}00025 \qquad (9.15)$$

as approximate values, probably correct to the number of significant figures given except in the case of b_4, whose value, however, is not required to any great degree of accuracy in the subsequent calculations.

The value of M is unity [*cf.* (4.11), in which n has to be made infinite] and J is found above to be $\frac{1}{3}\pi\sigma^2$, but it will be convenient to retain the general notation. $B_{11}^{(1)}, C_{11}^{(1)}, B_{12}^{(1)}, \ldots$ can be written down at once by symmetry by replacing $\partial h/\partial z$ by $\partial h/\partial x$ to get $B_{11}^{(1)}, B_{12}^{(1)}, \ldots$, and by $\partial h/\partial y$ to get $C_{11}^{(1)}, C_{12}^{(1)}, \ldots$.

Similarly, the coefficient N_{0n_2r} of (7.3) is

$$\int_0^\infty \{F(\chi)-F(0)\}\,a\,da,$$

* Burnett, *loc. cit.*, 410.

where $F(\chi)$ is the coefficient of $t^{n_2}u^r z^2$ in

$$\frac{(hm)^5}{\pi^3}\, n_2!\, \tfrac{5}{2}\pi^{\frac{1}{2}}\, 2\, \frac{\Gamma(\frac{5}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{2})}\, \frac{2\pi^3}{(hm)^{\frac{11}{2}}\,\Gamma(r+\frac{7}{2})}\, (1-t)^{-2}\,(1-u)^{-2}$$

$$\times \frac{\{2-u-t-z(1-u)(1-t)\}^{\frac{1}{2}}}{\{1-ut\sin^2\frac{1}{2}\chi - z(1-u)(1-t)\sin^2\frac{1}{2}\chi\}^2}$$

$$\times\left[1-\tfrac{3}{8}\,\frac{z^2(1-u)^2\,(1-t)^2\sin^2\chi}{\{1-ut\sin^2\frac{1}{2}\chi - z(1-u)(1-t)\sin^2\frac{1}{2}\chi\}^2}+\ldots\right]$$

plus the same expression with χ replaced by $\pi-\chi$; *i.e.* it is the coefficient of $t^{n_2}u^r$ in

$$\frac{n_2!}{\Gamma(r+\frac{7}{2})}\,\frac{4\pi}{(hm)^{\frac{1}{2}}}\left[3\,\frac{(2-u-t)^{\frac{1}{2}}\,(\sin^4\frac{1}{2}\chi-\frac{1}{2}\sin^2\frac{1}{2}\chi\cos^2\frac{1}{2}\chi)}{(1-ut\sin^2\frac{1}{2}\chi)^4}\right.$$

$$\left.-\frac{(2-u-t)^{-\frac{1}{2}}\sin^2\frac{1}{2}\chi}{(1-ut\sin^2\frac{1}{2}\chi)^3}-\tfrac{1}{8}\,\frac{(2-u-t)^{-\frac{3}{2}}}{(1-ut\sin^2\frac{1}{2}\chi)^2}\right]. \qquad (9.21)$$

Hence

$$N_{000}=\frac{1}{\Gamma(\frac{7}{2})}\left(\frac{2}{hm}\right)^{\frac{1}{2}}4\pi\int_0^\infty 3(\sin^4\tfrac{1}{2}\chi+\cos^4\tfrac{1}{2}\chi-\sin^2\tfrac{1}{2}\chi\cos^2\tfrac{1}{2}\chi-1)\,\alpha\,d\alpha$$

$$=-\tfrac{24}{5}\left(\frac{2}{\pi hm}\right)^{\frac{1}{2}}J, \qquad (9.22)$$

and the other coefficients are readily found to be

$$N_{010}=-\frac{1}{4}N_{000},\quad N_{020}=-\frac{1}{16}N_{000},\quad N_{030}=-\frac{3}{64}N_{000},\quad \ldots,$$

$$N_{001}=-\frac{1}{14}N_{000},\quad N_{011}=\frac{205}{12.14}N_{000},$$

$$N_{021}=-\frac{163}{14.16}N_{000},\quad N_{031}=-\frac{287}{14.64}N_{000},\quad \ldots,$$

$$N_{002}=-\frac{1}{7.8.9}N_{000},\quad N_{012}=-\frac{163}{4.7.8.9}N_{000},$$

$$N_{022}=-\frac{11889}{16.7.8.9}N_{000},\quad N_{032}=-\frac{2149}{24.64}N_{000},\quad \ldots,$$

$$N_{003}=-\frac{1}{16.7.9.11}N_{000},\quad N_{013}=-\frac{287}{12.16.7.9.11},$$

$$N_{023}=-\frac{2149}{16.16.9.11},\quad N_{033}=\frac{1532075}{7.9.11.1024},\quad \ldots. \qquad (9.23)$$

The equations for B_{20}, B_{21}, ..., are, therefore, to the required order of approximation,

$$M\nu N_{000}\left\{B_{20}-\frac{1}{4}B_{21}-\frac{1}{16}B_{22}-\frac{3}{64}B_{23}+\ldots\right\}=\frac{2}{3}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right),$$

$$M\nu N_{000}\left\{-\frac{1}{14}B_{20}+\frac{205}{12.14}B_{21}-\frac{163}{14.16}B_{22}-\frac{287}{14.64}B_{23}+\ldots\right\}=0, \quad \ldots, \tag{9.24}$$

from which it follows that

$$\frac{B_{20}}{b_0}=\frac{B_{21}}{b_1}=\frac{B_{22}}{b_2}=\frac{B_{23}}{b_3}=\ldots=-\frac{5}{72}\frac{(2\pi hm)^{\frac{1}{2}}}{M\nu J}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) \tag{9.25}$$

with $M=1$ and $J=\frac{1}{3}\pi\sigma^2$; if three rows and columns of each determinant are taken,

$$b_0=1{\cdot}0159, \quad b_1=0{\cdot}06232, \quad b_2=0{\cdot}00535,$$

while, if four rows and columns are taken,

$$b_0=1{\cdot}01598, \quad b_1=0{\cdot}06259, \quad b_2=0{\cdot}00509, \quad b_3=0{\cdot}0003.$$

Accordingly,

$$b_0=1{\cdot}016, \quad b_1=0{\cdot}063, \quad b_2=0{\cdot}0050, \quad b_3=0{\cdot}0003 \tag{9.26}$$

will be taken as the approximate values of these constants. By a change of axes it is found, as in (8.23), that $B^{(1)}_{20}$, $B^{(1)}_{21}$, ... are got from these by replacing $\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)$ by $3\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$; $C^{(1)}_{20}$, $C^{(1)}_{21}$, ... by replacing it by $3\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial y}\right)$; and $B^{(2)}_{20}$, $C^{(2)}_{20}$, ... by replacing it by $6\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)$ and $6\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$ respectively. Hence, as in (8.24), the gas behaves like a viscous fluid whose coefficient of viscosity is

$$\mu=\frac{5}{96MJ}\left(\frac{2\pi m}{h}\right)^{\frac{1}{2}}1{\cdot}016. \tag{9.27}$$

Terms of the second order.

The expression (5.12) (divided by ν) when

$$Q=\tfrac{5}{2}\pi^{\frac{1}{2}}hm\,Y_2\,S_{\frac{5}{2}}r(hm\,C^2)$$

is, so far as it depends on second derivatives and products of first derivatives,

$$\frac{d}{dt}B_{2r}+\{B_{2,r-1}-(r+1)B_{2r}\}\frac{1}{h}\frac{dh}{dt}$$

$$-\tfrac{2}{3}(hm)^{\frac{1}{2}}\left[2\left(Z-\frac{dw}{dt}\right)B_{1r}-\left(X-\frac{du}{dt}\right)B^{(1)}_{1r}-\left(Y-\frac{dv}{dt}\right)C^{(1)}_{1r}\right]$$

$$-\frac{1}{h(hm)^{\frac{1}{2}}}\left[\frac{2r+1}{3}\left\{2B_{1r}\frac{\partial h}{\partial z}-B^{(1)}_{1r}\frac{\partial h}{\partial x}-C^{(1)}_{1r}\frac{\partial h}{\partial y}\right\}\right.$$

$$-\frac{(r+1)^2}{3}\left\{2B_{1,r+1}\frac{\partial h}{\partial z}-B^{(1)}_{1,r+1}\frac{\partial h}{\partial x}-C^{(1)}_{1,r+1}\frac{\partial h}{\partial y}\right\}$$

$$\left.-\tfrac{1}{3}\left\{2B_{1,r-1}\frac{\partial h}{\partial z}-B^{(1)}_{1,r-1}\frac{\partial h}{\partial x}-C^{(1)}_{1,r-1}\frac{\partial h}{\partial y}\right\}\right]$$

$$-\frac{1}{3\nu}\left[2\frac{\partial}{\partial z}\frac{\nu}{(hm)^{\frac{1}{2}}}\{(r+1)B_{1,r+1}-B_{1r}\}-\frac{\partial}{\partial x}\frac{\nu}{(hm)^{\frac{1}{2}}}\{(r+1)B^{(1)}_{1,r+1}-B^{(1)}_{1r}\}\right.$$

$$\left.-\frac{\partial}{\partial y}\frac{\nu}{(hm)^{\frac{1}{2}}}\{(r+1)C^{(1)}_{1,r+1}-C^{(1)}_{1r}\}\right]$$

$$+\tfrac{1}{2}\tfrac{1}{1}\left[2\{(2r+\tfrac{7}{2})B_{2r}-2B_{2,r-1}\}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\right.$$

$$+\{(2r+\tfrac{7}{2})B^{(1)}_{2r}-2B^{(1)}_{2,r-1}\}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\{(2r+\tfrac{7}{2})C^{(1)}_{2r}-2C^{(1)}_{2,r-1}\}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)$$

$$\left.-\{(2r+\tfrac{7}{2})B^{(2)}_{2r}-2B^{(2)}_{2,r-1}\}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)-\{(2r+\tfrac{7}{2})C^{(2)}_{2r}-2C^{(2)}_{2,r-1}\}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]$$

$$+\tfrac{2}{3}\{(r+1)B_{2r}-B_{2,r-1}\}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)-\tfrac{1}{2}B^{(1)}_{2r}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)-\tfrac{1}{2}C^{(1)}_{2r}\left(\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\right).$$

(9.31)

Now, to the order of approximation adopted,

$$X-\frac{du}{dt}=\frac{1}{2\nu m}\frac{\partial}{\partial x}\frac{\nu}{h},\quad\ldots$$

from (5.2) and

$$\frac{1}{h}\frac{dh}{dt}=\tfrac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

from (5.31); and when these values are substituted in (9.31) it becomes

$$\frac{d}{dt}B_{2r}-\tfrac{1}{3}\frac{(hm)^{\frac{1}{2}}}{\nu m}\left\{2B_{1r}\frac{\partial}{\partial z}\frac{\nu}{h}-B^{(1)}_{1r}\frac{\partial}{\partial x}\frac{\nu}{h}-C^{(1)}_{1r}\frac{\partial}{\partial y}\frac{\nu}{h}\right\}$$

$$-\frac{1}{3h(hm)^{\frac{1}{2}}}\left[2r\left\{2B_{1r}\frac{\partial h}{\partial z}-B^{(1)}_{1r}\frac{\partial h}{\partial x}-C^{(1)}_{1r}\frac{\partial h}{\partial y}\right\}\right.$$

$$-r(r+1)\left\{2B_{1,r+1}\frac{\partial h}{\partial z}-B^{(1)}_{1,r+1}\frac{\partial h}{\partial x}-C^{(1)}_{1,r+1}\frac{\partial h}{\partial y}\right\}$$

$$\left.-\left\{2B_{1,r-1}\frac{\partial h}{\partial z}-B^{(1)}_{1,r-1}\frac{\partial h}{\partial x}-C^{(1)}_{1,r-1}\frac{\partial h}{\partial y}\right\}\right]$$

$$-\frac{h}{3\nu}\left[2\frac{\partial}{\partial z}\frac{\nu}{h(hm)^{\frac{1}{2}}}\{(r+1)B_{1,r+1}-B_{1r}\}-\frac{\partial}{\partial x}\frac{\nu}{h(hm)^{\frac{1}{2}}}\{(r+1)B^{(1)}_{1,r+1}-B^{(1)}_{1r}\}\right.$$

$$\left.-\frac{\partial}{\partial y}\frac{\nu}{h(hm)^{\frac{1}{2}}}\{(r+1)C^{(1)}_{1,r+1}-C^{(1)}_{1r}\}\right]$$

$$+\tfrac{1}{2}\tfrac{1}{1}\left[2\{2r+\tfrac{7}{2})B_{2r}-2B_{2,r-1}\}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\right.$$

$$+\{(2r+\tfrac{7}{2})B^{(1)}_{2r}-2B^{(1)}_{2,r-1}\}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\{(2r+\tfrac{7}{2})C^{(1)}_{2r}-2C^{(1)}_{2,r-1}\}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)$$

$$\left.-\{(2r+\tfrac{7}{2})B^{(2)}_{2r}-2B^{(2)}_{2,r-1}\}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)-\{(2r+\tfrac{7}{2})C^{(2)}_{2r}-2C^{(2)}_{2,r-1}\}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]$$

$$-\tfrac{1}{2}B^{(1)}_{2r}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)-\tfrac{1}{2}C^{(1)}_{2r}\left(\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\right). \qquad (9.32)$$

In view of the variety of terms involved, it will be simpler to deal with each type separately. Taking first the terms $d\,B_{2r}/dt$, it appears that the corresponding parts of B_{20}, B_{21}, ... are given by

$$M\nu N_{000}\left\{B_{20}-\frac{1}{4}B_{21}-\frac{1}{16}B_{22}+\ldots\right\}=\frac{d}{dt}B_{20}$$

$$=-\frac{2}{3}\frac{d}{dt}\frac{\mu}{p}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right),$$

$$M\nu N_{000}\left\{-\frac{1}{14}B_{20}+\frac{205}{12.14}B_{21}-\frac{163}{14.16}B_{22}+\ldots\right\}$$

$$=-\frac{2}{3}\frac{0{\cdot}063}{1{\cdot}016}\frac{d}{dt}\frac{\mu}{p}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right),$$

and so on, the coefficients on the left of each equation being the same as in (9.23) and the terms on the right being got by substituting the first approximations for B_{20}, B_{21}, ... from (9.24) with the value of μ given in (9.25) and $p = \nu/2h$. Hence

$$B_{20} = \tfrac{2}{3} K \frac{\mu}{p} \frac{d}{dt} \frac{\mu}{p} \left(2 \frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right), \tag{9.331}$$

where

$$K \begin{vmatrix} 1 & -\frac{1}{4} & -\frac{1}{16} & \cdots \\ -\frac{1}{14} & \frac{205}{12.14} & -\frac{163}{14.16} & \cdots \\ -\frac{1}{7.8.9} & -\frac{163}{4.7.8.9} & \frac{11889}{16.7.8.9} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

$$= \frac{1}{1{\cdot}016^2} \begin{vmatrix} 1{\cdot}016 & -\frac{1}{4} & -\frac{1}{16} & \cdots \\ 0{\cdot}063 & \frac{205}{12.14} & -\frac{163}{14.16} & \cdots \\ 0{\cdot}0050 & -\frac{163}{4.7.8.9} & \frac{11889}{16.7.8.9} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

The value of K found by taking the first two rows and columns of each determinant is 1·012; when three rows and columns are taken it is 1·0138, while four rows and columns give 1·0141, so that

$$K = 1{\cdot}014. \tag{9.332}$$

is probably not in error by as much as one part in a thousand*.

This is in satisfactory agreement with the result obtained by Jones†. The terms of type

$$-\tfrac{1}{3} \frac{(hm)^{\frac{1}{2}}}{\nu m} \left\{ 2B_{1r} \frac{\partial}{\partial z} \frac{\nu}{h} - B_{1r}^{(1)} \frac{\partial}{\partial x} \frac{\nu}{h} - C_{1r}^{(1)} \frac{\partial}{\partial y} \frac{\nu}{h} \right\} \quad (r = 0, 1, 2, \ldots)$$

* It has been verified here and in the subsequent calculations that the error due to the possible inaccuracies in the first column of the determinant on the right is sufficiently small not to affect the results to the number of significant figures given.

† *Phil. Trans. Royal Soc.* (A), 223 (1923), 27, 29.

give

$$M\nu N_{000}\left\{B_{20}-\frac{1}{4}B_{21}-\frac{1}{16}B_{22}\ldots\right\}=0,$$

$$M\nu N_{000}\left\{-\frac{1}{14}B_{20}+\frac{205}{12.14}B_{21}-\frac{163}{14.16}B_{22}\ldots\right\}$$

$$=\frac{1{\cdot}025}{1{\cdot}016}\frac{\mu}{\nu hmp}\left\{2\frac{\partial h}{\partial z}\frac{\partial p}{\partial z}-\frac{\partial h}{\partial x}\frac{\partial p}{\partial x}-\frac{\partial h}{\partial y}\frac{\partial p}{\partial y}\right\},$$

$$M\nu N_{000}\left\{-\frac{1}{7.8.9}B_{20}-\frac{163}{4.7.8.9}B_{21}+\frac{11889}{16.7.8.9}B_{22}\ldots\right\}$$

$$=\frac{0{\cdot}049}{1{\cdot}016}\frac{\mu}{\nu hmp}\left\{2\frac{\partial h}{\partial z}\frac{\partial p}{\partial z}-\frac{\partial h}{\partial x}\frac{\partial p}{\partial x}-\frac{\partial h}{\partial y}\frac{\partial p}{\partial y}\right\},$$

and similar equations. Hence this part of B_{20} is

$$-K\frac{\mu^2}{\nu hmp^2}\left\{2\frac{\partial h}{\partial z}\frac{\partial p}{\partial z}-\frac{\partial h}{\partial x}\frac{\partial p}{\partial x}-\frac{\partial h}{\partial y}\frac{\partial p}{\partial y}\right\},$$

where

$$K\begin{vmatrix} 1 & -\frac{1}{4} & -\frac{1}{16} & \ldots \\ -\frac{1}{14} & \frac{205}{12.14} & -\frac{163}{14.16} & \ldots \\ -\frac{1}{7.8.9} & -\frac{163}{4.7.8.9} & \frac{11889}{16.7.8.9} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$=\frac{1}{1{\cdot}016^2}\begin{vmatrix} 0 & -\frac{1}{4} & -\frac{1}{16} & \ldots \\ 1{\cdot}025 & \frac{205}{12.14} & -\frac{163}{14.16} & \\ 0{\cdot}049 & -\frac{163}{4.7.8.9} & \frac{11889}{16.7.8.9} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}.$$

The first two rows and columns of these determinants give $K=0{\cdot}213$; three rows and columns give 0·2238, while four rows and columns give

0·2258, so that $K = 0{\cdot}227$

is probably not in error by as much as $\frac{1}{2}$ per cent.

The terms

$$-\frac{h}{3\nu}\Big[2\frac{\partial}{\partial z}\frac{\nu}{h(hm)^{\frac{1}{2}}}\{(r+1)B_{1,r+1}-B_{1r}\} - \frac{\partial}{\partial x}\frac{\nu}{h(hm)^{\frac{1}{2}}}\{(r+1)B^{(1)}_{1,r+1}-B^{(1)}_{1r}\}+\ldots\Big]$$

in (9.32) give equations of the type

$$M\nu N_{000}\Big\{B_{20}-\frac{1}{4}B_{21}-\frac{1}{16}B_{22}+\ldots\Big\} = \frac{h}{\nu m}A_0\Big\{2\frac{\partial}{\partial z}\frac{\mu}{h^2}\frac{\partial h}{\partial z}-\frac{\partial}{\partial x}\frac{\mu}{h^2}\frac{\partial h}{\partial x}-\frac{\partial}{\partial y}\frac{\mu}{h^2}\frac{\partial h}{\partial y}\Big\},$$

$$M\nu N_{000}\Big\{-\frac{1}{14}B_{20}+\frac{205}{12\,.\,14}B_{21}+\ldots\Big\} = \frac{h}{\nu m}A_1\Big\{2\frac{\partial}{\partial z}\frac{\mu}{h^2}\frac{\partial h}{\partial z}-\frac{\partial}{\partial x}\frac{\mu}{h^2}\frac{\partial h}{\partial x}-\frac{\partial}{\partial y}\frac{\mu}{h^2}\frac{\partial h}{\partial y}\Big\},$$

where

$$A_0 = \frac{1{\cdot}025}{1{\cdot}016},\quad A_1 = -\frac{1}{1{\cdot}016}(1{\cdot}025-0{\cdot}098) = -\frac{0{\cdot}927}{1{\cdot}016},$$

$$A_2 = -\frac{1}{1{\cdot}016}(0{\cdot}049-0{\cdot}012) = -\frac{0{\cdot}037}{1{\cdot}016},$$

$$A_3 = -\frac{1}{1{\cdot}016}(0{\cdot}004-0{\cdot}001) = -\frac{0{\cdot}003}{1{\cdot}016},$$

and so on, from (9.14). Hence

$$B_{20} = -K\frac{\mu h}{\nu m p}\Big\{2\frac{\partial}{\partial z}\frac{\mu}{h^2}\frac{\partial h}{\partial z}-\frac{\partial}{\partial x}\frac{\mu}{h^2}\frac{\partial h}{\partial x}-\frac{\partial}{\partial y}\frac{\mu}{h^2}\frac{\partial h}{\partial y}\Big\},$$

where K is got, as before, as the ratio of two infinite determinants. The value found by taking the first two rows and columns of each determinant is 0·821; when three rows and columns of each are taken, it is 0·8074, and when four rows and columns are taken it is 0·8060, so that $K = 0{\cdot}806$ is probably correct to the third place of decimals. The coefficient given by Jones, viz.

$$\frac{4}{15}\frac{\mu\vartheta}{p^2}\,0{\cdot}80012,$$

where $$\vartheta = \frac{15}{4}\frac{1{\cdot}025}{1{\cdot}016},$$

corresponds to that got by taking three rows and columns, and, since $0{\cdot}8001 \times 1{\cdot}025 \div 1{\cdot}016 = {\cdot}8073$, the two results are in satisfactory agreement.

The terms of type

$$-\tfrac{1}{2}B^{(1)}_{2r}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)-\tfrac{1}{2}C^{(1)}_{2r}\left(\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\right)$$

give

$$M\nu N_{000}\left\{B_{20}-\frac{1}{4}B_{21}-\frac{1}{16}B_{22}+\ldots\right\}=\frac{\mu}{p}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2-\left(\frac{\partial w}{\partial x}\right)^2-\left(\frac{\partial w}{\partial y}\right)^2\right\},$$

$$M\nu N_{000}\left\{-\frac{1}{14}B_{20}+\frac{205}{12\,.\,14}B_{21}+\ldots\right\}$$

$$=\frac{0{\cdot}063}{1{\cdot}016}\frac{\mu}{p}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2-\left(\frac{\partial w}{\partial x}\right)^2-\left(\frac{\partial w}{\partial y}\right)^2\right\},$$

and similar equations, the numerical coefficients on the right being the same as occur above in the equations for the part of B_{20} which depends on

$$\frac{d}{dt}\frac{\mu}{p}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right).$$

Hence this part of B_{20} is

$$-1{\cdot}014\frac{\mu^2}{p^2}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2-\left(\frac{\partial w}{\partial x}\right)^2-\left(\frac{\partial w}{\partial y}\right)^2\right\}.$$

The remaining terms in (9 . 32) involve squares of first derivatives and with them must be taken the corresponding terms in (7 . 3), viz. those of types

$$\Sigma P_{1n_1n_2r}B_{1n_1}B_{1n_2},\quad \Sigma P_{2n_1n_2r}B_{2n_1}B_{2n_2},\quad \ldots.$$

Now $$P_{1n_1n_2r}=-2P^{(1)}_{1n_1n_2r}=\int_0^\infty\{F(\chi)-F(0)\}\,a\,da,$$

where $F(\chi)$ is the coefficient of $s^{n_1}t^{n_2}u^r$ in

$$\frac{(hm)^5}{\pi^3}n_1!\,n_2!\,\tfrac{5}{2}\pi^{\frac{1}{2}}\frac{3\pi^4}{4(hm)^{\frac{11}{2}}}\frac{2}{\Gamma(\frac{7}{2})}\frac{6}{\Gamma(r+\frac{7}{2})}\left[\frac{\sin^2\frac{1}{2}\chi\cos^2\frac{1}{2}\chi(2-u-s-t+ust)^{\frac{1}{2}}}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^4}\right.$$

$$-\tfrac{1}{9}\frac{(1-s\cos^2\tfrac{1}{2}\chi-t\sin^2\tfrac{1}{2}\chi)(2-u-s-t+ust)^{-\frac{1}{2}}}{(1-us\cos^2\tfrac{1}{2}\chi-ut\sin^2\tfrac{1}{2}\chi)^3}$$

$$\left.-\tfrac{1}{36}(1-s)(1-t)\frac{(2-u-s-t+ust)^{-\frac{3}{2}}}{(1-us\cos^2\tfrac{1}{2}\chi-ut\sin^2\tfrac{1}{2}\chi)^2}\right]$$

plus a similar expression with χ replaced by $\pi-\chi$, except when $n_1 = n_2$ [*cf.* (6.72) and (6.62)].

Hence

$$P_{1110} = -\frac{1}{5(2\pi hm)^{\frac{1}{2}}} J, \quad P_{1120} = \frac{3}{2} P_{1110},$$

$$P_{1130} = \frac{15}{8} P_{1110}, \quad P_{1220} = \frac{15}{16} P_{1110},$$

$$P_{1111} = \frac{89}{42} P_{1110}, \quad P_{1121} = \frac{173}{84} P_{1110}, \quad P_{1131} = \frac{345}{112} P_{1110}, \quad P_{1221} = \frac{169}{224} P_{1110},$$

$$P_{1112} = -\frac{77}{24} P_{1110}, \quad P_{1122} = \frac{417}{112} P_{1110}, \quad P_{1113} = \frac{7705}{6.7.8.9.11} P_{1110}, \quad \ldots.$$

When these values are substituted and the first approximations for B_{1n} taken from (9.14) and (9.15), it is found that

$$M\nu\Sigma P_{1n_1n_20} B_{1n_1} B_{1n_2} = -\frac{1}{5} \frac{M\nu J}{(2\pi hm)^{\frac{1}{2}}} \frac{25}{32^2} \frac{2\pi}{(M\nu J)^2} \frac{1}{h^2} \left(\frac{\partial h}{\partial z}\right)^2 . 1{\cdot}136$$

$$= -\frac{3}{32} \frac{1{\cdot}136}{1{\cdot}016} \frac{\mu}{\nu m h^2} \left(\frac{\partial h}{\partial z}\right)^2,$$

so that

$$M\nu\Sigma\{P_{1n_1n_20} B_{1n_1} B_{1n_2} + P^{(1)}_{1n_1n_20}(B^{(1)}_{1n_1} B^{(1)}_{1n_2} + C^{(1)}_{1n_1} C^{(1)}_{1n_2})\}$$

$$= -\frac{3}{64} \frac{1{\cdot}136}{1{\cdot}016} \frac{\mu}{\nu m h^2} \left\{2\left(\frac{\partial h}{\partial z}\right)^2 - \left(\frac{\partial h}{\partial x}\right)^2 - \left(\frac{\partial h}{\partial y}\right)^2\right\}.$$

Also

$$M\nu\Sigma\{P_{1n_1n_21} B_{1n_1} B_{1n_2} + P^{(1)}_{1n_1n_21}(B^{(1)}_{1n_1} B^{(1)}_{1n_2} + C^{(1)}_{1n_1} C^{(1)}_{1n_2})\}$$

$$= -\frac{3}{64} \frac{2{\cdot}344}{1{\cdot}016} \frac{\mu}{\nu m h^2} \left\{2\left(\frac{\partial h}{\partial z}\right)^2 - \left(\frac{\partial h}{\delta x}\right)^2 - \left(\frac{\partial h}{\partial y}\right)^2\right\},$$

$$M\nu\Sigma\{P_{1n_1n_22} B_{1n_1} B_{1n_2} + P^{(1)}_{1n_1n_12}(B^{(1)}_{1n_1} B^{(1)}_{1n_2} + C^{(1)}_{1n_1} C^{(1)}_{1n_2})\}$$

$$= +\frac{3}{64} \frac{3{\cdot}19}{1{\cdot}016} \frac{\mu}{\nu m h^2} \left\{2\left(\frac{\partial h}{\partial z}\right)^2 - \left(\frac{\partial h}{\partial x}\right)^2 - \left(\frac{\partial h}{\partial y}\right)^2\right\},$$

$$M\nu\Sigma\{P_{1n_1n_23} B_{1n_1} B_{1n_2} + P^{(1)}_{1n_1n_23}(B^{(1)}_{1n_1} B^{(1)}_{1n_2} + C^{(1)}_{1n_1} C^{(1)}_{1n_2})\}$$

$$= -\frac{3}{64} \frac{0{\cdot}243}{1{\cdot}016} \frac{\mu}{\nu m h^2} \left\{2\left(\frac{\partial h}{\partial z}\right)^2 - \left(\frac{\partial h}{\partial x}\right)^2 - \left(\frac{\partial h}{\partial y}\right)^2\right\},$$

and so on. The corresponding terms in (9.32) are

$$\frac{1}{3h(hm)^{\frac{1}{2}}}\Big[2B_{1,r-1}\frac{\partial h}{\partial z}-B^{(1)}_{1,r-1}\frac{\partial h}{\partial x}-C^{(1)}_{1,r-1}\frac{\partial h}{\partial y}$$

$$-2r\left\{2B_{1r}\frac{\partial h}{\partial z}-B^{(1)}_{1r}\frac{\partial h}{\partial x}-C^{(1)}_{1r}\frac{\partial h}{\partial y}\right\}$$

$$+r(r+1)\left\{2B_{1,r+1}\frac{\partial h}{\partial z}-B^{(1)}_{1,r+1}\frac{\partial h}{\partial x}-C^{(1)}_{1,r+1}\frac{\partial h}{\partial y}\right\}\Big].$$

When $r=0$, this is zero; when $r=1$ it is, to the required order of magnitude,

$$\frac{\mu}{\nu mh^2}\,\frac{1{\cdot}952}{1{\cdot}016}\left\{2\left(\frac{\partial h}{\partial z}\right)^2-\left(\frac{\partial h}{\partial x}\right)^2-\left(\frac{\partial h}{\partial y}\right)^2\right\};$$

when $r=2$, it is the same expression with a numerical factor $-0{\cdot}853$ instead of $1{\cdot}952$; when $r=3$, the corresponding factor is $-0{\cdot}028$, ... Hence the equations for the corresponding part of B_{20} are of the type

$$M\nu N_{000}\left\{B_{20}-\frac{1}{4}\,B_{21}\ldots\right\}=\frac{\mu}{\nu mh^2}\,\frac{A_0}{1{\cdot}016}\left\{2\left(\frac{\partial h}{\partial z}\right)^2-\left(\frac{\partial h}{\partial x}\right)^2-\left(\frac{\partial h}{\partial y}\right)^2\right\},$$

$$M\nu N_{000}\left\{-\frac{1}{14}\,B_{20}+\frac{205}{12\,.\,14}\,B_{21}\ldots\right\}=\frac{\mu}{\nu mh^2}\,\frac{A_1}{1{\cdot}016}\left\{2\left(\frac{\partial h}{\partial z}\right)^2-\left(\frac{\partial h}{\partial x}\right)^2-\left(\frac{\partial h}{\partial y}\right)^2\right\},$$

where

$$A_0=\tfrac{3}{64}\times 1{\cdot}136=0{\cdot}0533,$$

$$A_1=1{\cdot}952+\tfrac{3}{64}\times 2{\cdot}344=2{\cdot}062,$$

$$A_2=-0{\cdot}853-\tfrac{3}{64}\times 3{\cdot}19=-1{\cdot}003,$$

$$A_3=-0{\cdot}028+\tfrac{3}{64}\times 0{\cdot}243=-0{\cdot}017,\ \ldots.$$

And so this part of B_{20} is

$$-K\,\frac{\mu^2}{\nu mh^2 p}\left\{2\left(\frac{\partial h}{\partial z}\right)^2-\left(\frac{\partial h}{\partial x}\right)^2-\left(\frac{\partial h}{\partial y}\right)^2\right\},$$

where K is the ratio of two infinite determinants, as before. When two rows and columns of each are taken, K is found to be $0{\cdot}468$: when three rows and columns are taken it is $0{\cdot}342$, while with four rows and columns it is $0{\cdot}3317$. Since the convergence here is considerably less rapid than in the other calculations (on account of the relative smallness of A_0) the next approximation, taking five rows and columns of each determinant, was also computed and

found to be 0·3302, so that $K = 0{\cdot}330$ is probably not in error by as much as 0·001.

From (6.63), (6.72), and (6.73) it follows that

$$P_{2n_1n_2r} = 6P^{(1)}_{2n_1n_2r} = -12P^{(2)}_{2n_1n_2r}$$
$$= \int_0^\infty \{F(\chi) - F(0)\}\, a\, da,$$

where $F(\chi)$ is the coefficient of $s^{n_1}t^{n_2}u^r$ in

$$\frac{(hm)^6}{\pi^3}\, n_1!\, n_2!\, \tfrac{5}{2}\pi^{\frac{1}{2}}\, \frac{3\pi^4}{8(hm)^{\frac{13}{2}}}\, \frac{2}{\Gamma(\frac{9}{2})}\, \frac{24}{\Gamma(r+\frac{7}{2})}$$
$$\times\Big[\frac{u\sin^2\frac{1}{2}\chi\cos^2\frac{1}{2}\chi(2-u-s-t+ust)^{\frac{1}{2}}}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^5}$$
$$-\tfrac{1}{8}\,\frac{\sin^2\frac{1}{2}\chi\cos^2\frac{1}{2}\chi(3-u-us-ut)(2-u-s-t+ust)^{-\frac{1}{2}}}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^4}$$
$$-\tfrac{1}{24}\,\frac{(1-u)(1-s\cos^2\frac{1}{2}\chi-t\sin^2\frac{1}{2}\chi)(2-u-s-t+ust)^{-\frac{3}{2}}}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^3}$$
$$-\tfrac{1}{32}\,\frac{(1-s)(1-t)(1-u)(2-u-s-t+ust)^{-\frac{5}{2}}}{(1-us\cos^2\frac{1}{2}\chi-ut\sin^2\frac{1}{2}\chi)^2}\Big]$$

plus a similar expression with χ replaced by $\pi-\chi$, except when $n_1 = n_2$.

Hence

$$P_{2000} = -\frac{12}{35}\,\frac{1}{(2\pi hm)^{\frac{1}{2}}}\,J,$$

$$P_{2010} = \frac{1}{2}\,P_{2000},\quad P_{2020} = \frac{3}{8}\,P_{2000},\quad P_{2030} = \frac{15}{32}\,P_{2000},$$
$$P_{2110} = \frac{3}{16}\,P_{2000},\quad P_{2120} = \frac{15}{32}\,P_{2000},$$

$$P_{2001} = -\frac{65}{42}\,P_{2000},\quad P_{2011} = \frac{149}{84}\,P_{2000},$$
$$P_{2021} = \frac{107}{112}\,P_{2000},\quad P_{2111} = \frac{145}{672}\,P_{2000},$$

$$P_{2002} = \frac{43}{7.8.9}\,P_{2000},\quad P_{2012} = -\frac{563}{6.7.8}\,P_{2000},$$
$$P_{2022} = \frac{633}{7.64}\,P_{2000},\quad P_{2112} = \frac{537}{7.128}\,P_{2000},$$

$$P_{2003} = \frac{67}{3.7.9.11.16}\,P_{2000},\quad P_{2013} = \frac{5341}{3.7.9.11.32}\,P_{2000},\quad \ldots.$$

When these values are substituted and the first approximations for B_{2n} taken from (9.25) and (9.26), it is found that

$$M\nu\Sigma P_{2n_1n_20}B_{2n_1}B_{2n_2} = -\frac{12}{35}\frac{MvJ}{(2\pi hm)^{\frac{1}{2}}}\frac{5^2}{72^2}\frac{2\pi hm}{(MvJ)^2}\,1{\cdot}067\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2$$

$$= -\frac{1}{63}\frac{1{\cdot}067}{1{\cdot}016}\frac{\mu}{p}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2,$$

so that

$$M\nu\{\Sigma P_{2n_1n_20}B_{2n_1}B_{2n_2}+\Sigma P^{(1)}_{2n_1n_20}(B^{(1)}_{2n_1}B^{(1)}_{2n_2}+C^{(1)}_{2n_1}C^{(1)}_{2n_2})$$

$$+\Sigma P^{(2)}_{2n_1n_20}(B^{(2)}_{2n_1}B^{(2)}_{2n_2}+C^{(2)}_{2n_1}C^{(2)}_{2n_2})\}$$

$$= -\frac{1}{126}\frac{1{\cdot}067}{1{\cdot}016}\frac{\mu}{p}\left\{2\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2+3\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+3\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right.$$

$$\left.-6\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2-6\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2\right\}.$$

Also $$M\nu\{\Sigma P_{2n_1n_2r}B_{2n_1}B_{2n_2}+\ldots\} \quad (r=1,\,2,\,3,\,\ldots)$$

is the same expression with the factor 1·067 replaced by $-1{\cdot}477$ when $r=1$, by $-0{\cdot}010$ when $r=2$, by 0·0084 when $r=3$, ...

The corresponding terms in (9.32) are

$$\tfrac{1}{2^{\frac{1}{2}}}\left[2\{(2r+\tfrac{7}{2})B_{2r}-2B_{2,r-1}\}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\right.$$

$$+\{(2r+\tfrac{7}{2})B^{(1)}_{2r}-2B^{(1)}_{2,r-1}\}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\{(2r+\tfrac{7}{2})C^{(1)}_{2r}-2C^{(1)}_{2,r-1}\}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)$$

$$\left.-\{(2r+\tfrac{7}{2})B^{(2)}_{2r}-2B^{(2)}_{2,r-1}\}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)-\{(2r+\tfrac{7}{2})C^{(2)}_{2r}-2C^{(2)}_{2,r-1}\}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]$$

and this reduces as follows on substitution of the first approximations for the B's; when $r=0$, it gives

$$-\frac{\mu}{9p}\left[2\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2+3\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+3\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right.$$

$$\left.-6\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2-6\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2\right];$$

when $r=1$, it is the same expression multiplied by a factor $-(0{\cdot}482/1{\cdot}016)$; when $r=2$, it has to be multiplied by $-(0{\cdot}025/1{\cdot}016)$; and when $r=3$, it has to be multiplied by $-(0{\cdot}002/1{\cdot}016)$.

The corresponding parts of B_{20}, B_{21}, ... are therefore given by

$$M\nu N_{000}\{B_{20}-\tfrac{1}{4}B_{21}\ldots\}=-\frac{\mu}{9p}\frac{A_0}{1{\cdot}016}\left[2\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2\right.$$
$$\left.+3\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+3\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2-6\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2-6\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2\right]$$

and similar equations, where

$$A_0=1{\cdot}016-\tfrac{1}{14}\times 1{\cdot}067=0{\cdot}940,$$
$$A_1=-0{\cdot}482+\tfrac{1}{14}\times 1{\cdot}477=-0{\cdot}376,$$
$$A_2=-0{\cdot}025+\tfrac{1}{14}\times 0{\cdot}010=-0{\cdot}024,$$
$$A_3=-0{\cdot}002-\tfrac{1}{14}\times 0{\cdot}0084=-0{\cdot}003.$$

Hence this part of B_{20} is

$$K\frac{\mu^2}{9p^2}\left[2\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2+3\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+3\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right.$$
$$\left.-6\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2-6\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2\right],$$

where K is the ratio of two determinants, as before. When two rows and columns of each determinant are taken the value found for K is $0{\cdot}848$; when three rows and columns are taken it is $0{\cdot}8423$, and this same value is obtained when four rows and columns are taken, so that

$$K=0{\cdot}842$$

is probably correct to the third place of decimals.

This completes the determination of B_{20} to the required order of magnitude, and $B_{20}^{(1)}$, $C_{20}^{(1)}$, $B_{20}^{(2)}$, $C_{20}^{(2)}$ can be deduced by a transformation of axes as before. The components of stress in the gas are

$$P_{zz}-p=pB_{20},\quad P_{zx}=\tfrac{1}{2}p\,B_{20}^{(1)},\quad P_{zy}=\tfrac{1}{2}p\,C_{20}^{(1)},$$

and similar expressions; so that, in general,

$$P_{zz}-p=-\frac{2\mu}{3}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\tfrac{2}{3}K_1\mu\frac{d}{dt}\frac{\mu}{p}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)$$
$$-K_1\frac{\mu^2}{p}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2-\left(\frac{\partial w}{\partial x}\right)^2-\left(\frac{\partial w}{\partial y}\right)^2\right\}$$
$$-K_2\frac{\mu^2}{\nu hmp}\left\{2\frac{\partial h}{\partial z}\frac{\partial p}{\partial z}-\frac{\partial h}{\partial x}\frac{\partial p}{\partial x}-\frac{\partial h}{\partial y}\frac{\partial p}{\partial y}\right\}$$
$$-K_3\frac{\mu h}{\nu m}\left\{2\frac{\partial}{\partial z}\frac{\mu}{h^2}\frac{\partial h}{\partial z}-\frac{\partial}{\partial x}\frac{\mu}{h^2}\frac{\partial h}{\partial x}-\frac{\partial}{\partial y}\frac{\mu}{h^2}\frac{\partial h}{\partial y}\right\}$$
$$-K_4\frac{\mu^2}{\nu mh^2}\left\{2\left(\frac{\partial h}{\partial z}\right)^2-\left(\frac{\partial h}{\partial x}\right)^2-\left(\frac{\partial h}{\partial y}\right)^2\right\}$$
$$+K_5\frac{\mu^2}{9p}\left\{2\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2+3\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+3\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right.$$
$$\left.-6\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2-6\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2\right\},$$

$$P_{zx}=-\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+K_1\mu\frac{d}{dt}\frac{\mu}{p}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$
$$-K_1\frac{\mu^2}{p}\left\{\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial z}-\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right)\right\}$$
$$-K_2\frac{3\mu^2}{2\nu hmp}\left(\frac{\partial h}{\partial x}\frac{\partial p}{\partial z}+\frac{\partial h}{\partial z}\frac{\partial p}{\partial x}\right)-K_3\frac{3\mu h}{2\nu m}\left(\frac{\partial}{\partial z}\frac{\mu}{h^2}\frac{\partial h}{\partial x}+\frac{\partial}{\partial x}\frac{\mu}{h^2}\frac{\partial h}{\partial z}\right)$$
$$-K_4\frac{3\mu^2}{\nu mh^2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial z}+K_5\frac{\mu^2}{3p}\left\{3\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right.$$
$$\left.-2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\left(2\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}-\frac{\partial w}{\partial z}\right)\right\},$$

$$P_{yz}=-\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+K_1\mu\frac{d}{dt}\frac{\mu}{p}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)$$
$$-K_1\frac{\mu^2}{p}\left\{\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\frac{\partial w}{\partial z}-\left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\right)\right\}$$
$$-K_2\frac{3\mu^2}{2\nu hmp}\left(\frac{\partial h}{\partial y}\frac{\partial p}{\partial z}+\frac{\partial h}{\partial z}\frac{\partial p}{\partial y}\right)-K_3\frac{3\mu h}{2\nu m}\left(\frac{\partial}{\partial z}\frac{\mu}{h^2}\frac{\partial h}{\partial y}+\frac{\partial}{\partial y}\frac{\mu}{h^2}\frac{\partial h}{\partial z}\right)$$
$$-K_4\frac{3\mu^2}{\nu mh^2}\frac{\partial h}{\partial y}\frac{\partial h}{\partial z}+K_5\frac{\mu^2}{3p}\left\{3\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial z}\right)\right.$$
$$\left.-2\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\left(2\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}-\frac{\partial w}{\partial z}\right)\right\}.$$

The values of the constants K_1, K_2, K_3, K_4, K_5, for Maxwellian molecules and molecules which are elastic spheres, are

	K_1	K_2	K_3	K_4	K_5
Maxwellian molecules	1	0	1	0	1
Elastic spheres	1·014	0·227	0·806	0·330	0·842

10. It is of some interest to examine the circumstances in which the usual approximations for the stresses cease to be valid. It may readily be verified that, when

$$Q = \tfrac{5}{2}\pi^{\frac{1}{2}} hm Y_2 S_{\frac{5}{2}}^r(hmC^2),$$

the expression (5.12) (divided by ν) is

$$\begin{aligned}
&\tfrac{2}{3}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\frac{d}{\partial t}\,B_{20}-\frac{B_{20}}{h}\frac{dh}{dt}\\
&\quad-\tfrac{1}{6}\left[2B_{20}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+2B_{20}^{(1)}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+2C_{20}^{(1)}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right.\\
&\qquad\left.+B_{20}^{(2)}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+C_{20}^{(2)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\\
&\quad+2B_{20}\frac{\partial w}{\partial z}+B_{20}^{(1)}\frac{\partial w}{\partial x}+C_{20}^{(1)}\frac{\partial w}{\partial y}\\
&\quad+\frac{h}{\nu m^{\frac{1}{2}}}\left[\frac{\partial}{\partial x}\frac{\nu}{h^{\frac{3}{2}}}\left(\tfrac{1}{3}B_{11}^{(1)}+\tfrac{1}{2}B_{30}^{(1)}\right)+\frac{\partial}{\partial y}\frac{\nu}{h^{\frac{3}{2}}}\left(\tfrac{1}{3}C_{11}^{(1)}+\tfrac{1}{2}C_{30}^{(1)}\right)-\frac{\partial}{\partial z}\frac{\nu}{h^{\frac{3}{2}}}\left(2B_{11}-3B_{30}\right)\right],
\end{aligned}$$

which becomes, on substitution from (5.3),

$$\begin{aligned}
&\frac{d}{dt}\,B_{20}-\frac{1+B_{20}}{h}\frac{dh}{dt}+2(1+B_{20})\frac{\partial w}{\partial z}+B_{20}^{(1)}\frac{\partial w}{\partial x}+C_{20}^{(1)}\frac{\partial w}{\partial y}\\
&-\frac{h}{2\nu m^{\frac{1}{2}}}\left[\frac{\partial}{\partial x}\frac{\nu}{h^{\frac{3}{2}}}\left(B_{11}^{(1)}-B_{30}^{(1)}\right)+\frac{\partial}{\partial y}\frac{\nu}{h^{\frac{3}{2}}}\left(C_{11}^{(1)}-C_{30}^{(1)}\right)+3\frac{\partial}{\partial z}\frac{\nu}{h^{\frac{3}{2}}}\left(B_{11}-B_{30}\right)\right]. \quad (10.1)
\end{aligned}$$

If, in the expression for Q, Y_2 is replaced by $Y_2^{(1)}$, $Y_2^{(2)}$, $Z_2^{(1)}$, $Z_2^{(2)}$, in succession, four other expressions are obtained involving B_{20}, $B_{20}^{(1)}$, $B_{20}^{(2)}$, $C_{20}^{(1)}$, $C_{20}^{(2)}$, and coefficients with suffixes 11 and 30. The previous approximations were obtained by assuming that the parts of these expressions involving B's and C's were of smaller order than the other terms, which were equated to the part of Boltzmann's equation arising from the effect of collisions [$M\nu N_{000}B_{20}$ plus other terms, in (10.1)].

Another type of solution is that which occurs in problems on the motion of stellar clusters where the effect of collisions is negligible, so that the stresses are got by equating to zero (10.1) and the similar expressions. In the simple case when $u = v = 0$, $\partial w/\partial x = \partial w/\partial y = 0$, the solutions are $B_{20} = -1$, $B_{20}^{(1)} = C_{20}^{(1)} = 0$, $B_{11} = B_{30}$, ..., and the equation of mean motion reduces to

$$\frac{dw}{dt} = Z,$$

but, in general, the mean motion at any point is not the same as the motion of an individual of the cluster at that point.

We may also consider the case when terms arising from the effect of collisions, though not negligible, are of the same order as $B_{20}\partial w/\partial z$, and similar quantities in (10.1) and the corresponding expressions, *i.e.* when the velocity gradients are of the same order as $M\nu N_{000}$ (actually, when the eliminations are performed in order to determine B_{20}, ..., it is found that

$$M\nu N_{000} \quad \text{and} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

have to be of the same order). Taking for simplicity the case when $u = v = 0$, it appears that $\partial w/\partial z$ must be of the same order as $M\nu N_{000}$, which [*cf.* (8.21), (8.24), (9.22), (9.26)] is of the order of magnitude of p/μ. Now, for air at N.T.P., μ is 0·00017 and p is 10^6 c.g.s. units approximately, so that $p/\mu = 6 \cdot 10^9$ sec^{-1} units, roughly, and we want w to change by this amount per unit distance, *i.e.* w must change by an amount equal to the velocity of sound ($\frac{1}{3} \cdot 10^5$ cm. per sec.) in a distance of about $5{\cdot}6 \cdot 10^{-6}$ cm., which is of the order of magnitude of the mean free path of the molecules. It would appear, therefore, that, while the Navier-Stokes equations give a satisfactory first approximation to the motion of a gas under ordinary conditions, they may not be valid in cases where almost discontinuous motion is involved, as, for example, in the front of the "discontinuous" or "shock" wave which accompanies a projectile moving with a velocity greater than that of sound. Rayleigh's discussion of this problem is based on the Navier-Stokes equations, but it is easily shown that the relations which he obtains between the states of the medium in front of and behind the shock wave are independent of the forms assumed for the stresses in the medium and are therefore correct; but the conclusions which he draws from his discussion of the state of the medium while the shock wave is passing cannot be regarded as established.

Rhodes University College,
Grahamstown, S. Africa.

ON THE REGION OF SUMMABILITY (A, λ, α) $(\alpha > 1)$ AND $(A, \lambda \log \lambda, 1)$ OF DIRICHLET'S SERIES OF TYPE λ_n

By ERIC G. PHILLIPS.

[Received 26 January, 1935.—Read 14 February, 1935.]

1.1. Suppose that the series

$$\phi(y) = \sum_1^\infty a_n e^{-y\lambda_n{}^\alpha}$$

is convergent when $y > 0$ and that $\phi(y) \to l$ as $y \to 0$. Then we may say that the series Σa_n is summable (A, λ, α) to the sum l. This definition is due to Hardy*. It is an obvious extension of the definition to say that, if

$$\psi(y) = \sum_1^\infty a_n e^{-y\lambda_n \log \lambda_n}$$

is convergent when $y > 0$ and tends to a limit as $y \to 0$, the series Σa_n is summable $(A, \lambda \log \lambda, 1)$.

Following suggestions from Prof. Hardy and Miss M. L. Cartwright, I have been able to apply these methods of summation to the Dirichlet series

$$\sum_1^\infty a_n e^{-\lambda_n s},$$

where $0 < \lambda_n < \lambda_{n+1}$, $\lambda_n \to \infty$, and the series is absolutely convergent for $\sigma > \sigma_0$. The function defined by the series is denoted by $f(s)$. The results of this paper are expressed in the following two theorems.

THEOREM I. *The series is summable* (A, λ, α) *in the region* **A** *defined by the inequalities*

$$|\arg(s - s_n)| < \tfrac{1}{2}\pi(1 + 1/\alpha),$$

where s_n $(n = 1, 2, \ldots)$ *are the singularities of the function* $f(s)$, *and* $\alpha > 1$.

* G. H. Hardy, "The application of Abel's method of summation to Dirichlet series", *Quart. J. of Math.*, 47 (1916), 178–180.

THEOREM II. *The series is summable* $(A, \lambda \log \lambda, 1)$ *in the horizontal star domain of* $f(s)$, *which is defined by the inequalities*

$$|\arg(s-s_n)| < \pi.$$

1.2. These results are in contrast to those obtained by the arithmetical methods like typical means and Cesàro summation, which are valid only in half planes. But, on the other hand, they resemble closely those obtained by M. Riesz* in his application to Dirichlet series of Mittag Leffler's method for power series. Theorem II corresponds to M. Riesz's result, in which $1/\Gamma(1+y\lambda_n)$ is used as the summation factor, whereas we use

$$\exp(-y\lambda_n \log \lambda_n).$$

My results may, however, prove of interest in bridging the gap between M. Riesz's method, where the region of summability is related to the singularities of $f(s)$, and the other methods in which the region of summability is a half plane related to the order of $f(s)$. Hardy's results on Abel summation are of the latter type.

1.3. The double integrals used in the proof of Theorem I, I have taken from a paper† on Abel summation by Miss M. L. Cartwright, whom I should like to thank for her valuable suggestions and criticisms.

2.1. LEMMA‡ 2.1.

$$(2.11)\quad 2\Gamma(w)\cos\gamma w \sum_1^\infty a_n e^{-\lambda_n s}\lambda_n^{-w} = \int_0^\infty \{f(s+ve^{i\gamma})+f(s+ve^{-i\gamma})\}v^{w-1}dv,$$

provided that $\sigma > \sigma_0$, $R(w) > 0$, *and* $0 < \gamma < \frac{1}{2}\pi$.

When $R(w) > 0$, we have

$$\Gamma(w)\lambda_n^{-w} = \int_0^\infty e^{-\lambda_n v} v^{w-1}dv.$$

---

* M. Riesz, "Sur la représentation analytique des fonctions définies par des séries de Dirichlet", *Acta Math.*, 35 (1912), 253–270. There is an account of this method in V. Bernstein's *Leçons sur les séries de Dirichlet* (Paris, 1933).

† M. L. Cartwright, "On the relation between the different types of Abel summation", *Proc. London Math. Soc.* (2), 31 (1931), 81–96.

‡ M. L. Cartwright, *ibid.*, Lemma 2.1.

Multiplying each side by $a_n e^{-\lambda_n s}$ and summing, we obtain

$$\Gamma(w)\sum_1^\infty a_n e^{-\lambda_n s}\lambda_n^{-w} = \int_0^\infty v^{w-1} f(s+v)\,dv. \tag{2.12}$$

Term by term integration is justified by the fact that the series

$$\Sigma\left|a_n e^{-\lambda_n s}\right| \int_0^\infty v^{\kappa-1} e^{-\lambda_n v}\,dv,$$

where $\kappa = R(w)$, is convergent for $\kappa > 0$, $\sigma > \sigma_0$. Now, since

$$f(s+re^{i\theta}) = \sum_1^\infty a_n e^{-\lambda_n(s+r\cos\theta+ir\sin\theta)} = O(e^{-\lambda_1 r\cos\theta})$$

as $r\to\infty$, $|\theta| < \frac{1}{2}\pi$, we can show by a simple contour integration that

$$\int_0^\infty v^{w-1} f(s+v)\,dv = e^{i\gamma w}\int_0^\infty v^{w-1} f(s+ve^{i\gamma})\,dv$$

$$= e^{-i\gamma w}\int_0^\infty v^{w-1} f(s+ve^{-i\gamma})\,dv,$$

provided that $0 < \gamma < \frac{1}{2}\pi$. Multiplying the first equation by $e^{-i\gamma w}$, the second by $e^{i\gamma w}$, adding and substituting in (2.12), we obtain (2.11).

2.2. LEMMA* 2.2.

$$\sum_1^\infty a_n e^{-\lambda_n s} e^{-(y\lambda_n)^\alpha} \tag{2.21}$$

$$= \frac{1}{4\alpha\pi i}\int_0^\infty \{f(s+ve^{i\gamma})+f(s+ve^{-i\gamma})\}\frac{dv}{v}\int_{\kappa-i\infty}^{\kappa+i\infty}\left(\frac{y}{v}\right)^{-w}\frac{\Gamma(w/\alpha)}{\Gamma(w)\cos\gamma w}\,dw,$$

provided that $\sigma > \sigma_0$, $y > 0$, $\alpha > 1$, $1 > \kappa > 0$ *and* $\frac{1}{2}\pi(1-1/\alpha) < \gamma < \frac{1}{2}\pi$.

By Mellin's integral we have

$$e^{-y^\alpha} = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} y^{-\alpha w}\Gamma(w)\,dw = \frac{1}{2\alpha\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} y^{-w}\Gamma(w/\alpha)\,dw,$$

---

* *Cf.* M. L. Cartwright, *ibid.*, Lemma 2.3.

provided that $y>0$, $\kappa>0$. Writing $y\lambda_n$ for y, multiplying by $a_n e^{-\lambda_n s}$, and summing, we obtain

$$(2.22)\qquad \sum_1^\infty a_n e^{-\lambda_n s-(y\lambda_n)^z}=\frac{1}{2a\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} y^{-w}\Gamma(w/a)\sum_1^\infty a_n e^{-\lambda_n s}\lambda_n^{-w}\,dw.$$

Term by term integration is justified by the fact that

$$\Sigma\,|a_n e^{-\lambda_n s}\lambda_n^{-w}|$$

is convergent for $\kappa>0$, $\sigma>\sigma_0$, so that

$$\int_{\kappa-i\infty}^{\kappa+i\infty}|y^{-w}|\,|\Gamma(w/a)|\,\Sigma|a_n e^{-\lambda_n s}\lambda_n^{-w}|\,|dw|$$

is convergent. Substituting from Lemma 2.1 in (2.22), we obtain (2.21). To justify the inversion, we observe that the double integral

$$\int_{\kappa-i\infty}^{\kappa+i\infty}\int_0^\infty\left|\frac{y^{-w}\Gamma(w/a)\,v^{w-1}}{\Gamma(w)\cos\gamma w}\{f(s+ve^{i\gamma})+f(s+ve^{-i\gamma})\}\right||dw|\,dv$$

is convergent for $\sigma>\sigma_0$, $y>0$, $\frac{1}{2}\pi(1-1/a)<\gamma<\frac{1}{2}\pi$, since it can be compared with

$$\int_{-\infty}^{\infty}\int_0^\infty\exp\{(-\tfrac{1}{2}\pi/a+\tfrac{1}{2}\pi-\gamma+\epsilon)\,|t|\}\cdot\left(\frac{v}{y}\right)^\kappa\frac{e^{-\lambda_1 v\cos\gamma}}{v}\,|dt|\,dv.$$

It may be noted that we restrict κ by the condition $\kappa<1$ to avoid difficulty with the poles due to the factor $\cos\gamma w$ in the denominator.

3.1. *Proof of Theorem I.*

Let $\mathbf{A}_1$ be the region defined by the inequalities

$$(3.11)\qquad\begin{cases}|\arg(s-s_n-\epsilon)|<\pi-\gamma\quad(n=1,2,\ldots),\\ |\arg(s+R)|<\gamma,\\ \sigma<R,\end{cases}$$

where $\epsilon>0$, $R>\sigma_0$, $\frac{1}{2}\pi(1-1/a)<\gamma<\frac{1}{2}\pi$.

Then $\pi-\gamma<\frac{1}{2}\pi(1+1/a)$, so that $\mathbf{A}_1$ lies in $\mathbf{A}$; and $f(s)$ is analytic and therefore bounded in $\mathbf{A}_1$. Also we can choose the constants ϵ, γ, R such that $\mathbf{A}_1$ contains any given point of $\mathbf{A}$. From the Argand diagram it is

easily seen that the inequalities

$$|\arg(s+ve^{\pm i\gamma}-s_n-\epsilon)|<\pi-\gamma,$$

$$|\arg(s+ve^{\pm i\gamma}+R)|<\gamma \quad (v\geqslant 0)$$

are satisfied whenever s satisfies (3.11), *i.e.* whenever s lies in $\mathbf{A}_1$. When $\sigma+v\cos\gamma<R$, therefore, the points $s+ve^{\pm i\gamma}$ lie in $\mathbf{A}_1$, and

$$|f(s+ve^{\pm i\gamma})|\leqslant M, \tag{3.12}$$

where M is the maximum modulus of $f(s)$ in $\mathbf{A}_1$. When $\sigma+v\cos\gamma\geqslant R>\sigma_0$, we can use the Dirichlet series, and, by taking R large enough, we can make

$$|f(s+ve^{\pm i\gamma})|<Ke^{-\lambda_1 v\cos\gamma}, \tag{3.13}$$

where K is independent of s and v. It follows from these two inequalities that we can find a K, independent of s and v, such that (3.13) is satisfied for every s in $\mathbf{A}_1$ and every $v\geqslant 0$.

It follows that the double integral in (2.21) is uniformly convergent, and therefore is an analytic function of s throughout $\mathbf{A}_1$, so that the equation (2.21) holds throughout $\mathbf{A}_1$.

If now we take any point s in $\mathbf{A}$ we can choose the constants ϵ, γ, R so that $\mathbf{A}_1$ contains s and the equation (2.21) with this value of γ holds at the point s.

We have now to prove that the double integral in (2.21) tends to $f(s)$ as y tends to zero. If we change the variable v into vy the integral becomes

$$\frac{1}{4a\pi i}\int_0^\infty \{f(s+yve^{i\gamma})+f(s+yve^{-i\gamma})\}\frac{dv}{v}\int_{\kappa-i\infty}^{\kappa+i\infty} v^w\frac{\Gamma(w/a)}{\Gamma(w)\cos\gamma w}\,dw. \tag{3.14}$$

Consider the function*

$$I(v,\kappa)=\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} v^w\frac{\Gamma(w/a)}{\Gamma(w)\cos\gamma w}\,dw.$$

We integrate

$$\frac{1}{2\pi i}\int v^w\frac{\Gamma(w/a)}{\Gamma(w)\cos\gamma w}\,dw$$

round a rectangular contour with corners at the points κ_1-iT, κ_2-iT, κ_2+iT, κ_1+iT. The integrals along the horizontal parts tend to zero as

* *Cf.* M. L. Cartwright, *ibid.*, 87–89.

$T \to \infty$, and, provided that

$$-\min(\tfrac{1}{2}\pi/\gamma, \alpha) < \kappa_1 < \kappa_2 < \tfrac{1}{2}\pi/\gamma,$$

there is no pole inside the contour. Hence

$$I(v, \kappa_1) = I(v, \kappa_2).$$

Further, $I(v, \kappa) = O(v^\kappa)$ when $\kappa > 0$, $v \geqslant 0$, or $\kappa < 0$, $v > 0$.

We have, therefore,

$$\int_0^\infty |I(v, \kappa)| \frac{dv}{v} = \int_0^1 |I(v, \kappa_2)| \frac{dv}{v} + \int_1^\infty |I(v, \kappa_1)| \frac{dv}{v},$$

and, taking $\kappa_1 < 0 < \kappa_2$, we see that this integral converges. It follows from (3.12) and (3.13) that

$$|f(s + yve^{\pm i\gamma})| < K,$$

where $y \geqslant 0$, $v \geqslant 0$, and K is independent of y and v. Therefore the double integral (3.14) is uniformly convergent for $y \geqslant 0$ and we can make $y \to 0$ under the sign of integration. Thus the limit of the expression (3.14) is

$$\frac{f(s)}{\alpha} \int_0^\infty I(v, \kappa) \frac{dv}{v}.$$

To prove, finally, that $\int_0^\infty I(v, \kappa) \frac{dv}{v}$ is equal to α, we write it as*

$$\int_0^1 \frac{dv}{v} \int_{\kappa_2 - i\infty}^{\kappa_2 + i\infty} v^w \frac{\Gamma(w/\alpha)}{\Gamma(w) \cos \gamma w} dw + \int_1^\infty \frac{dv}{v} \int_{\kappa_1 - i\infty}^{\kappa_1 + i\infty} v^w \frac{\Gamma(w/\alpha)}{\Gamma(w) \cos \gamma w} dw$$

$$= S_1 + S_2.$$

The double integrals are absolutely convergent and we may calculate them by inverting the order of integration, obtaining

$$S_1 + S_2 = \frac{1}{2\pi i} \int_{\kappa_2 - i\infty}^{\kappa_2 + i\infty} - \int_{\kappa_1 - i\infty}^{\kappa_1 + i\infty} \frac{1}{w} \frac{\Gamma(w/\alpha)}{\Gamma(w) \cos \gamma w} dw$$

$$= \lim_{T \to \infty} \frac{1}{2\pi i} \int \frac{1}{w} \frac{\Gamma(w/\alpha)}{\Gamma(w) \cos \gamma w} dw,$$

* *Cf.* M. L. Cartwright, *ibid.*, 90.

where the integral is taken round the same rectangle as before. There is a pole at the origin with residue a. Thus

$$S_1+S_2=a,$$

and therefore $$\lim_{y\to 0}\sum_1^\infty a_n e^{-\lambda_n s}e^{-(y\lambda_n)^\alpha}=f(s).$$

This completes the proof of Theorem I.

4.1. In proving Theorem II it is necessary to use the function

$$f_N(s)=f(s)-\sum_1^{N-1}a_n e^{-\lambda_n s}.$$

It has the same singularities as $f(s)$ and, when $\sigma>\sigma_0$,

$$f_N(s)=\sum_N^\infty a_n e^{-\lambda_n s}.$$

LEMMA 4.1.

$$\phi_N(s,y)=\sum_N^\infty a_n e^{-\lambda_n s}e^{-yx\lambda_n\log\lambda_n}$$

$$=\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}(yx)^{-w}\Gamma(w)\sum_N^\infty a_n e^{-\lambda_n s}(\lambda_n\log\lambda_n)^{-w}\,dw,$$

provided that $0<y<1$, $x=1/\log(1/y)$, $\kappa>0$, *and* $\sigma>\sigma_0$.

This is proved in exactly the same way as (2.22).

4.2. LEMMA 4.2.

$$\sum_N^\infty a_n e^{-\lambda_n s}(\lambda_n\log\lambda_n)^{-w}=\frac{i}{2\Gamma(w)\sin\pi w}\int_{U_\gamma}(-u)^{w-1}\sum_N^\infty a_n e^{-\lambda_n s}\lambda_n^{-w-u}\,du,$$

provided that $R(w)>0$, $\sigma>\sigma_0$, *where* U_γ *is a modified Hankel contour going to infinity along the lines* $\arg u=\pm\gamma$ $(0<\gamma<\frac{1}{2}\pi)$.

This follows by summation from the formula

$$(\log\lambda_n)^{-w}=\frac{i}{2\Gamma(w)\sin\pi w}\int_{U_\gamma}(-u)^{w-1}\lambda_n^{-u}\,du,$$

the inversion of summation and integration presenting no difficulty. It is necessary to specify more particularly the contour U_γ. We

shall make it consist of the two lines $u = re^{\pm i\gamma}$ $(r \geqslant r_0)$, denoted by U_γ', and the larger arc of the circle with centre at the origin which joints the points $r_0 e^{\pm i\gamma}$, denoted by U_γ''. By choosing r_0 small enough, we can ensure that $R(w+u) > 0$ on U_γ when $R(w) > 0$.

4.3. LEMMA 4.3.

$$\psi(s, w, u) = \sum_N^\infty a_n e^{-\lambda_n s} \lambda_n^{-w-u} = \frac{1}{2\cos\delta w\, \Gamma(w+u)} \int_0^\infty v^{w+u-1} F(s, v)\, dv,$$

provided that $R(w+u) > 0$, $0 < \delta < \frac{1}{2}\pi$, $\sigma > \sigma_0$, *where*

$$F(s, v) = e^{i\delta u} f_N(s+ve^{i\delta}) + e^{-i\delta u} f_N(s+ve^{-i\delta}).$$

As in Lemma 2.1, we can show that

$$\Gamma(w+u)\psi(s, w, u) = e^{i\delta(w+u)} \int_0^\infty v^{w+u-1} f_N(s+ve^{i\delta})\, dv$$

$$= e^{-i\delta(w+u)} \int_0^\infty v^{w+u-1} f_N(s+ve^{-i\delta})\, dv.$$

Multiplying the first by $e^{-i\delta w}$, the second by $e^{i\delta w}$, and adding, we prove the lemma.

4.4. LEMMA 4.4. *Let the conditions of the three previous lemmas be satisfied and let* $\frac{1}{2}\pi - \delta < \gamma < \frac{1}{2}\pi$. *Then we can choose* N, *depending on* δ *and* γ, *such that*

$$\phi_N(s, y) = J_N(\delta, y) + J_N(-\delta, y), \tag{4.41}$$

where

$$J_N(\delta, y) = \frac{1}{8\pi} \int_0^\infty f_N(s+ve^{i\delta}) \frac{dv}{v}$$

$$\times \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{yx}{v}\right)^{-w} \frac{dw}{\cos\delta w \sin\pi w} \int_{U_\gamma} \frac{(-u)^{w-1} e^{i\delta u} v^u}{\Gamma(w+u)}\, du.$$

This follows by successive substitution from Lemmas 4.1, 4.2, and 4.3, provided that we can justify the inversion of the integration. This we do by showing that the triple integral is absolutely convergent when

N is taken large enough. We shall need a majorant for $1/\Gamma(w+u)$ valid when $R(w) > -\frac{1}{2}$ and $R(w+u) > -\frac{1}{2}$. We have

$$\left| \frac{1}{\Gamma(w+u)} \right| = \left| \frac{w+1+u}{\Gamma(w+1+u)} \right|.$$

With the above conditions $R(w+u+1) > \frac{1}{2}$, and we can use Binet's* result that

$$|\log \Gamma(w+u+1) - (w+u+\tfrac{1}{2}) \log (w+u+1) + w+u+1 - \tfrac{1}{2} \log (2\pi)| < 2K_1,$$

where K_1 is an absolute constant. If we write $w+u+1 = X+iY = Re^{i\theta}$, then $X-\frac{1}{2} > 0$, $|\theta| < \frac{1}{2}\pi$, and we deduce from the above that

$$\begin{aligned} |1/\Gamma(w+u+1)| &< K_2 \exp\{Y\theta + X - (X-\tfrac{1}{2}) \log R\} \\ &< K_2 \exp\{\tfrac{1}{2}\pi |Y| + X - (X-\tfrac{1}{2}) \log (X-\tfrac{1}{2})\}. \end{aligned}$$

Hence, when $w = \kappa + it$ and u is on U_γ', *i.e.* $u = re^{i\gamma}$,

$$1/\Gamma(w+u) = O\{\exp [-\rho \log \rho + a\rho + (\tfrac{1}{2}\pi + \epsilon)|t|]\},$$

as $|t| \to \infty$ and $r \to \infty$, where $\rho = \kappa + \frac{1}{2} + r \cos \gamma$ and a is a constant greater than $1 + \frac{1}{2}\pi \tan \gamma$. And, when u is on U_γ'', *i.e.* $|u| = r_0 < \kappa + \frac{1}{2}$,

$$1/\Gamma(w+u) = O\{\exp (\tfrac{1}{2}\pi + \epsilon)|t|\}$$

as $|t| \to \infty$. As for the other elements of the integrand, we have

$$(-u)^{w-1} = O\{r^{\kappa-1} e^{(\pi-\gamma)|t|}\}, \quad \sin \pi w = O\{e^{\pi|t|}\}, \quad \ldots$$

and $f_N(s+ve^{i\delta}) = O(e^{-\lambda_N v \cos \delta})$ as $v \to \infty$, the last result being obtained from the expression of f_N by its Dirichlet series. So far, therefore, as concerns the integral at its infinite limits, we can compare it with

$$\int_1^\infty dv \int_{-\infty}^\infty dt \int_{\rho_0}^\infty d\rho v^{-\frac{3}{2}} r^{\kappa-1} \exp\{-\lambda_N v \cos \delta - \rho \log \rho + c\rho + \rho \log v + (\pi - \gamma + \tfrac{1}{2}\pi + \epsilon - \pi - \delta)|t|\},$$

where $c = a + \delta > 1 + \delta + \frac{1}{2}\pi \tan \gamma$.

---

* See Whittaker and Watson, *Modern analysis*, 4 ed., 249.

The t-integral is convergent since $\gamma+\delta > \frac{1}{2}\pi$. It is easily verified that

$$-\lambda_N v\cos\delta-\rho\log\rho+c\rho+\rho\log v<-\rho$$

if $\lambda_N > e^c/\cos\delta$. Choosing N so that this is satisfied, we can compare the rest of the triple integral with

$$\int_1^\infty dv\int_{\rho_0}^\infty d\rho\, v^{-\frac{3}{2}}e^{-\rho}$$

which is convergent.

The absolute convergence of the integral at its other limits presents no difficulties and the proof of the lemma is thus complete.

5.1. *Proof of Theorem II.* Let **B** be the region defined by the inequalities

$$|\arg(s-s_n-\epsilon)|<\pi-\delta \quad (n=1, 2, \ldots),$$

$$|\arg(s+R)|<\delta,$$

$$\sigma<R,$$

where $\epsilon>0$, $0<\delta<\frac{1}{2}\pi$, $R>\sigma_0$.

We can choose the constants ϵ, δ, R so that the region **B**, while still lying inside the horizontal star domain of $f(s)$, will yet contain any given point of the latter. Let γ be chosen so that $\frac{1}{2}\pi-\delta<\gamma<\frac{1}{2}\pi$ and then let N be chosen so that (4.41) of Lemma 4.4 is true. We show that (4.41) with these values of δ, γ, and N remains true throughout **B**. Exactly as in the proof of Theorem I, we can show that

$$|f_N(s+ve^{\pm i\delta})|<Ke^{-\lambda_N v\cos\delta} \tag{5.101}$$

for every s in **B** and every $v\geqslant 0$, where K is a constant independent of s and v.

It follows that the triple integral in (4.41) is uniformly convergent and therefore is an analytic function of s throughout **B**, so that the equation (4.41) holds throughout **B**. If now we take any point s in the horizontal star domain of $f(s)$, we can choose the constants ϵ, δ, R so that **B** contains s and then the constants γ, N so that the equation (4.41) holds at the point s. If we can prove that

$$J_N(\delta, y)+J_N(-\delta, y)\to f_N(s) \tag{5.11}$$

as $y \to 0$, then

$$\lim_{y\to 0} \sum_1^\infty a_n e^{-\lambda_n s} e^{-yx\lambda_n \log \lambda_n} = \lim_{y\to 0} \left(\sum_1^{N-1} + \sum_N^\infty \right)$$

$$= \sum_1^{N-1} a_n e^{-\lambda_n s} + f_N(s)$$

$$= f(s),$$

and the theorem will be proved.

Consider the integral $J_N(\delta, y)$ and change the variables v and u into vy and ux respectively. The integral becomes

$$J_N(\delta, y) = \tfrac{1}{2} \int_0^\infty f_N(s + yve^{i\delta})\, I(\kappa, v)\, \frac{dv}{v}, \tag{5.12}$$

where

$$I(\kappa, v) = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \frac{v^w}{\cos \delta w}\, H(w, v)\, dw, \tag{5.13}$$

and

$$H(w, v) = \frac{i}{2 \sin \pi w} \int_{U_\gamma} \frac{(-u)^{w-1} e^{i\delta xu} e^{-u} v^{xu}}{\Gamma(w + xu)}\, du. \tag{5.14}$$

It is easily seen that $H(w, v)$ is an analytic function of w if $\kappa > -1$. If $\kappa > -\frac{1}{2}$, we can choose r_0 so small that $R(w + xu) > -\frac{1}{2}$ on the contour U_γ and then, using the majorant for $1/\Gamma(w + xu)$, we shall have

$$H(w, v)/\cos \delta w = O\{K(v) \exp(\tfrac{1}{2}\pi + \epsilon - \gamma - \delta)|t|\}$$

as $|t| \to \infty$, where $K(v)$ is independent of t. Hence we can integrate round a rectangular contour as in § 3.1 to obtain

$$I(\kappa_1, v) = I(\kappa_2, v),$$

where $-\frac{1}{2} < \kappa_1 < \kappa_2 < 1$. Choosing $\kappa_1 < 0 < \kappa_2$, we write

$$J_N(\delta, y) = \tfrac{1}{2} \int_0^1 f_N(s + yve^{i\delta})\, I(\kappa_2, v)\, \frac{dv}{v} + \tfrac{1}{2} \int_1^\infty f_N(s + yve^{i\delta})\, I(\kappa_1, v)\, \frac{dv}{v}. \tag{5.15}$$

Now, when $w = \kappa_2 + it$, $v \leqslant 1$, we have

$$v^w H(w, v)/\cos \delta w = O\{v^{\kappa_2 - xr_0} \exp(\tfrac{1}{2}\pi + \epsilon - \gamma - \delta)|t|\},$$

and, since $\kappa_2 > 0$, we can choose r_0 so small that $\kappa_2 - xr_0 > \frac{1}{2}\kappa_2$ for $0 \leqslant x \leqslant 1$. Hence

$$I(\kappa_2, v) = O(v^{\frac{1}{2}\kappa_2}) \tag{5.16}$$

uniformly for $0 \leqslant y \leqslant 1/e$. Also, when $w = \kappa_1 + it$, $v \geqslant 1$, we have

$$v^w e^{i\delta xu - u} v^{xu}/\Gamma(w+xu)$$

$$= O[v^{\kappa_1} \exp\{-\xi \log \xi + c\xi + (\xi - \kappa_1 - \tfrac{1}{2}) \log yv + (\tfrac{1}{2}\pi + \epsilon)|t|\}]$$

on U_γ', where $\xi = \kappa_1 + \frac{1}{2} + xr\cos\gamma$. When $x \geqslant 0$, $r \geqslant r_0$, then $\xi \geqslant \kappa_1 + \frac{1}{2} > 0$, and it is easily seen that for such values of x and r we have

$$\exp\{-\xi \log \xi + c\xi + (\xi - \kappa_1 - \tfrac{1}{2}) \log yv\} < A \exp(yve^{c-1}),$$

where A is independent of x. Thus

$$\frac{v^w e^{i\delta xu - u} v^{xu}}{\Gamma(w+xu)} = O[v^{\kappa_1} \exp\{yve^{c-1} + (\tfrac{1}{2}\pi + \epsilon)|t|\}]$$

on U_γ', uniformly for $0 \leqslant y \leqslant 1/e$.

Since $\kappa_1 < 0$, $v \geqslant 1$, the same expression is

$$O[v^{\kappa_1 + xr_0\cos\gamma} \exp\{(\tfrac{1}{2}\pi + \epsilon)|t|\}] = O[v^{\frac{1}{2}\kappa_1} \exp\{(\tfrac{1}{2}\pi + \epsilon)|t|\}]$$

on U_γ'', uniformly for $0 \leqslant y \leqslant 1/e$; for we can choose r_0 so small that

$$R(w+xu) \leqslant \kappa_1 + xr_0\cos\gamma < \tfrac{1}{2}\kappa_1 < 0$$

on U_γ'', with $0 \leqslant x \leqslant 1$.

Hence, when $\kappa_1 < 0$, $v \geqslant 1$,

$$I(\kappa_1, v) = O(v^{\frac{1}{2}\kappa_1}) + O\left\{v^{\kappa_1} \exp(yve^{c-1}) \int_{r_0}^{\infty} r^{-1+\kappa_1}\, dr\right\} \tag{5.17}$$

$$= O\{v^{\frac{1}{2}\kappa_1} \exp(yve^{c-1})\}$$

uniformly for $0 \leqslant y \leqslant 1/e$.

Now it follows from (5.101) that

$$|f_N(s + yve^{\pm i\delta})| < Ke^{-\lambda_N yv\cos\delta}$$

for $y \geqslant 0$, $v \geqslant 0$, where K is independent of y and v. Remembering that $-\lambda_N \cos\delta < -e^c < -e^{c-1}$, we see from (5.15), (5.16), and (5.17) that

the integral $J_N(\delta, y)$ is uniformly convergent for $0 \leqslant y \leqslant 1/e$. For it can be compared with

$$\int_0^1 v^{\frac{1}{2}\kappa_2 - 1}\, dv + \int_1 v^{\frac{1}{2}\kappa_1 - 1}\, dv,$$

which is convergent since $\kappa_1 < 0 < \kappa_2$. We can, therefore, make $y \to 0$ under the sign of integration in $J_N(\delta, y)$. We find that $H(w, v) \to 1$ and

$$\lim_{y \to 0} J_N(\delta, y) = \tfrac{1}{2} f_N(s) \int_0^1 \frac{dv}{v} \int_{\kappa_2 - i\infty}^{\kappa_2 + i\infty} \frac{v^w\, dw}{\cos \delta w}$$

$$+ \tfrac{1}{2} f_N(s) \int_1^\infty \frac{dv}{v} \int_{\kappa_1 - i\infty}^{\kappa_1 + i\infty} \frac{v^w\, dw}{\cos \delta w}$$

$$= \tfrac{1}{2} f_N(s),$$

as in the proof of Theorem I.

In a similar way we can show that

$$\lim_{y \to 0} J_N(-\delta, y) \to \tfrac{1}{2} f_N(s).$$

This completes the proof of (5.11) and therefore of the theorem.

Magdalen College,
Oxford.

SOME RELATIONS BETWEEN THE DE LA VALLEE POUSSIN AND ABEL METHODS OF SUMMABILITY

By J. M. HYSLOP.

[Received 6 April, 1935.—Read 25 April, 1935.]

1. *Introduction.*

The $(V\text{-}P)$ method of summability was introduced by de la Vallée Poussin*. According to his definition, the series Σa_λ is summable $(V\text{-}P)$, to the sum s, if

$$\lim_{n\to\infty} \sum_0^n \frac{(n!)^2}{(n-\lambda)!\,(n+\lambda)!}\, a_\lambda = s.$$

I shall have occasion to use an auxiliary method of summability which I denote by $(G\text{-}V\text{-}P)$. The series Σa_λ will be said to be summable $(G\text{-}V\text{-}P)$, to the sum s, if

$$\lim_{x\to 0} \sum_0^\infty a_\lambda K(x, \lambda) = s,$$

where x tends to zero through any positive values and

$$K(x, \lambda) = \frac{\{\Gamma(x^{-1}+1)\}^2}{\Gamma(x^{-1}+\lambda+1)\,\Gamma(x^{-1}-\lambda+1)} \qquad (0 \leqslant \lambda \leqslant x^{-1}),$$

$$= 0 \qquad (\lambda > x^{-1}).$$

It will be observed that, when x tends to zero through the values n^{-1}, the $(G\text{-}V\text{-}P)$ definition reduces to the $(V\text{-}P)$ definition.

The series Σa_λ is said to be summable $(A, 2a)$ to the sum s if

(i) $\sum_0^\infty a_\lambda \exp(-\lambda^{2a}x)$ converges when x is positive,

(ii) $$\lim_{x\to 0} \sum_0^\infty a_\lambda \exp(-\lambda^{2a}x) = s.$$

* Ch. de la Vallée Poussin, *Bull. de l'acad. r. de Belgique* (1908), 230–233.

Two particular cases of this method will be required here, namely those corresponding to $\alpha = 1$ and $\alpha = \frac{1}{2}$. The former will be denoted by (W) and the latter, being the ordinary Abel power series method, by (A).

It has been proved by Gronwall* that, if Σa_λ is summable by Cesàro's means of any order, it is also summable $(V\text{-}P)$ to the same sum. The object of the present paper is to prove that, if $a_\lambda = O(\lambda^k)$, the methods $(V\text{-}P)$ and (W) are equivalent, in the sense that summability $(V\text{-}P)$ implies summability (W) and conversely. These results are proved in Theorems III and VI. In §8 I prove that, if Σa_λ is summable $(V\text{-}P)$ and if $\Sigma a_\lambda r^\lambda$ converges for $0 \leqslant r < 1$, then Σa_λ is summable (A) to the same sum.

I should like to take this opportunity of thanking Prof. Hardy and Prof. Littlewood for their helpful suggestions and criticisms and also Mr. G. W. Morgan for the careful way in which he read the rough manuscript.

2. *Preliminary lemmas.*

LEMMA 1. *If ρ is any positive number,*

$$\sum_0^\infty \lambda^\rho \exp(-\lambda^2 x) = O\{x^{-\frac{1}{2}(\rho+1)}\}$$

as x tends to zero.

By considering the derivative with respect to λ of the function $\lambda^\rho \exp(-\lambda^2 x)$, it is easy to see that the function takes its maximum value when $\lambda^2 x = \frac{1}{2}\rho$. Hence

$$\begin{aligned}\sum_0^\infty \lambda^\rho \exp(-\lambda^2 x) &= \int_0^\infty y^\rho \exp(-y^2 x)\, dy + O\{x^{-\frac{1}{2}\rho}\} \\ &= \tfrac{1}{2} x^{-\frac{1}{2}(\rho+1)} \int_0^\infty u^{\frac{1}{2}(\rho-1)} \exp(-u)\, du + O\{x^{-\frac{1}{2}\rho}\} \\ &= O\{x^{-\frac{1}{2}(\rho+1)}\}\end{aligned}$$

as x tends to zero.

LEMMA 2. *If ρ, δ are positive, then*

$$\sum_{\delta/x}^\infty \lambda^\rho \exp(-\lambda^2 x) = o(1)$$

as x tends to zero.

---

* T. H. Gronwall, *Journal für Math.*, 147 (1916–1918), 16–27.

As in Lemma 1,

$$\begin{aligned}\sum_{\delta/x}^{\infty} \lambda^\rho \exp(-\lambda^2 x) &= \int_{\delta/x}^{\infty} y^\rho \exp(-y^2 x)\, dy + o(1)\\ &= \tfrac{1}{2} x^{-\frac{1}{2}(\rho+1)} \int_{\delta^2/x}^{\infty} u^{\frac{1}{2}(\rho-1)} \exp(-u)\, du + o(1)\\ &= O\{\exp(-\delta^2/x)\, x^{-\rho}\} + o(1)\\ &= o(1).\end{aligned}$$

LEMMA 3. *If $\phi(x)$ and $\psi(x)$ increase as x decreases and at least one of the functions tends to infinity as x tends to zero; and if $f^{(r)}(x)$ is continuous and*

$$f(x) = o\{\phi(x)\},$$

$$f^{(r)}(x) = O\{\psi(x)\};$$

then

$$f^{(s)}(x) = o\{\phi^{(r-s)/r}\, \psi^{s/r}\} \quad (0 \leqslant s < r)$$

as x tends to zero.

This result has been proved by Hardy and Littlewood*.

LEMMA 4. *If $a_\lambda = O(\lambda^k)$, p is a positive integer, and*

$$\sum_{0}^{\infty} a_\lambda \exp(-\lambda^2 x) = o(1)$$

as x tends to zero, then

$$\sum_{0}^{\infty} \lambda^{2p} a_\lambda \exp(-\lambda^2 x) = o(x^{-p-\eta})$$

as x tends to zero, for every positive η.

Choose η and let

$$r = \left[\frac{p(k+1)}{2\eta}\right] + 1.$$

If

$$f(x) = \sum_{0}^{\infty} a_\lambda \exp(-\lambda^2 x),$$

then

$$f^{(p)}(x) = (-1)^p \sum_{0}^{\infty} \lambda^{2p} a_\lambda \exp(-\lambda^2 x)$$

and

$$f^{(r)}(x) = (-1)^r \sum_{0}^{\infty} \lambda^{2r} a_\lambda \exp(-\lambda^2 x).$$

* G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 11 (1912), 417–427.

When x is positive, $f^{(r)}(x)$ is continuous and

$$|f^{(r)}(x)| = O\left\{\sum_0^\infty \lambda^{2r+k} \exp(-\lambda^2 x)\right\} = O\{x^{-\frac{1}{2}(2r+k+1)}\}$$

by Lemma 1. Thus, by Lemma 3,

$$f^{(p)}(x) = o\{x^{-p(2r+k+1)/2r}\} = o\{x^{-p-\eta}\}.$$

The lemma is therefore proved.

LEMMA 4*a*. *If the conditions of Lemma* 4 *are satisfied and* δ *is positive, then*

$$\sum_0^{\delta/x} \lambda^{2p} a_\lambda \exp(-\lambda^2 x) = o(x^{-p-\eta})$$

as x *tends to zero.*

This follows immediately from Lemmas 4 and 2.

LEMMA 5. *If* $0 < \delta < 1$, $0 \leqslant \lambda \leqslant \delta x^{-1}$, *and* μ *is any positive integer, then, for sufficiently small values of* x,

$$K(x, \lambda) = \exp\{-\lambda^2 x - u(x, \lambda) - R(x, \lambda)\},$$

where

$$R(x, \lambda) = O\{\lambda^{2\mu+2} x^{2\mu+1}\},$$

and $u(x, \lambda)$ *is of the form*

$$\lambda^2 \sum_{\beta=2}^{2\mu} A_{2,\beta} x^\beta + \sum_{\alpha=3}^{2\mu} \sum_{\beta=\alpha-1}^{2\mu} A_{\alpha,\beta} \lambda^\alpha x^\beta,$$

where the $A_{\alpha,\beta}$ *are constants.*

Since

$$\log K(x, \lambda) = 2\log\Gamma(x^{-1}+1) - \log\Gamma(x^{-1}-\lambda+1) - \log\Gamma(x^{-1}+\lambda+1),$$

we have, by a direct application of Stirling's theorem,

$$\begin{aligned}\log K(x, \lambda) = &-(x^{-1}-\lambda+\tfrac{1}{2})\log(1-\lambda x) - (x^{-1}+\lambda+\tfrac{1}{2})\log(1+\lambda x)\\ &+\frac{B_1}{1.2}\left\{\frac{2}{x^{-1}} - \frac{1}{x^{-1}-\lambda} - \frac{1}{x^{-1}+\lambda}\right\}\\ &-\frac{B_2}{3.4}\left\{\frac{2}{x^{-3}} - \frac{1}{(x^{-1}-\lambda)^3} - \frac{1}{(x^{-1}+\lambda)^3}\right\}\\ &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\ &+\frac{(-1)^{\mu-1} B_\mu}{(2\mu-1)\,2\mu}\left\{\frac{2}{x^{-(2\mu-1)}} - \frac{1}{(x^{-1}-\lambda)^{2\mu-1}} - \frac{1}{(x^{-1}+\lambda)^{2\mu-1}}\right\}\\ &+O(x^{2\mu+1}),\end{aligned}$$

where $B_1, B_2, \ldots, B_\mu$ are the Bernoulli numbers. Expanding each of these terms in a power series, we obtain

$$\log K(x,\lambda) = -\lambda^2 x - \frac{2}{3.4}\lambda^4 x^3 - \ldots - \frac{2}{(2\mu-1)\,2\mu}\lambda^{2\mu}x^{2\mu-1} - R''(x,\lambda)$$

$$+\frac{1}{2}\lambda^2 x^2 + \frac{1}{4}\lambda^4 x^4 + \ldots + \frac{1}{2\mu}\lambda^{2\mu}x^{2\mu} + R'(x,\lambda)$$

$$-B_1(\lambda^2 x^3 + \lambda^4 x^5 + \ldots + \lambda^{2\mu-2}x^{2\mu-1}) - R_1(x,\lambda)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$+\frac{(-1)^{\mu-1}2B_{\mu-1}}{(2\mu-2)!}\,\frac{(2\mu-2)!}{2!}\,\lambda^2 x^{2\mu-1}$$

$$+(-1)^{\mu-1}R_{\mu-1}(x,\lambda) + (-1)^{\mu}R_\mu(x,\lambda),$$

where

$$0 < R'' < C''\lambda^{2\mu+2}x^{2\mu+1},$$

$$0 < R' < C'\lambda^{2\mu+2}x^{2\mu+2},$$

$$0 < R_q < C_q\lambda^{2\mu+2-2q}x^{2\mu+1},$$

C'', C', C_q being constants and $q = 1, 2, \ldots, \mu$. The result therefore follows.

LEMMA 5*a*. *If* $0 < \delta < 1$, *and* $u(x,\lambda)$, $R(x,\lambda)$ *are the functions defined in Lemma* 5, *then, for sufficiently small values of* x,

(i) $\quad |u+R| < 1 \quad$ *when* $\quad 0 \leqslant \lambda \leqslant 2x^{-\frac{1}{2}}$,

(ii) $\quad u+R > 0 \quad$ *when* $\quad 2x^{-\frac{1}{2}} \leqslant \lambda \leqslant \delta x^{-1}$,

(iii) $\quad u > 0 \quad$ *when* $\quad \lambda \geqslant 2x^{-\frac{1}{2}}$,

(iv) $\quad e^{-u} = O(1) \quad$ *when* $\quad \lambda \geqslant 0$.

It is obvious that (i) is true, since $u = o(1)$, $R = O(x^\mu)$ when

$$0 \leqslant \lambda \leqslant 2x^{-\frac{1}{2}}.$$

To prove (ii), put $\lambda = bx^{-\alpha}$ in $u+R$, b being any positive constant. Then, for $\frac{1}{2} \leqslant \alpha < \frac{3}{4}$,

$$u+R = (\tfrac{1}{6}b^4 x^{3-4\alpha} - \tfrac{1}{2}b^2 x^{2-2\alpha}) + O(x^{5-6\alpha}),$$

which is positive for any value of b if $\frac{1}{2} < \alpha < \frac{3}{4}$. If $\alpha = \frac{1}{2}$, it is certainly positive if $b \geqslant 2$.

When $\frac{3}{4} \leqslant \alpha < 1$, we have

$$u+R = \tfrac{1}{6} b^4 x^{3-4\alpha} + O(x^{5-6\alpha}),$$

which is positive for all values of b.

When $\alpha = 1$ and $0 < b \leqslant \delta < 1$, we have

$$u+R = \left\{ \frac{2}{3.4} b^4 + \frac{2}{5.6} b^6 + \ldots + \frac{2}{(2\mu-1)\,2\mu} b^{2\mu} \right\} x^{-1} + R''(x, bx^{-1}) + O(1),$$

which is also positive.

It is clear that (iii) may be proved similarly. Those values of λ for which $\lambda = bx^{-\alpha}$, $\alpha > 1$, present no difficulty, since in this case

$$u = \frac{2}{(2\mu-1)\,2\mu} b^{2\mu} x^{2\mu(1-\alpha)-1} + o\{x^{2\mu(1-\alpha)-1}\},$$

which is positive.

By (iii), u is positive when $\lambda \geqslant 2x^{-\frac{1}{2}}$. Also $u = o(1)$ when $0 \leqslant \lambda \leqslant 2x^{-\frac{1}{2}}$. Hence, in both cases, $e^{-u} = O(1)$, which proves (iv).

LEMMA 5*b*. *If ρ is positive and $0 < \delta < 1$, then*

$$\sum_{0}^{\delta/x} \lambda^\rho \exp(-\lambda^2 x - u) = O\{x^{-\frac{1}{2}(\rho+1)}\}$$

as x tends to zero.

This follows immediately from Lemma 1 and (iv) of Lemma 5*a*.

LEMMA 5*c*. *If ρ is positive and $0 < \delta < 1$, then*

$$\sum_{\delta/x}^{\infty} \lambda^\rho \exp(-\lambda^2 x - u) = o(1)$$

as x tends to zero.

This follows immediately from (iv) of Lemma 5*a* and Lemma 2.

3. *Summability (W) implies summability (V-P).*

THEOREM I. *If $a_\lambda = O(\lambda^k)$ and Σa_λ is summable (W), then it is summable (G-V-P) to the same sum.*

We may suppose, without loss of generality, that the sum of the series in the (*W*) sense is zero.

If x is sufficiently small and $0<\delta<1$ we have, by Lemma 5,

$$\left|\sum_0^{\delta/x}\{\exp(-\lambda^2 x)-K(x,\lambda)\}a_\lambda\right| = \left|\sum_0^{\delta/x}\{1-e^{-u-R}\}\exp(-\lambda^2 x)a_\lambda\right|$$

$$\leqslant \left|\sum_{\lambda=0}^{\delta/x} a_\lambda \exp(-\lambda^2 x)\sum_{q=1}^{2\nu}(-1)^{q-1}\frac{(u+R)^q}{q!}\right|$$

$$+\left|\sum_{\lambda=0}^{\delta/x} a_\lambda \exp(-\lambda^2 x)\sum_{q=2\nu+1}^{\infty}(-1)^{q-1}\frac{(u+R)^q}{q!}\right|$$

$$= S_1+S_2,$$

where ν and the μ of Lemma 5 are at our disposal.

We now write $S_1 \leqslant |S_{1,1}|+S_{1,2}$,

where $$S_{1,1}=\sum_{q=1}^{2\nu}\left\{\sum_{\lambda=0}^{\delta/x}(-1)^{q-1}\frac{u^q}{q!}a_\lambda\exp(-\lambda^2 x)\right\},$$

$$S_{1,2}=\left|\sum_{\lambda=0}^{\delta/x} a_\lambda\exp(-\lambda^2 x)\left\{\sum_{q=1}^{2\nu}(-1)^{q-1}\frac{(u+R)^q-u^q}{q!}\right\}\right|.$$

Referring to the definition of u in Lemma 5, we see that the general term of u^q is of the form $D\lambda^m x^n$, where m and n satisfy

$$n\geqslant 2q, \quad m\leqslant q+n.$$

By Lemma 4a,

$$\sum_0^{\delta/x}\lambda^m x^n\exp(-\lambda^2 x)a_\lambda = o(x^{n-\frac{1}{2}m-\eta})$$

and $$n-\tfrac{1}{2}m\geqslant\tfrac{1}{2}(n-q)\geqslant\tfrac{1}{2}q\geqslant\tfrac{1}{2}.$$

Hence each term in $S_{1,1}$ is $o(1)$ as x tends to zero, for any fixed value of ν.

$S_{1,2}$ is majorized by a sum of terms such as

$$\sigma=\sum_0^{\delta/x}|R|^s|u|^t|a_\lambda|\exp(-\lambda^2 x),$$

where $s\geqslant 1$, $1\leqslant s+t\leqslant 2\nu$. Since $R=O(\lambda^{2\mu+2}x^{2\mu+1})$, $a_\lambda=O(\lambda^k)$ and $u=O(\lambda^4x^3)$ or $O(\lambda^2x^2)$, according as λ is greater or less than some number of the form $Ex^{-\frac{1}{2}}$, we have

$$\sigma = O\left\{x^{(2\mu+1)s+3t}\sum_0^{\delta/x}\lambda^{(2\mu+2)s+4t+k}\exp(-\lambda^2 x)\right\}$$

$$+O\left\{x^{(2\mu+1)s+2t}\sum_0^{\delta/x}\lambda^{(2\mu+2)s+2t+k}\exp(-\lambda^2 x)\right\}$$

$$= O\{x^{\mu s+t-\frac{1}{2}(k+1)}\},$$

by Lemma 1. If μ is chosen larger than $\frac{1}{2}(k+1)$ it follows that $\sigma = o(1)$. Thus, subject to this choice of μ, $S_1 = o(1)$ as x tends to zero.

We now write

$$S_2 \leqslant \sum_{0}^{2/\sqrt{x}} + \sum_{2/\sqrt{x}}^{\delta/x} |a_\lambda| \exp(-\lambda^2 x) \left| \sum_{2\nu+1}^{\infty} (-1)^{q-1} \frac{(u+R)^q}{q!} \right|$$

$$= S_{2,1} + S_{2,2}.$$

In $S_{2,1}$ we have $|u+R| < 1$, by Lemma 5*a*, so that

$$S_{2,1} = O\left\{ \sum_{0}^{2/\sqrt{x}} |u+R|^{2\nu+1} \lambda^k \exp(-\lambda^2 x) \right\}$$

$$= O\left\{ x^{4\nu+2} \sum_{0}^{2/\sqrt{x}} \lambda^{4\nu+2+k} \exp(-\lambda^2 x) \right\}$$

$$= O\{x^{2\nu-\frac{1}{2}(k-1)}\} = o(1),$$

by Lemma 1, if ν is chosen larger than $\frac{1}{4}(k-1)$.

It may be shown by repeated differentiation that, if y is positive,

$$1 - y + \frac{y^2}{2!} - \ldots + \frac{y^{2\nu}}{(2\nu)!} - e^{-y} < \frac{y^{2\nu+1}}{(2\nu+1)!}.$$

Now $u+R$ is positive in $S_{2,2}$, by Lemma 5*a*, so that, on applying this inequality, we have

$$S_{2,2} = O\left\{ \sum_{2/\sqrt{x}}^{\delta/x} (u+R)^{2\nu+1} \lambda^k \exp(-\lambda^2 x) \right\} = o(1),$$

by Lemma 1, if ν is chosen larger than $\frac{1}{4}(k-1)$.

It follows that

$$\sum_{0}^{\delta/x} \{\exp(-\lambda^2 x) - K(x, \lambda)\} a_\lambda = o(1)$$

as x tends to zero.

For each fixed value of x, $K(x, \lambda)$ is a decreasing function of λ, so that

$$\left| \sum_{\delta/x}^{1/x} \{\exp(-\lambda^2 x) - K(x, \lambda)\} a_\lambda \right| \leqslant x^{-k-1} \max_{\delta \leqslant \lambda x \leqslant 1} |\exp(-\lambda^2 x) - K(x, \lambda)|$$

$$\leqslant x^{-k-1} \{\exp(-\delta^2/x) + K(x, \delta/x)\} = o(1),$$

by Lemmas 5 and 5*a*. We therefore have

$$\sum_{0}^{1/x} \{\exp(-\lambda^2 x) - K(x, \lambda)\} a_\lambda = o(1).$$

But, by the hypothesis and Lemma 2,

$$\sum_0^{1/x} a_\lambda \exp(-\lambda^2 x) = o(1).$$

The theorem is therefore proved.

THEOREM II. *If the series Σa_λ is summable (G-V-P), it is summable (V-P) to the same sum.*

This is obvious from the definitions of (*V-P*) and (*G-V-P*) summability.

THEOREM III. *If $a_\lambda = O(\lambda^k)$ and Σa_λ is summable (W), then the series is summable (V-P) to the same sum.*

This follows immediately from Theorems I and II.

4. *Some further lemmas.*

LEMMA 6. *If ρ is positive and $0 < \delta < 1$, then*

$$\sum_0^{\delta/x} \lambda^\rho K(x, \lambda) = O\{x^{-\frac{1}{2}(\rho+1)}\}$$

as x tends to zero.

This follows from Lemmas 5, 5*a*, and 1; for

$$\sum_0^{\delta/x} \lambda^\rho K(x, \lambda) = \sum_0^{\delta/x} \lambda^\rho \exp(-u - R - \lambda^2 x)$$

$$\leqslant e \sum_0^{\delta/x} \lambda^\rho \exp(-\lambda^2 x) = O\{x^{-\frac{1}{2}(\rho+1)}\}.$$

LEMMA 7. *If ρ is positive and $0 < \delta < 1$, then*

$$\sum_{\delta/x}^{\infty} \lambda^\rho K(x, \lambda) = o(1)$$

as x tends to zero.

$K(x, \lambda)$ is a decreasing function of λ and is equal to zero when $\lambda > 1/x$. Hence

$$\sum_{\delta/x}^{\infty} \lambda^\rho K(x, \lambda) = \sum_{\delta/x}^{1/x} \lambda^\rho K(x, \lambda)$$

$$\leqslant x^{-\rho-1} K(x, \delta/x) = o(1),$$

by Lemma 5.

LEMMA 8. *If* $a_\lambda = O(\lambda^k)$, $0 < \delta < 1$, *n is a positive integer or zero,* $\gamma \geqslant n$, *and u, R are defined as in Lemma* 5, *with* $\mu = [\frac{1}{2}(k+1)]+1$, *then*

$$\sum_0^{\delta/x} \lambda^{2n} a_\lambda \exp(-\lambda^2 x - u - R) = o(x^{-\gamma})$$

implies

$$\sum_0^{\delta/x} \lambda^{2n} a_\lambda \exp(-\lambda^2 x - u) = o(x^{-\gamma})$$

and conversely.

If $0 \leqslant \lambda \leqslant x^{-\frac{1}{2}}$, then $R = O(\lambda^{2\mu+2} x^{2\mu+1}) = o(1)$, so that

$$1 - e^{-R} = O\{|R|\}.$$

Moreover, $\exp(-u)$ and $\exp(-u-R)$ are bounded for $0 \leqslant \lambda \leqslant \delta/x$. Hence

$$\left| \sum_0^{\delta/x} \lambda^{2n} a_\lambda \{\exp(-\lambda^2 x - u) - \exp(-\lambda^2 x - u - R)\} \right|$$
$$= O\left\{x^{2\mu+1} \sum_0^{x^{-\frac{1}{2}}} \lambda^{2\mu+2+k+2n} \exp(-\lambda^2 x - u)\right\} + O\left\{\sum_{x^{-\frac{1}{2}}}^{\delta/x} \lambda^{k+2n} \exp(-\lambda^2 x)\right\}.$$

When we use Lemma 5*b*, this becomes

$$O\{x^{-n+\mu-\frac{1}{2}(k+1)}\} + O\{x^{-k-2n-1} \exp(-x^{-\frac{1}{2}})\} = o(x^{-n}) = o(x^{-\gamma}),$$

since $\gamma \geqslant n$. The lemma is therefore proved.

LEMMA 9. *If* $a_\lambda = o(\lambda^k)$, $0 < \delta < 1$, *and*

$$\sum_0^\infty a_\lambda K(x, \lambda) = o(1)$$

as x tends to zero, then

(i) $$\sum_0^{\delta/x} \lambda^{2p} a_\lambda \exp(-\lambda^2 x - u) = o(x^{-p-\eta}),$$

(ii) $$\sum_0^\infty \lambda^{2p} a_\lambda K(x, \lambda) = o(x^{-p-\eta}),$$

where u is defined as in Lemma 5 *with* $\mu = [\frac{1}{2}(k+1)]+1$, *p is a positive integer, and* η *is any positive number.*

We have, by hypothesis,

$$\sum_0^{\delta/x} a_\lambda K(x, \lambda) + \sum_{\delta/x}^\infty a_\lambda K(x, \lambda) = o(1).$$

By Lemma 7, the second term tends to zero with x, and therefore, by Lemmas 5 and 8,

$$\sum_0^{\delta/x} a_\lambda \exp(-\lambda^2 x - u) = o(1).$$

It follows, by Lemma 5*c*, that

$$\sum_0^\infty a_\lambda \exp(-\lambda^2 x - u) = o(1).$$

Choose a positive η and let

$$r = \left[\frac{(p+\mu)(k+1)}{2\eta}\right] + 1.$$

If $f(x)$ denotes the function

$$\sum_0^\infty a_\lambda \exp(-\lambda^2 x - u),$$

it is easy to verify by direct differentiation that $f^{(s)}(x)$ is of the form

$$\sum_0^\infty a_\lambda \exp(-\lambda^2 x - u) \times \{\lambda^2 b_{s,1} + \lambda^4 b_{s,2} + \ldots + \lambda^{2s} b_{s,s} + \lambda^{2s+2} x^2 b_{s,s+1} + \ldots + \lambda^{2s\mu} x^{2s(\mu-1)} b_{s,s\mu}\},$$

where $b_{s,1}, b_{s,2}, \ldots, b_{s,s\mu}$ are polynomials in x alone and $b_{s,s}$ is not zero for small values of x. $f^{(r)}(x)$ is clearly continuous and is majorized by a sum of terms of the form

$$\sigma = \sum_0^\infty |a_\lambda| \exp(-\lambda^2 x - u) \lambda^{2\rho} x^t |b_{r,\rho}|,$$

where $1 \leqslant \rho \leqslant r\mu$, $t = \max\{0, 2(\rho - r)\}$. Now

$$\sigma = O\left\{x^t \sum_0^\infty \lambda^{2\rho+k} \exp(-\lambda^2 x - u)\right\} = O\{x^{t-\rho-\frac{1}{2}(k+1)}\},$$

by Lemmas 5*b* and 5*c*. The dominant term is given by $\rho = r$, $t = 0$, so that

$$f^{(r)}(x) = O\{x^{-r-\frac{1}{2}(k+1)}\}.$$

Hence, by Lemma 3,

$$f^{(s)}(x) = o\{x^{-s-(k+1)s/2r}\} \quad (s < r)$$
$$= o\{x^{-s-\eta}\} \quad (s = 1, 2, \ldots, p+\mu-1).$$

Denoting by $\phi_t(x)$ the function

$$\sum_0^\infty \lambda^{2t} a_\lambda \exp(-\lambda^2 x - u),$$

we then have $p+\mu-1$ equations of the form

$$b_{s,1}\phi_1 + \ldots + b_{s,s}\phi_s + b_{s,s+1} x^2 \phi_{s+1} + \ldots + b_{s,s\mu} x^{2s(\mu-1)} \phi_{s\mu} = o(x^{-s-\eta}),$$

where $s = 1, 2, \ldots, p+\mu-1$. Now, by Lemmas 5*b* and 5*c*,

$$x^{2\rho}\phi_{s+\rho} = O\{x^{\rho-s-\frac{1}{2}(k+1)}\} = o(x^{-s-\eta})$$

if $\rho \geqslant \mu$. We may therefore omit certain terms from each of the equations and rewrite them as

$$b_{s,1}\phi_1+\ldots+b_{s,s}\phi_s+b_{s,s+1}x^2\phi_{s+1}+\ldots+b_{s,s+\mu-1}x^{2\mu-2}\phi_{s+\mu-1} = o(x^{-s-\eta}).$$

Omit* now from this set of equations the ϕ_t's for which $t \geqslant p+\mu$. We then have the following $p+\mu-1$ equations in $p+\mu-1$ unknowns:

$$b_{s,1}\phi_1+\ldots+b_{s,s}\phi_s+b_{s,s+1}x^2\phi_{s+1}+\ldots+b_{s,p+\mu-1}x^{2\mu-2-2\sigma}\phi_{p+\mu-1} = o(x^{-p-2\tau-\eta}),$$

where $\sigma = \max\{0, s-p\}$ and $s = 1, 2, \ldots, p+\mu-1$.

The discriminant of the equations does not vanish when $x = 0$, since all the elements on one side of the principal diagonal vanish and no element in the principal diagonal does so. It does not vanish, therefore, in some neighbourhood of $x = 0$. The equations can thus be solved and, on examining the determinant obtained, we see at once that $\phi_p = o(x^{-p-\eta})$.

Now, from Lemma 5*c*,

$$\sum_{\delta/x}^{\infty} \lambda^{2p} a_\lambda \exp(-\lambda^2 x-u) = o(1),$$

and the first part of the lemma is therefore proved.

Since $$\sum_{0}^{\delta/x} \lambda^{2p} a_\lambda \exp(-\lambda^2 x-u) = o(x^{-p-\eta}),$$

it follows, from Lemma 8, that

$$\sum_{0}^{\delta/x} \lambda^{2p} a_\lambda \exp(-\lambda^2 x-u-R) = o(x^{-p-\eta}).$$

Hence, by Lemma 5,

$$\sum_{0}^{\delta/x} \lambda^{2p} a_\lambda K(x, \lambda) = o(x^{-p-\eta}),$$

and therefore, by Lemma 7,

$$\sum_{0}^{\infty} \lambda^{2p} a_\lambda K(x, \lambda) = o(x^{-p-\eta}).$$

---

* I am indebted to a Referee for suggesting this method of proof. It is much shorter than my original method.

5. *The converse of Theorem I.*

THEOREM IV. *If $a_\lambda = O(\lambda^k)$ and Σa_λ is summable (G-V-P), then it is summable (W) to the same sum.*

Without loss of generality we may suppose that the sum of the series in the (*G-V-P*) sense is zero.

By Lemma 8, we may take as our hypothesis

$$\sum_0^{\delta/x} a_\lambda \exp(-\lambda^2 x - u) = o(1),$$

where the μ associated with the function u is equal to $[\frac{1}{2}(k+1)]+1$ and $0 < \delta < 1$.

We therefore consider

$$\begin{aligned}
&\left| \sum_0^{\delta/x} \{\exp(-\lambda^2 x) - \exp(-\lambda^2 x - u)\} a_\lambda \right| \\
&\quad = \left| \sum_0^{\delta/x} (e^u - 1) a_\lambda \exp(-\lambda^2 x - u) \right| \\
&\quad \leqslant \left| \sum_{q=1}^{2\nu} \left\{ \sum_0^{\delta/x} \frac{u^q}{q!} a_\lambda \exp(-\lambda^2 x - u) \right\} \right| + \left| \sum_0^{\delta/x} \left\{ \sum_{q=2\nu+1}^{\infty} \frac{u^q}{q!} a_\lambda \exp(-\lambda^2 x - u) \right\} \right| \\
&\quad = |S_1| + S_2.
\end{aligned}$$

We can use Lemma 9 to show, much as in the proof of Theorem I, that $S_1 = o(1)$, as x tends to zero, for any value of ν.

We now write

$$S_2 = \sum_0^{2/\sqrt{x}} + \sum_{2/\sqrt{x}}^{\delta/x} = S_{2,1} + S_{2,2}.$$

Clearly $u = O(x)$ when $0 \leqslant \lambda \leqslant 2/\sqrt{x}$. Hence

$$\begin{aligned}
S_{2,1} &= O\left\{ \sum_0^{2/\sqrt{x}} |u|^{2\nu+1} \lambda^k \exp(-\lambda^2 x - u) \right\} \\
&= O\left\{ x^{2\nu+1} \sum_0^{\delta/x} \lambda^k \exp(-\lambda^2 x - u) \right\} = O\{x^{2\nu+1-\frac{1}{2}(k+1)}\}
\end{aligned}$$

by Lemma 5*b*. Thus, if ν is chosen greater than $\frac{1}{4}(k-1)$, we have $S_{2,1} = o(1)$.

It may easily be verified by differentiation that, if y is positive, and ρ is a positive integer,

$$\left(\frac{y^\rho}{\rho!} + \frac{y^{\rho+1}}{(\rho+1)!} + \ldots \right) e^{-y} < \frac{y^\rho}{\rho!}.$$

Applying this result to $S_{2,2}$, we obtain

$$S_{2,2} = O\left\{\sum_{2/\sqrt{x}}^{\delta/x} \lambda^k u^{2\nu+1} \exp(-\lambda^2 x)\right\}$$

$$= O\left\{x^{3(2\nu+1)} \sum_{0}^{\infty} \lambda^{4(2\nu+1)+k} \exp(-\lambda^2 x)\right\} = O\{x^{2\nu+1-\frac{1}{2}(k+1)}\}$$

by Lemma 1. Thus $S_{2,2}$ is also $o(1)$ if ν is larger than $\frac{1}{4}(k-1)$.

Combining these results, we have

$$\sum_{0}^{\delta/x} a_\lambda \exp(-\lambda^2 x) = o(1).$$

But, by Lemma 2,

$$\sum_{\delta/x}^{\infty} a_\lambda \exp(-\lambda^2 x) = o(1),$$

so that the theorem is proved.

6. *The converse of Theorem III.*

LEMMA 10. *If $a_\lambda = O(\lambda^k)$, p is a positive integer, and*

$$\sum_{0}^{n} a_\lambda \exp\{-\lambda^2 n^{-1} - u(n^{-1}, \lambda)\} = o(1)$$

as n tends to infinity through integral values, then

$$n^{-2p} \sum_{0}^{n} \lambda^{2p} a_\lambda \{\exp(-\lambda^2 n^{-1} - u(n^{-1}, \lambda)\} = o(1)$$

as n tends to infinity, if u is defined as in Lemma 5 and $\mu = [\frac{1}{2}(k+1)]+1$.

Let
$$f(n) = \sum_{0}^{n} a_\lambda \exp(-\lambda^2 n^{-1} - u),$$

$$\psi_\rho(n) = n^{-2\rho} \sum_{0}^{n} \lambda^{2\rho} a_\lambda \exp(-\lambda^2 n^{-1} - u)$$

$$(\rho = 1, 2, \ldots, p, \ldots).$$

Since $a_\lambda = O(\lambda^k)$ it follows, by Lemmas 5*b* and 5*c*, that $\psi_\rho(n) = o(1)$ when $\rho \geqslant \mu$.

It may be verified by direct calculation that the difference $f(n) - f(n-1)$ is of the form

$$c_{1,1}\psi_1 + c_{1,2}\psi_2 + \ldots + c_{1,\mu}\psi_\mu + o(1),$$

where $c_{1,1}, c_{1,2}, \ldots$ are polynomials in n^{-1} and $c_{1,1}$ is not zero for large values of n. More generally, if $\Delta^\rho f(n)$ denotes the expression

$$f(n)-\binom{\rho}{1}f(n-1)+\binom{\rho}{2}f(n-2)-\ldots+(-1)^\rho f(n-\rho),$$

then $\Delta^\rho f(n)$ is of the form

$$c_{\rho,1}n^{1-\rho}\psi_1+c_{\rho,2}n^{2-\rho}\psi_2+\ldots+c_{\rho,\rho}\psi_\rho+\ldots+c_{\rho,\mu}\psi_\mu+o(1),$$

where $c_{\rho,1}, c_{\rho,2}, \ldots$ are polynomials in n^{-1} and $c_{\rho,\rho}$ is not zero for large values of n.

By the hypothesis, each term in $\Delta^\rho f(n)$ tends to zero as n tends to infinity. We therefore obtain μ equations of the form

$$c_{\rho,1}n^{1-\rho}\psi_1+c_{\rho,2}n^{2-\rho}\psi_2+\ldots+c_{\rho,\rho}\psi_\rho+\ldots+c_{\rho,\mu}\psi_\mu=o(1),$$

where $\rho=1, 2, \ldots, \mu$.

Proceeding from this point as in Lemma 9, we easily see that $\psi_p=o(1)$ for any positive integer p.

THEOREM V. *If $a_\lambda=O(\lambda^k)$ and Σa_λ is summable (V-P), then it is summable (G-V-P) to the same sum.*

We may assume, without loss of generality, that the sum of the series in the (V-P) sense is zero.

It is clear that the results of Lemma 8 apply in the case of (V-P) summability, so that we may take as our hypothesis

$$\sum_0^n a_\lambda \exp\{-\lambda^2 n^{-1}-u(n^{-1},\lambda)\}=o(1)$$

as n tends to infinity, provided that the μ of Lemma 5 is chosen larger than $\frac{1}{2}(k+1)$.

Let $y=n+\theta$, where $0<\theta<1$. Then

$$\left|\sum_0^y a_\lambda \exp\{-\lambda^2 y^{-1}-u(y^{-1},\lambda)\}-\sum_0^n a_\lambda \exp\{-\lambda^2 n^{-1}-u(n^{-1},\lambda)\}\right|$$

$$=\left|\sum_0^n a_\lambda \exp\{-\lambda^2 n^{-1}-u(n^{-1},\lambda)\}\right.$$
$$\left.\times\sum_{t=1}^\infty \frac{1}{t!}\left[\frac{\theta\lambda^2}{n(n+\theta)}+u(n^{-1},\lambda)-u\{(n+\theta)^{-1},\lambda\}\right]^t\right|$$

$$\leqslant|A_1\psi_1+A_2\psi_2+\ldots+A_{2\mu\nu}\psi_{2\mu\nu}|$$
$$+A\sum_0^n\left|\frac{\theta\lambda^2}{n(n+\theta)}+u(n^{-1},\lambda)-u\{(n+\theta)^{-1},\lambda\}\right|^{2\nu+1}\lambda^k\exp\{-\lambda^2 n^{-1}-u(n^{-1},\lambda)\},$$

where $A_1, A_2, \ldots$ are polynomials or power series in n^{-1} and A is independent of n. The first term tends to zero, by Lemma 10; and if ν is chosen larger than $\frac{1}{4}(k-1)$, then the second term also tends to zero, by Lemmas 5*b* and 5*c*.

Hence, for any value of y such that $n \leqslant y \leqslant n+1$,

$$\sum_0^y a_\lambda \exp\{-\lambda^2 y^{-1} - u(y^{-1}, \lambda)\} = o(1)$$

as n tends to infinity. Thus

$$\sum_0^{1/x} a_\lambda \exp\{-\lambda^2 x - u(x, \lambda)\} = o(1)$$

as x tends to zero continuously. It then follows from Lemmas 5*c*, 8, and 7 that

$$\sum_0^\infty a_\lambda K(x, \lambda) = o(1).$$

The theorem is therefore proved.

THEOREM VI. *If $a_\lambda = O(\lambda^k)$ and Σa_λ is summable (V-P), then it is summable (W) to the same sum.*

This follows immediately from Theorems IV and V.

7. *The necessity of the condition $a_\lambda = O(\lambda^k)$.*

Gronwall* has proved the following result.

The series Σz^λ is summable (V-P) for any point z lying within the curve

$$|(1+z)/2\sqrt{z}| = 1$$

and is not summable (V-P) for any point on or outside this curve.

In a recent paper†, E. G. Phillips has proved a theorem which includes the following as a particular case.

The series Σz^λ is summable (W) for any point z lying within the curve

$$r = e^{|\theta|} \quad (-\pi < \theta \leqslant \pi).$$

It is perhaps worth mentioning that the series Σz^λ is not summable (*W*) on or outside this curve. This, however, is not required for our purpose, and we shall not give a proof here.

---

* T. H. Gronwall, *loc. cit.*

† E. G. Phillips, "On the region of summability (A, λ, α), $(\alpha > 1)$, and $(A, \lambda \log \lambda, 1)$ of Dirichlet's series of type λ_n", *Proc. London Math. Soc.* (2), 40 (1936), 436–448.

THEOREM VII. *There exist series* Σa_λ *with* $a_\lambda = O(e^{A\lambda})$, $A > 0$, *which are summable* (W) *but which are not summable* $(V\text{-}P)$.

This follows immediately from the above results, since the first curve lies wholly inside the second.

While Theorem VII does not by any means completely solve the question of the necessity of the condition $a_\lambda = O(\lambda^k)$ in Theorems III and VI, it points to the fact that the condition might be dispensed with in Theorem VI, but that very little if any generalization will be possible in the case of Theorem III.

8. *Summability* $(V\text{-}P)$ *and summability* (A).

It has been proved by Miss Cartwright*, as a particular case of a more general theorem, that, if Σa_λ is summable (W) and $\Sigma a_\lambda r^\lambda$ converges for $0 \leqslant r < 1$, then Σa_λ is summable (A) to the same sum. Combining this result with that of Theorem VI, we see that summability $(V\text{-}P)$ implies summability (A) subject to the condition $a_\lambda = O(\lambda^k)$. We shall show that this condition may be removed, and that the convergence of $\Sigma a_\lambda r^\lambda$, for $0 \leqslant r < 1$, is all that we require.

Suppose that $f(x)$ is a function integrable in the range $-\pi \leqslant x \leqslant \pi$ and that $f(x)$ has the Fourier series

$$\tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \dots .$$

Let h_n be defined by means of the relation

$$h_n^{-1} = \int_0^\pi (\cos \tfrac{1}{2}u)^{2n}\, du = \pi \frac{1 \cdot 3 \dots 2n-1}{2 \cdot 4 \dots 2n},$$

and let

$$I_n = \tfrac{1}{2}h_n \int_{-\pi}^{\pi} f(u)\, (\cos \tfrac{1}{2}u)^{2n}\, du.$$

If g_n denotes the expression

$$(2n)!/2^{2n-1}(n!)^2,$$

then, by expanding $(\cos \frac{1}{2}u)^{2n}$ in terms of cosines of multiples of u, we obtain

$$I_n = \tfrac{1}{2}h_n g_n \left[\int_{-\pi}^{\pi} f(u) \left\{ \tfrac{1}{2} + \frac{n}{n+1} \cos u + \dots + \frac{(n!)^2}{(2n)!} \cos nu \right\} du \right]$$

$$= \tfrac{1}{2}\pi h_n g_n \left\{ \tfrac{1}{2}a_0 + \frac{n}{n+1}\, a_1 + \dots + \frac{(n!)^2}{(2n)!}\, a_n \right\} = V_n,$$

where

$$V_n = \tfrac{1}{2}a_0 + \sum_{\lambda=1}^{n} a_\lambda K(n^{-1}, \lambda).$$

* M. L. Cartwright, *Proc. London Math. Soc.* (2), 31 (1930), 81–96.

This result occurs in the paper of de là Vallée Poussin* already mentioned and leads to his definition of $(V\text{-}P)$ summability.

Let $P(r)$ denote the power series

$$\tfrac{1}{2}a_0+\sum_{\lambda=1}^{\infty} a_\lambda r^\lambda,$$

convergent when $0 \leqslant r < 1$.

Then

$$P(r)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,\frac{1-r^2}{1-2r\cos u+r^2}\,du$$

$$=\frac{1}{2\pi}\,\frac{1-r^2}{(1+r)^2}\int_{-\pi}^{\pi} f(u)\left[1+\sum_{n=1}^{\infty}\left\{\frac{4r}{(1+r)^2}\right\}^n(\cos\tfrac{1}{2}u)^{2n}\right]du$$

$$=\frac{1-r}{1+r}\,\tfrac{1}{2}a_0+\frac{1}{2\pi}\,\frac{1-r}{1+r}\sum_{n=1}^{\infty}\left\{\frac{4r}{(1+r)^2}\right\}^n\frac{2I_n}{h_n}$$

$$=\frac{1-r}{1+r}\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2\,2^{2n}}\left\{\frac{4r}{(1+r)^2}\right\}^n V_n,$$

where $V_0=\frac{1}{2}a_0$. This formula has been proved on the assumption that a_λ is a Fourier constant. We proceed to show that it holds for all series Σa_λ for which $P(r)$ is convergent when $0 \leqslant r < 1$.

LEMMA 11. *If $0 \leqslant r < 1$ and m is a positive integer, then*

$$\frac{(1+r)^{2m+1}}{1-r}=\sum_{k=0}^{\infty}\binom{2m+2k}{k}\left\{\frac{r}{(1+r)^2}\right\}^k.$$

This is a case of Lagrange's expansion.

LEMMA 12. *If $P(r)$ is convergent when $0 \leqslant r < 1$, then*

$$P(r)=\frac{1-r}{1+r}\sum_{0}^{\infty}\frac{(2n)!}{(n!)^2\,2^{2n}}\left\{\frac{4r}{(1+r)^2}\right\}^n V_n.$$

The factor multiplying a_m on the right-hand side of the relation is

$$\frac{1-r}{1+r}\sum_{k=0}^{\infty}\frac{(4r)^{m+k}}{(1+r)^{2m+2k}}\cdot\frac{(2m+2k)!}{\{(m+k)!\}^2\,2^{2m+2k}}\cdot\frac{\{(m+k)!\}^2}{k!\,(2m+2k)!}$$

$$=\frac{1-r}{1+r}\,\frac{r^m}{(1+r)^{2m}}\sum_{k=0}^{\infty}\binom{2m+2k}{k}\left\{\frac{r}{(1+r)^2}\right\}^k=r^m.$$

---

* Ch. de là Vallée Poussin, *loc. cit.*

The interchange of the order of summation is certainly valid if the double series

$$\frac{1-r}{1+r}\sum_{m=0}^{\infty}\frac{|a_m|r^m}{(1+r)^{2m}}\sum_{k=0}^{\infty}\binom{2m+2k}{k}\left\{\frac{r}{(1+r)^2}\right\}^k$$

converges; and this series, by Lemma 11, is

$$\sum_{0}^{\infty}|a_m|r^m.$$

The lemma is therefore proved.

THEOREM VIII. *If the series Σa_λ is summable (V-P) and the series $\Sigma a_\lambda r^\lambda$ has unit radius of convergence, then Σa_λ is summable (A) to the same sum.*

We may suppose that the sum of the series Σa_λ in the (V-P) sense is zero. By Stirling's theorem,

$$\frac{(2n)!}{(n!)^2\,2^{2n}} = O(n^{-\frac{1}{2}}).$$

Hence, given ϵ, we can find N, independent of r, such that, for $n \geqslant N$,

$$\{(2n)!/(n!)^2\,2^{2n}\}\,V_n < \epsilon n^{-\frac{1}{2}}.$$

We may therefore write

$$|P(r)| < |P_1(r)| + \epsilon P_2(r),$$

where

$$P_1(r) = \frac{1-r}{1+r}\sum_{0}^{N-1}\left\{\frac{4r}{(1+r)^2}\right\}^n\frac{(2n)!}{(n!)^2\,2^{2n}}\,V_n,$$

$$P_2(r) = \frac{1-r}{1+r}\sum_{N}^{\infty}\left\{\frac{4r}{(1+r)^2}\right\}^n n^{-\frac{1}{2}}.$$

As r tends to unity $P_1(r)$ tends to zero and

$$\begin{aligned}P_2(r) &= \frac{1-r}{1+r}\int_N^{\infty}\left\{\frac{4r}{(1+r)^2}\right\}^x x^{-\frac{1}{2}}\,dx + o(1)\\ &= O\left[\left(\frac{1-r}{1+r}\right)\Big/\left\{\log\frac{(1+r)^2}{4r}\right\}^{\frac{1}{2}}\right] + o(1)\\ &= O(1).\end{aligned}$$

Hence $P(r)$ tends to zero and the theorem is proved.

Mathematical Department,
The University, Glasgow.

ON A THEOREM OF FROBENIUS

By P. HALL.

[Received 22 June, 1935.—Read 14 November, 1935.]

1. *Outline.*

1.1. Frobenius proved that *the number of elements X of a group of finite order G which satisfy the equation*

$$X^n = 1 \tag{1}$$

(that is, whose orders divide the positive integer n) is a multiple of the greatest common factor (n, g) of n and the order g of G.

This classical result may be found in the text-books†.

In a later paper‡, Frobenius gave the following theorem which includes that just quoted as a special case:

I. *The number of elements X of G which satisfy the equation*

$$X^n = A \tag{2}$$

is congruent to 0 mod (n, g_A), *where A is any given element of G, and g_A is the order of the normaliser§ of A in G.*

The main theme of the present paper is a three-fold generalisation of this theorem of Frobenius.

First, we may replace the simple equations (1) or (2) by the most general system of equations in one unknown. The "natural" extension of Theorem I to this case is still valid.

† Burnside, *Theory of groups*, 2nd edition, 49; Speiser, *Theorie der Gruppen von endlicher Ordnung*, 2nd edition, 44.

‡ G. Frobenius, "Ueber einen Fundamentalsatz der Gruppentheorie", *Berliner Sitzungsberichte* (1903), 987–991.

§ That is, the subgroup of all elements of G which are permutable with A.

Secondly, we may make the content of the theorem more precise by taking account of the way in which the solutions are distributed in G. The range in which the solutions are sought may be limited to certain suitably chosen subgroups or their cosets; and this may be done in the most favourable case without in any way weakening the modulus (n, g_A).

And, finally, we may (in general) greatly strengthen this modulus.

This last improvement is the most important of the three and the most troublesome to prove. It depends on the theory of prime-power groups and we devote § 2 to giving the required portions of that theory. The proof of the main theorem is the subject of § 3; it consists essentially in reducing the general system of equations under consideration to a special case of Frobenius's equation (2) above, and then using the results of § 2 to deal with this special case. Finally, in § 4, as an application of the main theorem, we have chosen the problem of the number of subgroups of a given group which have certain specified properties, a problem which has been frequently treated in the literature, in particular by Frobenius himself.

We now proceed in the remainder of § 1 to give a more detailed account of our conclusions, apart from those which concern the "subgroup-problem", for which the reader is referred to § 4.

1.2. By a *word* $f(X_1, X_2, \ldots, X_m)$ in the symbols X_i we understand as usual any finite product of the X_i and their inverses X_i^{-1}, the order of the factors being (of course) prescribed but arbitrary, while an arbitrary number of occurrences of any symbol is allowed. By the *degree* of f in one of the symbols, for example X_1, we shall mean the sum of the exponents with which X_1 occurs in f; that is, the number of factors equal to X_1, minus the number equal to X_1^{-1}. Thus X_1^n is of degree n; and commutators are all of degree 0 in all the symbols which they contain.

Given any group G of finite order, we shall consider now, in place of the equations (1) or (2), the most general system of equations of the form

$$\begin{aligned} f(X, A_1, A_2, \ldots) &= 1, \\ g(X, A_1, A_2, \ldots) &= 1, \\ &\ldots; \end{aligned} \tag{3}$$

here $A_1, A_2, \ldots$ are given elements of G and play the rôle of the single element A of (2); while $f, g, \ldots$ are arbitrary words in the symbols which they contain.

The equations (3) may be described as the most *general system of word-equations in one unknown*. The number of equations need not be finite, nor need a given A_i occur in all or any of the equations. The problem is to find

what can be said of the number of "solutions" of the equations (3), that is, the number of elements X of G which satisfy them all simultaneously.

The natural generalization of Theorem I may be expressed as follows:—

II. *If n divides the degree in X of each of the words $f, g, \ldots$, then the number of solutions of equations* (3) *in G is congruent to* 0 mod (n, g_A), *where g_A is the order of the centraliser of the A_i in G.*

The centraliser of the A_i in G is, by definition, the set of all elements of G which are permutable with every A_i. This is the analogue of the normaliser† of the single element A which figures in I. It is a subgroup, namely the meet of the normalisers in G of the separate A_i.

1.3. It will be observed that the modulus (n, g_A) of the congruence which occurs in Theorems I and II is in general inferior to the modulus (n, g) of the original theorem of Frobenius. This is because the equation (1) involves no "constants" A_i; and it lies in the nature of the case, as may be seen by considering the equation

$$X^{-1}A^{-1}XA = 1.$$

Here we may take $n = 0$, while the number of solutions is precisely g_A.

However, we may to some extent compensate for this loss if we take into account *the local distribution of the solutions in G*. The most suitable way of doing this is in terms of the dissection of G into double cosets, a concept introduced by Frobenius in another connection. If n is defined as in II, then we have

III. *If H is any subgroup of the centraliser G_A of the A_i in G, then the number of solutions of* (3) *which lie in any assigned‡ double coset HPH of H is congruent to* 0 mod (n, h), *where h is the order of H.*

The best modulus is still (n, g_A), which is obtained by taking $H = G_A$. Thus III contains II as an immediate corollary of this special case. But III gives more than II: for example, if G_A is self-conjugate in G, not only is the *total* number of solutions in G divisible by (n, g_A), but so also is the number of solutions contained in any (ordinary) coset of G_A.

---

† We cannot use the term normaliser for the present concept, owing to the probability of confusion with the normaliser of a subgroup.

‡ That is, P is an arbitrary but given element of G. If we take for H a subgroup of G *not* contained in G_A, then the result still holds provided that we replace h by the order of the meet of H with G_A. This follows at once from Theorem III itself.

Obviously the smaller H is chosen to be, the weaker the congruence of III is liable to become; but, on the other hand, the more narrowly the solutions are located. Thus by varying the choice of H different kinds of information may be obtained. This is the main point of the variable subgroup H, but it also greatly facilitates the proof.

In III, there is no reference to the order of G, and the result holds even when this is infinite; though it must be assumed that h, the order of H, is finite. We adopt the convention that ∞ like 0 is divisible by every integer.

1.4. It will be convenient to make here one further obvious remark: that the n of Theorem III may be supposed without loss of generality to be the power of a prime p. For if the theorem has been proved in this restricted form, we obtain it for general n by applying it in turn for each of the maximal prime-power factors of n.

Hence we shall in future take $n = p^k$ without further comment, and pass next to the problem of improving the modulus (p^k, h) which occurs in Theorem III.

That some such improvement is in general possible may be made plausible as follows. Suppose that G is of finite order, that J is a p-Sylow subgroup of G, that p^j is the order of J, and that p^k is the least common multiple of the orders of the elements of J. Except when J is cyclical, k will be less than j. But every solution of $X^{p^j} = 1$ in G automatically satisfies $X^{p^k} = 1$. Thus the number of solutions of the latter equation is (by the original Frobenius theorem) a multiple not merely of p^k but of p^j. Now it seems very likely from this that, if we take any positive $l < k$, the number of solutions of $X^{p^l} = 1$ will (except when J is cyclical) be divisible by some higher power of p than merely p^l. And this is in fact true except when $p = 2$, in which case the quaternion group is a *Gegenbeispiel*.

We consider first, for simplicity, a group G of order a power of p; and in place of (3) the simple equation

$$X^{p^k} = 1, \tag{4}$$

and ignore for the moment the localization of the solutions.

It is known† that when G is *regular* the number of solutions of (4) in G is an exact power of p; indeed, the solutions themselves in this case form a characteristic subgroup $\Omega_k = \Omega_k(G)$. Regular p-groups have a number of properties which ally them with Abelian groups: in particular, they have a *basis* and *invariants*; and the order of Ω_k may be written down in terms

† P. Hall, "A contribution to the theory of groups of prime-power order", *Proc. London Math. Soc.* (2), 36 (1933), 29–95; especially § 4. This paper will be quoted as **C.**

of the invariants of G by exactly the same rule as if G were Abelian. Thus, for regular G, the problem of the number of solutions of (4) may be regarded as solved.

Obviously then, in view of these special properties, it would be desirable to possess a simple criterion for the regularity of a prime-power group. Essentially, all that has been proved hitherto is that every group whose order divides p^p is regular; or, more generally, every p-group of class† less than p is regular.

The crucial result of the present paper is an improved criterion of regularity. If G is any p-group, we define $\mathfrak{V}_k = \mathfrak{V}_k(G)$ to be the characteristic subgroup of G which is generated by the p^k-th powers of the elements of G‡. We then have [*cf.* Theorem (2.3) for the best result]

IV. *G is regular whenever the order of $G/\mathfrak{V}_1$ is less than p^p.*

From Theorem IV, it is easy to conclude that *the number of solutions of $X^p = 1$ in an irregular p-group G is always divisible by p^{p-1}.* While for $k > 1$, either all the elements of G satisfy (4) or else the number of solutions of (4) is divisible by p^{p+k-2}.

In the case $k > 1$, a more elaborate argument gives an essentially better result [*cf.* Theorem (2.6)]:

V. *In any p-group, either the solutions of* (4) *form a characteristic subgroup Ω_k, or else the number of solutions of* (4) *is divisible by $p^{k(p-1)}$.*

Thus§ the number of solutions of (4) is either an exact power of p, or else is congruent to 0 mod $p^{k(p-1)}$.

1.5. Returning now to the general case, our main result may be enunciated as follows:

THEOREM 1. *Let G be any group; let $f, g, \ldots$ be any system of words in the symbols $X, A_1, A_2, \ldots$, where X is an unknown and $A_1, A_2, \ldots$ are given elements of G; let p be any prime and let p^k divide the degrees in X of each of the words $f, g, \ldots$; and let H be any subgroup of finite order of the centraliser G_A of the A_i in G.*

---

† If we denote the commutator $P^{-1}Q^{-1}PQ$ by $[P, Q]$ and define the r-fold commutator $[P_1, P_2, \ldots, P_r]$ by recurrence to mean $[[P_1, P_2, \ldots, P_{r-1}], P_r]$, then, in a p-group G, the r-fold commutators all equal 1 for sufficiently high r. The greatest integer k for which there exist $P_1, \ldots, P_k$ in G such that $[P_1, P_2, \ldots, P_k] \neq 1$ is the *class* of G.

‡ If G is regular, every element of $\mathfrak{V}_k$ is the p^k-th power of some element of G. (*Cf.* **C**, 4.21.)

§ I understand that Herr H. Wieland, of Berlin, has obtained some results in this direction.

Then the number of elements X in the double coset HPH of H in G for which

$$f = g = \ldots = 1$$

is divisible by p^m, where $m = m(H, G, k)$ is defined as follows:

(i) *if G is of finite order, and if the p-Sylow subgroups of G are regular, we may take $m = \rho_k$ where p^{ρ_k} is the order of $\Omega_k(J)$, and J is a p-Sylow subgroup of H;*

(ii) *if J is regular, we may take $m = \min_{j=0}^{k} (\rho_j + p^{k-j} - 1)$;*

(iii) *in any case, we may take $m = \min [\rho_k, k(p-1)]$ or $m = k(p-1)$, according as the solutions of the equation $X^{p^k} = 1$ in J form a subgroup $\Omega_k(J)$, of order p^{ρ_k}, or do not form a subgroup.*

These three cases are not mutually exclusive. But it is clear that in every case p^m is a multiple of (p^k, h); so that Theorem 1 implies the truth of Theorem III. It will be seen that we may assert very nearly as much (*mutatis mutandis*) for the general system of equations (3) as for the simple equation $X^{p^k} = 1$ considered in (1.4).

If J is irregular, then $\Omega_k(J)$, if it exists, must either coincide with J itself or else be of order at least p^{p+k-2}; hence, if J is irregular and has elements of order p^k, we may certainly take $m = p+k-2$. However, case (iii) will give in general a better value of m than this.

1.6. There is one corollary of Theorem 1 which is perhaps of special interest, since it is a direct and immediate generalization of the original theorem of Frobenius. Let H be any group of finite order h, and let θ be any automorphism of H whose order divides n. Denote by X^θ the effect of θ on the element X of H. Then

THEOREM 1.6. *The number of X of H for which the product*

$$X \cdot X^\theta \cdot X^{\theta^2} \cdot X^{\theta^3} \cdot \ldots \cdot X^{\theta^{n-1}} = 1 \tag{5}$$

is a multiple of (n, h); and if H is Abelian, it is also a divisor of h.

Obviously, if we take $\theta = 1$ to be the identical automorphism of H, we are reduced to the original theorem of Frobenius.

To prove this corollary, we have only to apply Theorem III to the group $G = \{H, \theta\}$ obtained by adjoining θ to H in the usual way, and consider the number of solutions of the equation $Y^n = 1$ in the coset $H\theta^{-1}$. Writing

$Y = X\theta^{-1}$ with X in H, we have

$$Y^n = X \,.\, \theta^{-1} X\theta \,.\, \theta^{-2} X\theta^2 \,.\,\ldots\,.\, \theta^{-n+1} X\theta^{n-1} \,.\, \theta^n$$
$$= X \,.\, X^\theta \,.\, X^{\theta^2} \,.\,\ldots\,.\, X^{\theta^{n-1}}$$

since $\theta^n = 1$. Thus the first part of the corollary follows from Theorem III.

Now suppose that H is Abelian, and consider the number of solutions in $H\theta^{-1}$ of the equation $Y^n = B$, with B in H. If Z is a particular solution, $Y = XZ$ with X in H will also be a solution if and only if

$$X \,.\, ZXZ^{-1} \,.\, Z^2XZ^{-2} \,.\,\ldots\,.\, Z^{n-1}XZ^{-n+1} = 1,$$

or, since $ZXZ^{-1} = X^\theta$, $Z^iXZ^{-i} = X^{\theta^i}$, if and only if X satisfies (5). Thus the number of elements of $H\theta^{-1}$ having a given n-th power is (if any) always the same. Thus this number must divide h if it is not equal to 0.

In proving this result, we have appealed only to Theorem III. A better result could obviously have been obtained by using Theorem 1 itself; this is left to the reader.

1.7. We mention in conclusion a generalization of Theorem 1 which is of considerable interest. In a continuation† of the paper previously referred to, Frobenius proved the following:

THEOREM VI. *If χ is any character of the group G of finite order and A any given element of G, then the algebraic integer*

$$\Sigma\chi(X),$$

where the sum is taken over all elements X of G which satisfy $X^n = A$, is divisible by (n, g_A), where g_A is the order of the normaliser of A in G.

This theorem of Frobenius can be extended to cover the general situation described in Theorem 1. The corresponding result here may be stated as

THEOREM 1.7. *If, in Theorem* 1, *G is of finite order and χ is any character of G, then the sum*

$$\Sigma\chi(X), \tag{1}$$

taken over all the solutions X which are there in question, is an algebraic integer congruent to 0 mod p^m.

Here p^m is defined as in Theorem 1.

If for χ in (1.7) we take the so-called principal character of G:

$$\chi(X) = 1,$$

† G. Frobenius, "Ueber einen Fundamentalsatz der Gruppentheorie", *Berliner Sitzungsberichte* (1907), 428–437.

for all X; then the sum (1) becomes equal to the total number of elements X with the specified properties. Thus (1.7) contains Theorem 1 as a particular case, at least when the order of G is finite.

The argument used by Frobenius to prove VI and based on an idea of Blichfeldt would probably lead to difficulties if applied to (1.7). However, we may argue as follows.

In the proof of the main theorem, we divide the relevant solutions into classes. As an inspection of § 3 will immediately show, each of these classes consists of a certain number of mutually exclusive cosets, the cosets in a given class being all conjugate in G; moreover, if XH is a typical one of these cosets, X transforms the subgroup H into itself†. Further, the number of cosets μ in the class being considered, and the order h of H are so related that

$$\mu h \equiv 0 \mod p^m. \tag{2}$$

So that the number of X *in each class* is divisible by p^m.

This is the essence of the proof; and it is easy to see, merely from this sketch, how we may deduce Theorem 1.7. For if S is any element of G, we have $\chi(X) = \chi(S^{-1}XS)$. Hence, considering a particular class of relevant solutions as described above, each of the μ cosets XH in this class will contribute the same quantity to the sum (1). And therefore, by (2), we need only prove that

$$\sum_{(Z)} \chi(XZ) \equiv 0 \mod h, \tag{3}$$

where the sum in (3) is taken over all elements Z of H.

To prove (3), let $\Gamma(X)$ be the matrix representation of G whose character is χ. Since H is self-conjugate in the group $K = \{H, X\}$, it easily follows that the expressions

$$\phi(X) \equiv \frac{1}{h} \sum_{(Z)} \Gamma(XZ) \tag{4}$$

form a representation of the factor group K/H. For if $X'H$ is any other coset of H in K, we get

$$\phi(X)\phi(X') \equiv \frac{1}{h^2} \sum_{(Z)} \Gamma(XZ) \sum_{(Z')} \Gamma(X'Z')$$

$$= \frac{1}{h^2} \sum_{(Z'', Z')} \Gamma(XX'Z''Z'),$$

where $Z'' = (X')^{-1}ZX'$; and Z'' and Z' go independently through H. This gives $\phi(X)\phi(X') \equiv \phi(XX')$, as required.

† This is not the H of § 3, but one of the subgroups $\Omega_k(H)$, M_m, or L_k used in § 3.9.

Hence the spur of $\phi(X)$, being a character, is also an algebraic integer. But this spur is simply

$$\frac{1}{h}\sum_{(Z)}\chi(XZ);$$

and the result (3) follows at once.

2. *Details of prime-power group theory.*

2.1. We shall frequently make use of a formula, proved in (C, 3.2), for the powers of a product. If P and Q are any two elements of a group G, we have, for any prime-power p^k,

$$(PQ)^{p^k} \equiv P^{p^k} Q^{p^k} R_1^{n_1} R_2^{n_2} \dots R_\alpha^{n_\alpha} \dots, \tag{2.1}$$

where the R_i are certain commutators in P and Q; and if the weight of R_i is w_i, where

$$w_i < p^l \leqslant p^k,$$

then n_i is divisible by p^{k-l+1}. The weights w_i in (2.1) may be taken to increase with i, and only a finite number of terms occur with any given weight. If G is a prime-power group of class w, then all the terms of weight greater than w in (2.1) will be 1 and the right-hand side is a finite product; in this case, therefore, we get an identity, and not merely a system of congruences.

2.2. *Definition.* If G is any p-group, we define the invariant $\omega = \omega(G)$ by the rule: if $\mathfrak{V}_1 = \mathfrak{V}_1(G)$ is the group generated by the p-th powers of the elements of G, then p^ω is the index of $\mathfrak{V}_1$ in G. Thus $\omega = 1$ if and only if G is cyclical. If G is generated by d independent elements, then $\omega \geqslant d$.

LEMMA (2.2). *If $\omega(G) < p$, then $\omega(H) \leqslant \omega(G)$ for every subgroup H of G.*

For regular G, $\omega(H)$ never exceeds $\omega(G)$ for any subgroup H of G, whatever value $\omega(G)$ may have. On the other hand, the inequality of the lemma cannot, in general, be replaced by $\omega(G) \leqslant p$. For example, the general† group of class 3 generated by two elements of order 3 is a group G of order 3^5 having $\omega(G) = 3$; but this G possesses a subgroup H of order 3^4 whose elements all satisfy the equation $X^3 = 1$; so that $\omega(H) = 4$.

† By the *general* group with certain specified properties we mean the group whose system of defining relations are those which express that it has the specified properties. Every group with these properties is isomorphic with some factor group of this general group. The term " general " can, of course, only be used in this sense, if the properties in question are such as can be expressed by relations.

In proving the lemma, an induction argument over the order of G allows us to assume that *H is of index p in G*; for every proper subgroup of G is contained in such an H.

Suppose that the lemma is not true for a certain H of index p. Then we can choose a self-conjugate subgroup L of G, containing $\mathfrak{V}_1(H)$ and of index p^2 in $\mathfrak{V}_1 = \mathfrak{V}_1(G)$. There will be no loss of generality in assuming $L = 1$. G cannot be regular, for then the theorem would be true for G. Hence its order must be at least p^{p+1}. But, since $L = 1$, its order is $p^{\omega+2}$, where $\omega = \omega(G) \leqslant p-1$. Hence $\omega = p-1$, and the order of G is exactly p^{p+1}. Further, G must be of class p, and be generated by two elements P and Q, of which Q may be chosen in H. Also the central Z of G must be of order p and is contained in $\mathfrak{V}_1$, which is of order p^2.

Since G/Z has elements of order p^2, there is no loss of generality in assuming that P^p lies in $\mathfrak{V}_1 - Z$. We now consider $(PQ)^p$. By the formula (2.1), this is of the form $P^p R$, where R lies in the group of p-fold commutators; since all the elements of H (and therefore all the commutators) are of order p. That is, in the present case R lies in Z. Hence $(PQ)^p$ also lies in $\mathfrak{V}_1 - Z$. This means that both P and PQ are permutable with every element of $\mathfrak{V}_1$. Thus $\mathfrak{V}_1$ belongs to the central Z of G, which is impossible. This proves the lemma.

2.3. THEOREM 2.3. *If G is irregular, then $\omega(H_i) > p-i$, for each $i = 1, 2, \ldots, p$; where $H_1 = G$, $H_2 = [G, G]$ is the derived group of G, and $H_i = [H_{i-1}, G]$ is the group of i-fold commutators of G.*

The cases $i = 1$ and $i = p$ may be specially mentioned. By the first of these, every group G with $\omega(G) < p$ is regular. The second merely expresses the known result (C, 4.13) that all groups of class less than p are regular.

The theorem is, in a certain sense, "best possible". If for G we take the p-Sylow subgroup of the symmetric group on p^2 symbols (which is an irregular group of order p^{p+1})†, we have, in fact,

$$\omega(H_i) = p-i+1$$

for each $i = 1, 2, \ldots, p$.

Proof of Theorem 2.3. We have to show that, if, for some value of i, $1 \leqslant i \leqslant p$, then

$$\omega(H_i) \leqslant p-i, \tag{1}$$

then G is regular. We may argue by induction over the order of G. By the lemma, if K is any subgroup of G, so that $H_i(K) \leqslant H_i = H_i(G)$, it follows

† From many points of view, this is the simplest irregular p-group.

that $\omega(H_i(K)) \leqslant p-i$. Hence we may assume that every proper subgroup of G is regular. Since a group is regular if all its subgroups of the form $\{P, Q\}$ (*i.e.* generated by two elements) are regular, we may further assume that G is of this form. (The trivial case $p = 2$ is neglected.)

We shall first prove that

$$H_{p^\alpha} \leqslant \mho_\alpha(H_2) \tag{2}$$

for all $\alpha = 1, 2, \ldots$.

Suppose first that $i \neq 1$. Then H_i is regular, and hence the order of $H_i/\mho_\alpha(H_i)$ is at most $p^{\alpha(p-i)}$, (**C**, 4.3). Since $\mho_\alpha(H_i)$ is self-conjugate in G, it follows that

$$H_{i+\alpha(p-1)} \leqslant \mho_\alpha(H_i).$$

Since $H_{p^\alpha} \leqslant H_{i+\alpha(p-i)}$ and $\mho_\alpha(H_i) \leqslant \mho_\alpha(H_2)$ because $i \geqslant 2$, the result (2) follows.

Next let $i = 1$. Then (1) gives $\omega(G) \leqslant p-1$. Hence, by the lemma, $\omega(H_2) \leqslant p-1$ and the preceding argument gives

$$H_{2+\alpha(p-1)} \leqslant \mho_\alpha(H_2);$$

whence (2) follows, except for the case $\alpha = 1$.

To deal with the remaining case $\alpha = 1$, we may suppose that $\mho_1(H_2) = 1$ and we then have to prove that the class of G is less than p.

If not, the order of H_2 must be exactly p^{p-1} and the class of G exactly p. Hence also, by the lemma, $\omega(G) = p-1$. Let $G = \{P, Q\}$ and let R be any element of G. Then

$$R^{-p} P^{-1} R^p P = R^{-p}(P^{-1} RP)^p = R^{-p}(RS)^p,$$

where $S = R^{-1} P^{-1} RP$. Now R and S belong to a proper subgroup of G, which is clearly of class less than p, since S is a commutator. Since all the commutators in G are of order p, in particular S, it follows from (2.1) that $(RS)^p = R^p$, and hence R^p is permutable with P. Similarly it is permutable with Q; and therefore with every element of G. Hence $\mho_1$ belongs to the central of G. But, G having only two independent generators, the join $\{\mho_1, H_2\}$ is of index p^2 in G. Hence, since $G/\mho_1$ and H_2 have the same order, the meet of $\mho_1$ and H_2 is of order p^2. Since this meet belongs to the central of G, it follows that $H_p = 1$, as required.

Thus (2) is established in every case.

To complete the proof of Theorem (2.3) we have to show, by the definition of *regular* (**C**, 4.1), that, for any two elements A and B of G and any positive λ,

$$(AB)^{p^\lambda} = A^{p^\lambda} B^{p^\lambda} S_3^{p^\lambda} S_4^{p^\lambda} \ldots S_r^{p^\lambda}$$

for suitably chosen $S_3, S_4, \ldots, S_r$ from the derived group of $\{A, B\}$. Owing to the induction hypothesis, we may assume that $\{A, B\} = G$. By the formula (2.1), we have, in any case, an equation of the form

$$(AB)^{p^\lambda} = A^{p^\lambda} B^{p^\lambda} R_3^{n_3} R_4^{n_4} \ldots R_s^{n_s},$$

where the R_i all belong to H_2 and the exponents obey the rule that either R_i belongs to H_{p^α} or else n_i is divisible by $p^{\lambda-\alpha+1}$. If then, for some particular i, n_i is divisible by $p^{\lambda-\alpha}$ but not by $p^{\lambda-\alpha+1}$, it follows that R_i belongs to H_{p^α}, and therefore, by (2), to $\mathfrak{V}_\alpha(H_2)$. H_2 being regular, this means that R_i is of the form $S_i^{p^\alpha}$, with S_i in H_2; this leads at once to the required form for $(AB)^{p^\lambda}$.

Theorem (2.3) can obviously be stated in a slightly different form, namely:

2.31. *If G is irregular and if for some $i = 1, 2, 3, \ldots$ the order of H_i is p^{p-i+k}, then the order of $\mathfrak{V}_1(H_i)$ is less than p^k; in particular, the order of an element of H_i cannot exceed p^k.*

2.4. It is now easy to deduce

THEOREM 2.4. *In an irregular group G the number of solutions of the equation $X^p = 1$ is always congruent to* 0 mod p^{p-1}.

First we need

LEMMA 2.41. *An irregular p-group G always possesses a self-conjugate subgroup H of order p^{p-1} whose elements are all of order p at most.*

Arguing by induction over the order of G, let K be a self-conjugate subgroup of G of order p and contained in $\mathfrak{V}_1$. (When $\mathfrak{V}_1 = 1$, the result is trivial.) If G/K is regular, there exists a subgroup L self-conjugate in G and such that L/K is of order p^{p-1} and $\mathfrak{V}_1(L) \leqslant K$; for in this case $\omega(G/K) = \omega(G)$ since $K \leqslant \mathfrak{V}_1$, and $\omega(G) \geqslant p$ by Theorem (2.3); so that $\Omega_1(G/K)$ is of order at least p^p, and for L we may take any self-conjugate subgroup of G of the right order for which $L/K \leqslant \Omega_1(G/K)$. On the other hand, if G/K is irregular, an L with these properties exists by the induction hypothesis.

L is of order p^p and is therefore regular. Also, since $\mathfrak{V}_1(L) \leqslant K$, L has at least $p-1$ invariants. Thus either its elements all satisfy $X^p = 1$, in which case we may take for H any self-conjugate subgroup of G of index p in L; or else L has a characteristic subgroup $H = \Omega_1(L)$ of index p in L whose elements all satisfy $X^p = 1$. This proves the lemma.

Now let Y be any element of order p in G. Then $\{H, Y\}$ is of order at most p^p and is therefore regular. Hence its elements all satisfy $X^p = 1$

and in particular all the elements congruent to Y mod H satisfy this equation. Thus the solutions of the equation constitute a number of cosets of H. Each coset contains p^{p-1} elements; whence the result.

This is, in a sense, "best possible". For the p-Sylow subgroup of the symmetric group on p^2 symbols has exactly $2p^p - p^{p-1}$ elements satisfying $X^p = 1$.

For a general p-group, we have the result that the number of solutions of $X^p = 1$ is either a power of p or else is divisible by p^{p-1}, according as the group is regular or irregular.

As a corollary† we have

THEOREM 2.42. *The number of subgroups of order p in an irregular group is always congruent to* $1+p+p^2+\ldots+p^{p-2}$ mod p^{p-1}.

For a regular group the number of subgroups of order p is, of course, exactly $1+p+p^2+\ldots+p^{\omega-1}$.

For the commutator subgroups of an irregular group we may state the following result, which follows immediately from what has already been proved with the help of Theorem (2.3): if G is irregular and K is any subgroup of G containing H_i, then the number of solutions of $X^p = 1$ in K is either one of the numbers p^k, where $p-i < k < p-1$, or else is congruent to 0 mod p^{p-1}.

2.5. Whereas the formula of (2.1) is sufficient, when taken with the argument of sections (3.1–3.8) omitting (3.6), to prove the result III of § 1, for the main theorem we need the following:

THEOREM 2.5. *Let G be any p-group and H any self-conjugate subgroup of G. Then there exists in H a sequence of self-conjugate subgroups of G,*

$$1 = L_0 \leqslant L_1 \leqslant L_2 \leqslant \ldots \leqslant H$$

with the properties:

(i) *L_i/L_{i-1} is of order less than or equal to p^{p-1} and its elements other than 1 are all of order p, and*

(ii) *either the order of L_i is exactly $p^{i(p-1)}$, or else L_i is a characteristic subgroup of H and consists of all the elements of H whose orders divide p^i.*

† *Cf.* M. Tazawa, *Science reports, Tôhoku University* (1), 23 (1934), 449–476, where by a different method the following theorem is proved: let G be of order p^n, $p > 3$, and let G have a self-conjugate cyclic subgroup D such that G/D is elementary Abelian of order p^d with $d \geqslant 3$; then the number of subgroups of order p in G is congruent to $1+p+p^2 \pmod{p^3}$.

To facilitate the proof of this theorem, it will be convenient to prove first the following corollary.

COROLLARY 2.51. *For any sequence L_i of self-conjugate subgroups of G which satisfy* (i), *but not necessarily* (ii), *we have: if P is any element of G and Q any element of L_i, then P and PQ have the same p^i-th power.*

For the p-fold commutators of P and Q all belong to L_{i-1} and so are of order p^{i-1} at most; more generally, the p^λ-fold commutators of P and Q all belong to $L_{i-\lambda}$, since the order of $L_i/L_{i-\lambda}$ is less than or equal to

$$\lambda(p-1) < p^\lambda,$$

and so they are of order $p^{i-\lambda}$ at most (for $i \geqslant \lambda$). Also $Q^{p^i} = 1$ since Q belongs to L_i. Hence, in formula (2.1) for $(PQ)^{p^i}$, all the terms equal 1 except possibly P^{p^i}.

Proof of (2.5). The theorem being trivial for $H = 1$, we may suppose that $H > 1$ and argue by induction over the order of H. Hence, if we choose for K a self-conjugate subgroup of G of order p contained in H and apply the theorem to G/K and H/K in place of G and H respectively, we deduce the existence in H of a sequence of self-conjugate subgroups L_i' of G, $L_i' \geqslant L_{i-1}'$, $L_0' = K$, L_i'/L_{i-1}' of order p^{p-1} at most and with elements of order p at most. Further, if the order of L_i' is less than $p^{1+i(p-1)}$, it consists of all the elements of H whose p^i-th powers lie in K. This is certainly true for i sufficiently large, since the order of L_i' cannot exceed that of H. Also, if true for any given value of i, it will be true for all greater values. We may suppose that i_0 is the greatest value of i for which the order of L_i' is exactly $p^{1+i(p-1)}$.

For $i > i_0$, therefore, every element of H whose order divides p^i will belong to L_i'. If for a certain i the converse is true, that every element of L_i' is of order less than or equal to p^i, then this will also be true for every greater value of i, since the order of an element of L_j' (mod L_i') is, for $j > i$, at most p^{j-i}. We may suppose that i_1 is the least value of $i > i_0$ for which the elements of L_i' are all of order less than or equal to p^i.

Then we may distinguish two cases, according as $i_1 > i_0+1$ or $i_1 = i_0+1$.

First, let $i_1 > i_0+1$.

Then the subgroups L_i may be constructed by recurrence, as follows. Suppose that, for some i satisfying $0 \leqslant i < i_1-1$, the L_j are already defined for $j = 0, 1, \ldots, i$ and satisfy the conditions of the theorem. Suppose also that for these values of j, L_j is of index p in L_j'. Now consider L_{i+1}'/L_i.

Since L_i is, by hypothesis, of index p in L_i' and L_i' is of index p^{p-1} at most in L_{i+1}', it follows that L_{i+1}'/L_i is of order p^p at most, and is therefore regular. Also, every p-th power in L_{i+1}' lies in L_i' and hence this regular group is either of type $(p^2, p, \ldots, p)$ or of type $(p, p, \ldots, p)$. In the former case, it possesses a characteristic subgroup L_{i+1}/L_i of index p whose elements are all of order p at most; L_{i+1} will therefore be self-conjugate in G, since both L_i and L_{i+1}' are. In the latter case, we may choose for L_{i+1} any self-conjugate subgroup of G containing L_i and of index p in L_{i+1}'. In this way, starting with $L_0 = 1$, we may define L_i for each $i \leqslant i_1 - 1$.

Clearly, for these values of i, the order of L_i/L_{i-1} is less than p^p, for it is the same as that of L_i'/L_{i-1}'. Also, the order of L_i is for the same reason only less than $p^{i(p-1)}$ when i exceeds i_0. But, for any $i \leqslant i_1 - 1$, it is evident that the orders of the elements of L_i cannot exceed p^i. Also, for $i > i_0$, every element of H whose order divides p^i must belong to L_i' as already mentioned. Hence, if for some i in the range $i_0 < i \leqslant i_1 - 1$ there is an element P in H whose order is less than or equal to p^i but which does not lie in L_i, we may apply Corollary (2.51) to L_i (in place of H), which is legitimate by our induction hypothesis, since in the circumstances we certainly have $L_i < H$. We deduce that every element in the join $\{L_i, P\} = L_i'$ is of order p^i at most. This implies that $i \geqslant i_1$, which is a contradiction. Hence, for $i_0 < i \leqslant i_1 - 1$, every element of H whose order divides p^i must lie in L_i.

For $i \geqslant i_1$, we define $L_i = L_i'$; thus the L_i so defined certainly satisfy (ii). And it now only remains, in the present case, to show that L_{i_1}/L_{i_1-1} is of order less than p^p and has all its elements of order p. To see this, we must note the

LEMMA. *H/L_{i_0} is regular and $L_{i_0+k}/L_{i_0} = \Omega_k(H/L_{i_0})$.*

For if H/L_{i_0} was irregular, it would have a subgroup of order p^{p-1} whose elements are of order p at most, by (2.41). The corresponding elements of H are therefore all of order p^{i_0+1} at most, and so must belong to L_{i_0+1} by what has already been proved. Hence, since L_{i_0} is of order $p^{i_0(p-1)}$, it follows that L_{i_0+1} is of order $p^{(i_0+1)(p-1)}$; which contradicts the definition of i_0, since L_{i_0+1} is of index p in L_{i_0+1}'. It follows then that not only is H/L_{i_0} regular, but it has fewer than $p-1$ invariants. That $L_{i_0+k}/L_{i_0} \geqslant \Omega_k(H/L_{i_0})$ is obvious from property (ii) of the L_i, which has already been seen to hold for all $i > i_0$. But, conversely, $L_{i_0+k}/L_{i_0} \leqslant \Omega_k(H/L_{i_0})$ is certainly true for $k = 1$, since property (i) has been proved for the L_i for which $i \leqslant i_1 - 1$ and $i_0 > i_0 + 1$. Suppose that it was not true for all k; we should have an element P of H of order p^{k+1} (mod L_{i_0}) yet contained in L_{i_0+k}. But this means that

$Q = P^{p^{k-1}}$ is of order at most p^{i_0+1} and is therefore contained in L_{i_0+1} whose elements are all of order at most p (mod L_{i_0}). This is a contradiction and proves the result $\Omega_k(H/L_{i_0}) = L_{i_0+k}/L_{i_0}$ for all k. Thus the L_i as defined satisfy both (i) and (ii) for all values of i.

Next, let $i_1 = i_0+1$.

We need not consider the case $L'_{i_0} = H$. For the method used above of constructing the L_i of index p in the corresponding L_i' gives all that is required. Hence we may suppose that $H > L'_{i_0}$. But if we define $L_i = L_i'$ for all $i > i_0$, then for these L_i all the conditions of the theorem are fulfilled: since by definition of i_1, for these values of i, L_i' consists of all the elements of H whose orders divide p^i. Hence there is no loss of generality in assuming that $H = L'_{i_0+1}$. The elements of H are then of order p^{i_0+1} at most.

Let H'' be any self-conjugate subgroup of G containing L'_{i_0} and of index p in H. By the induction hypothesis, we may assume the theorem true of H''. Let the L_i'' be the sequence of self-conjugate subgroups of G contained in H'' whose existence is thereby assured.

Now the order of L''_{i_0} must be exactly $p^{i_0(p-1)}$. For if it was less, then, by property (ii) of (2.5), H'' would have fewer than $p^{i_0(p-1)}$ elements whose orders divide p^{i_0}. Whereas, in fact, $H'' \geqslant L'_{i_0}$ and L'_{i_0} contains, by the argument used in the previous part, a subgroup of index p (and so of order $p^{i_0(p-1)}$) whose elements are all of order p^{i_0} at most.

Now $(H : L'_{i_0}) \leqslant p^{p-2}$ by definition of i_0, and the order of L'_{i_0} is $p^{1+i_0(p-1)}$, that is, just p times that of L''_{i_0}. Hence $(H : L''_{i_0}) \leqslant p^{p-1}$. Thus H/L''_{i_0} is regular and the p-th powers of H mod L''_{i_0} therefore form a subgroup $L^*_{i_0}/L''_{i_0}$ of H/L''_{i_0}, and, by Corollary (2.51) applied to L''_{i_0}, every element of $L^*_{i_0}$ is of order p^{i_0} at most. Also the elements of $H/L^*_{i_0}$ are of order p at most; and $L^*_{i_0}$, being a characteristic subgroup of H mod L''_{i_0}, is certainly self-conjugate in G.

We dismiss the trivial case $i_0 = 0$; then $\omega(H)$ is clearly at least $p-1$; if H is regular, because of the subgroup L_1' with at least $p-1$ invariants; if H is irregular, by Theorem (2.3). Hence, since $\mho_1(H) \leqslant L^*_{i_0}$, there exists a self-conjugate subgroup L_{i_0} of G satisfying $L^*_{i_0} \geqslant L_{i_0} \geqslant \mho_1(H)$ and whose order is precisely $p^{i_0(p-1)}$.

Now apply the theorem to L_{i_0}. Since the elements of this are all of order p^{i_0} at most, it follows that the sequence of self-conjugate subgroups L_i of G whose existence is thus assured must have exactly i_0 distinct terms other than 1, as indicated by the notation L_{i_0}. These terms satisfy all the conditions of the theorem, provided that we take $H = L_{i_0+1}$; for H/L_{i_0} has order p^{p-1} at most and since $L_{i_0} \geqslant \mho_1(H)$, all its elements are of order p at most.

This completes the proof of Theorem (2.5).

2.6. From Theorem (2.5) and its Corollary (2.51) we have immediately the following

THEOREM 2.6. *In any p-group G, either (a) the solutions of the equation*

$$X^{p^k} = 1 \tag{1}$$

form a characteristic subgroup Ω_k, so that the number of solutions is an exact power of p, and, further, if the order of Ω_k is less than $p^{k(p-1)}$, then, for every $l > k$, the solutions of $X^{p^l} = 1$ also form a characteristic subgroup Ω_l; or else (b) the number of solutions of (1) is congruent to 0 (mod $p^{k(p-1)}$).

Also, if A is any given element of G, then either we are in case (a) with Ω_k of order less than $p^{k(p-1)}$ and the number of solutions of

$$X^{p^k} = A \tag{2}$$

is a multiple of the order of Ω_k; or else the number of solutions of (2) is a multiple of $p^{k(p-1)}$.

For, applying (2.5) with $G = H$, we have by (2.51), for any Q of L_k and any X of G,

$$X^{p^k} = (XQ)^{p^k}. \tag{3}$$

Thus the solutions of $X^{p^k} = A$ constitute a certain number of cosets of L_k. Either their number is divisible by $p^{k(p-1)}$, or else the order of L_k is less than $p^{k(p-1)}$. In the latter case, the order of every L_l with $l \geqslant k$ is less than $p^{k(p-1)}$; and L_l consists, by (2.5), property (ii), of all the elements of G which satisfy $X^{p^l} = 1$, and no others. Thus $L_l = \Omega_l$ is, for $l \geqslant k$, a characteristic subgroup of G; and the number of solutions of (2) or (1) is in this case a multiple of the order of Ω_k.

It will be observed that, in contrast to Frobenius's Theorem I of § 1, the normaliser of A does not enter into this result. This is obviously owing to the fact that *the p^k-th power of every element of G is permutable with every element of L_k*. Equation (3) above shows, namely, that X^{p^k} is permutable both with X and also with XQ, and therefore with Q, which is an arbitrary element of L_k. This result may be expressed by saying that:

$$[L_k, \mho_k] = 1, \tag{2.61}$$

for all $k = 1, 2, \ldots$. Of course, L_k (unlike $\mho_k$) is not as a rule a characteristic subgroup of G, except when its order is less than $p^{k(p-1)}$, when, as we have just seen, it is identical with Ω_k. Indeed, the sequence L_i of (2.5) is by no means uniquely determined. (2.61) holds, however, for every choice of the L_i. It is clear, moreover, that if we operate on a given sequence L_i

satisfying the conditions of (2.5) with any automorphism of $G = H$, we shall get a new sequence still satisfying these conditions. It is therefore important to remark that we have

THEOREM (2.62). *The characteristic subgroup $\bar{\Omega}_k$ of the p-group G, formed by the join of all possible L_k which occur in some sequence satisfying the conditions of* (2.5) *with $G = H$, has all its elements of order p^k at most.*

For let the set of all possible L_k be, in any order, L_k, L_k', L_k'', ..., $L_k^{(\sigma)}$; and let $K^{(\rho)}$ be the join of L_k, L_k', ..., $L_k^{(\rho)}$; $K^{(0)} = L_k$. Suppose that for some ρ, $0 < \rho \leqslant \sigma$, we have already proved that the elements of $K^{(\rho-1)}$ all satisfy the equation $X^{p^k} = 1$. To prove the same for $K^{(\rho)}$, let P and Q be arbitrary elements of $K^{(\rho-1)}$ and $L_k^{(\rho)}$ respectively. Then, since $L_k^{(\rho)}$ belongs to a sequence satisfying the conditions of (2.5) with $G = H$, we may apply Corollary (2.51). This shows that P and PQ have the same p^k-th powers. But, by hypothesis, $P^{p^k} = 1$. Hence $(PQ)^{p^k} = 1$. Thus the elements of $K^{(\rho)} = K^{(\rho-1)} L_k^{(\rho)}$ are all of order p^k at most. It follows by induction that the characteristic subgroup $\bar{\Omega}_k = K^{(\sigma)}$ has all its elements of order p^k at most.

Thus every p-group G has a *characteristic* subgroup $\bar{\Omega}_k$ whose elements are all of order p^k or less, and such that either (a) $\bar{\Omega}_k = \Omega_k$ consists of *all* the elements of G with this property, or else (b) the order of $\bar{\Omega}_k$ is at least $p^{k(p-1)}$.

From (2.61) we deduce at once that

$$[\bar{\Omega}_k, \mathfrak{V}_k] = 1; \tag{2.63}$$

which is a substitute for the Theorem (4.461) of **C**, which holds for regular G.

We may conclude the present digression by proving

THEOREM 2.64. *If the p-group G is generated by elements of order p^k or less, and if the order of G is $p^{1+k(p-1)}$ or less, then every element of G is of order p^k or less.*

For if the order of L_k was less than $p^{k(p-1)}$, it would contain every element of G whose order divides p^k and this is clearly impossible unless $L_k = G$. Hence we may suppose the order of L_k to be $p^{k(p-1)}$ and that of G to be $p^{1+k(p-1)}$. Not all the generating elements of G of order p^k or less can be contained in L_k. If P is one of them not in L_k, then every element of G is of the form $P^\alpha Q$, with Q in L_k. By (2.51), since $P^{p^k} = 1$, every element of G also satisfies the equation $X^{p^k} = 1$.

It may be noted that if, under the hypotheses of (2.64), the order of G is actually less than $p^{1+k(p-1)}$, then it contains a characteristic

subgroup $\bar{\Omega}_k$, *of index less than* p^p, whose elements are all of order less than p^k.

3. *Proof of the main theorem.*

3.1. We may now proceed to the proof of Theorem 1, as enunciated in §1.5. We have tried to reduce to a minimum the number of appeals to the results of the preceding section.

If the order h of H is 1, then the theorem is obviously true, whatever the other circumstances may be; for then H is the identity, and $m = 0$ in every case, and there is nothing to prove. Hence we may argue by *induction over the order of* H.

If the order of H is not a power of p, then $H > J$, and the induction hypothesis allows us to assume the theorem for J. But $m = m(H, G, k)$ is the same as $m(J, G, k)$ in every case, and the theorem gives that the number of solutions in any double coset of J is divisible by p^m. But any double coset of H is the disjoint sum of a certain number of double cosets of J. Hence the theorem follows for H. Accordingly we may assume that $H = J$ *is a* p*-group.*

3.2. Next we need the

LEMMA. *Given any group* G *and any subgroup* H *of* G, *it is always possible to imbed* G *as a subgroup in a group* G', *such that, in* G', H *is a centraliser.*

There will be no loss of generality if we suppose that the index $l = (G : H)$ of H in G is greater than 2. (It may be infinite, of course.) Express G in its representation as a regular transitive permutation group, for example on its own elements as symbols. In this representation of G, H occurs in its own regular transitive representation repeated l times. Let G' be the "symmetric" group of *all* permutations on these symbols; and let H' and H'' be respectively the centralisers of H and H' in G'.

We shall show that $H'' = H$. For H' permutes the l transitive sets for H according to the symmetric group on l symbols; and therefore, since $l > 2$, H'' must leave each of these transitive sets invariant. Again, H' permutes the symbols of a given transitive set for H transitively; namely, according to the "transposed" regular transitive representation of H itself. Hence H' is transitive on the whole set of symbols, and therefore every permutation of H'' is regular. Since H'' certainly contains H, it follows that $H'' = H$; since if it had other permutations not in H, it would certainly possess irregular permutations.

Proceeding with the main argument, we replace G by G' and adjoin to the A_i all the elements of H'. This does not affect in any way the data of

the theorem. For, as we have already explained, it is not necessary to assume that a given A_i really occurs in the proposed equations. Nor does it affect the result to be proved, except that we now have to deal with certain additional double cosets of H, namely those that occur in G' but not in G; that is, we have rather more to prove. Hence we may assume both these steps to have been already carried out, so that from now on *H is the centraliser of the A_i in G.*

3.3. The next step which is necessary is to reduce the words $f, g, \ldots$, which form the left-hand sides of the given equations, to a certain normal form. These words are given as certain finite products of the A_i, X, and X^{-1}. We rearrange each word by moving each X and each X^{-1} it contains step by step steadily to the left of the word. In order to achieve this without altering the "meaning" of the word, commutators must be introduced. Thus a typical step in the process will consist in passing from

$$\ldots YX \ldots \quad \text{or} \quad \ldots YX^{-1} \ldots$$

to $\ldots XY[Y, X] \ldots$ or to $\ldots X^{-1}Y[Y, X^{-1}] \ldots$

respectively. Here $[P, Q]$ stands for $P^{-1}Q^{-1}PQ$.

Obviously, if we persist we shall arrive eventually, after a finite number of steps for each of the words, at an expression of the form

$$X^{\lambda p^k} Y_{j_1} Y_{j_2} \ldots Y_{j_s}, \tag{1}$$

where the Y_j are either A_i's or else commutators of the form

$$Y = [A_i, X_1, X_2, \ldots, X_t] \quad (t \geqslant 1), \tag{2}$$

with each X_k either equal to X or to X^{-1}. The exponent λp^k corresponds to the hypothesis that the degree in X of each of the words is a multiple of p^k.

We suppose then that $f, g, \ldots$ have all been reduced to the above normal form. The commutators Y_i thereby introduced are perfectly definite in form, but are probably infinite in number, if we have an infinite number of equations to deal with. Let them be, in any order,

$$Y_1, Y_2, \ldots. \tag{3}$$

We suppose that in the sequence (3) of the Y_i there occur all the $[A_i, X]$ and $[A_i, X^{-1}]$. But in writing the Y_i in the form of a sequence, we do not intend to imply that they are denumerable.

3.4. The next step is to *classify the solutions X of our equations according to the values which they give to X^{p^k} and the Y_i.* That is, we place in a single

class all the solutions X for which

$$X^{p^k} = B$$

and

$$Y_j = C_j \quad (j = 1, 2, \ldots), \tag{4}$$

where B and the C_j are arbitrary but fixed elements of G.

Obviously, every solution determines uniquely the class to which it belongs, for the Y_j are words which involve, beside the given elements A_i of G, only X and X^{-1}.

Consider now the solutions in a given class. These are simply the solutions of the system of equations formed by taking (4) together with our original equations

$$f = g = \ldots = 1. \tag{5}$$

In this new system (4)+(5), it is evident that p^k still divides the degree in X of each left-hand side; for the Y_i being commutators are all of degree 0. On the other hand, the new system has more " constants " than the old; viz. B and the C_j as well as the A_i. The centraliser of the set formed by all these constants will be denoted by K. Evidently $K \leqslant H$, since H is the centraliser of the A_i alone. We shall say that *a class of solutions is of the first or the second kind according as* $K < H$ *or* $K = H$.

3.5. We consider first a given class of solutions of the first kind. If the index of K in H is p^b, then the given class is one of p^b classes, distinct from one another and therefore with no member in common, viz., those obtained by transforming the given class in turn by the various elements of H. For the elements of H transform the A_i into themselves, and therefore always transform a solution into another solution.

Now $K < H$, and so the induction hypothesis allows us to apply the theorem with K for H and with (4)+(5) as our system of equations. Thus the number of solutions of the given class which lie in a given double coset of K is congruent to 0 (mod $p^{m(K)}$), where $m(K) = m(K, G, k)$ is the exponent appropriate with the subgroup K. But any double coset HPH of H is the disjoint sum of a certain number of double cosets of K. Hence the number of solutions of the given class which lie in HPH is also a multiple of $p^{m(K)}$. It follows that the number of solutions which lie in HPH and belong to one or other of the p^b classes which are conjugate to the given class under H is a multiple of $p^{b+m(K)}$; for on transforming any element X with an element of H we get an element from the *same* double coset of H.

If we can prove that in every case $b+m(K) \geqslant m = m(H)$, it will obviously follow that the total number of solutions of the first kind which

lie in HPH is *congruent to* 0 (mod p^m); so that *we may for the future ignore the solutions of the first kind.*

3.6. That $b+m(K) \geqslant m(H)$ is obvious in case (i) of Theorem 1 because

$$\Omega_k(K) = K \wedge \Omega_k(H),$$

and so the index of $\Omega_k(K)$ in $\Omega_k(H)$ cannot exceed the index of K in H. In case (ii), if $\rho_j(K)$ are the corresponding invariants of K, we have for the same reason $b+\rho_j(K) \geqslant \rho_j(H)$, and therefore

$$b+\min_{j=0}^{k}\{\rho_j(K)+p^{k-j}-1\} \geqslant \min_{j=0}^{k}\{\rho_j(H)+p^{k-j}-1\},$$

i.e. $b+m(K) \geqslant m(H)$. Finally, in case (iii), if $b+m(K)$ is less than $k(p-1)$, so is $m(K)$ *a fortiori*, and therefore the elements of K which satisfy $X^{p^k}=1$ form a subgroup $\Omega_k(K)$ of order $p^{m(K)}$. Now, if $\bar{\Omega}_k$ is *any* subgroup of H whose elements are of order p^k at most, clearly $K \wedge \bar{\Omega}_k \leqslant \Omega_k(K)$, and therefore the order of $\bar{\Omega}_k$ is at most $p^{b+m(K)}$, that is, it is less than $p^{k(p-1)}$. It follows from Theorem (2.6) that the elements of H of order less than or equal to p^k form a subgroup $\Omega_k(H)$ of order $p^{m(H)}$ with $m(H) \leqslant b+m(K)$. This proves the required result in all three cases.

3.7. We turn now to the solutions of the second kind. And it is at once clear that it is not merely sufficient but also necessary to prove that the number of solutions *of a given class* of the second kind which lie in HPH satisfy the congruence of Theorem 1. For in this case we have $K=H$, so that H is contained in the centraliser of the extended system of equations (4)+(5). We shall suppose then that B and the C_i have been chosen once for all as fixed elements of G, all permutable with every element of H. And by "solution" we understand in future a solution of (4)+(5).

It is evident, however, at this point that the original equations (5) are no longer relevant. For the values of $f, g, \ldots$ are completely determined by the values of X^{p^k} and the Y_i, since they are all of the form (1) above. Thus either every solution of (4) satisfies (5) as well, or none does; and we may suppose B and the C_i to have been chosen so that the former is the case. Thus *we need only consider the equations* (4).

For our restricted class of solutions, the commutators $[A_i, X]$ and $[A_i, X^{-1}]$, being equal to certain of the C_j, are permutable with every element of H. The A_i themselves also have this property, and therefore

$$X^{-1}A_iX = A_i[A_i, X]$$

is also permutable with every element of H. But for arbitrary X, the

centraliser of the $X^{-1}A_iX$ is $X^{-1}HX$, since H is the centraliser of the A_i. Hence, if X is a solution, $X^{-1}HX$ contains and therefore coincides with H. Thus all the solutions that still concern us belong to the normaliser of H in G. Since the "constants" all centralise H, nothing is lost if we suppose that *H is self-conjugate in G.*

We shall now show that not only the original equations (5) but also the equations $Y_i = C_i$ are irrelevant; so that we shall be left with the simple equation

$$X^{p^k} = B. \tag{6}$$

This will be obvious if we can show that the Y_i are all unaffected when X is replaced by an element of the form XZ with Z in H. For we are in any case only concerned with the solutions which lie in an assigned coset of H.

The conclusion required is easily proved by induction over the length t of the commutators Y, which are all, it will be recalled, of the form (2) above. Note first that X and X^{-1} transform H into itself, and therefore transform into itself the centraliser of H. But all the A_i belong to this centraliser, and hence, step by step, it is seen that all the Y_j do so too. That is, the Y_i are permutable with every element of H, whether X is a solution or not.

Suppose now that we have already proved for a certain $Y_i = Y$ that it is invariant under the transformation $X \to XZ$ with Z in H. Then we have $[Y, XZ] = Y^{-1}Z^{-1}X^{-1}YXZ = Y^{-1}X^{-1}YX$, since Z is permutable with Y and with $[Y, X]$. Similarly $[Y, (XZ)^{-1}] = [Y, X^{-1}]$. The result required now follows by induction for all Y.

3.8. It remains only to consider the problem of the number of solutions of (6) which lie in a given coset of H. Suppose that X_0 is such a solution. Then, since B is permutable with every element of H, it follows that X_0 transforms H according to an automorphism whose order is a power of p. If we denote this automorphism by θ^{-1}, so that $X_0 Z X_0^{-1} = Z^\theta$ for Z in H, then $X_0 Z$ will be a solution if and only if

$$B = X_0^{p^k} = (X_0 Z)^{p^k} = X_0 Z X_0^{-1} . X_0^2 Z X_0^{-2} \ldots . X_0^{p^k} Z X_0^{-p^k} . X_0^{p^k},$$

that is, if and only if

$$Z^\theta . Z^{\theta^2} \ldots . Z^{\theta^{p^k}} = 1. \tag{7}$$

Thus the number of solutions depends only on θ.

3.9. We must now take the three cases of Theorem 1 separately. In case (i), G, and therefore X_0, is of finite order. Also, since θ is of order a power of p, we can find a certain power X_0' of X_0, of order a power of p,

which transforms H in accordance with θ. Now $\{X_0', H\}$ is of order a power of p, and therefore belongs to some Sylow subgroup L of G. By hypothesis, L is regular. Hence X_0' and $X_0' Z$ for Z in H have, by (C, 4.28), the same p^k-th powers if and only if Z belongs to $\Omega_k(H)$. But the above calculation shows that this will also happen if and only if Z satisfies equation (7). Thus the number of solutions of (7) is precisely the order p^{ρ_k} of $\Omega_k(H)$. But this is also the number of solutions of (6) which are congruent to $X_0 \pmod H$. Thus Theorem 1 is completely proved in case (i).

Next suppose, as in case (ii), that H is regular. Since θ is of order a power of p, we can construct a sequence of self-conjugate subgroups M_i of H,

$$M_0 = 1 < M_1 < M_2 < \ldots < M_\sigma = \Omega_k(H),$$

leading from the identity to $\Omega_k(H)$, each of which is of index p in the next and each of which is invariant under θ. Further, since the $\Omega_j(H)$ are characteristic subgroups of H, we may legitimately assume the M_i to be so chosen that, for each $j = 1, 2, \ldots, k$, $\Omega_j(H)$ occurs as an M_i. Now consider M_m, where $m = \min_{j=0}^{k}(\rho_j + p^{k-j} - 1)$; so that the order of M_m is precisely equal to p^m. We shall show that every element of the coset $X_0 M_m$ satisfies equation (6) if, as before, X_0 does so. For if Q lies in M_m, then $Q^{p^k} = 1$ since $m \leqslant \sigma = \rho_k$. Also, every r-fold commutator of Q with X_0 lies in M_{m-r+1}, since X_0 transforms H in accordance with θ. Thus we see that the p^j-fold commutators of Q with X_0 lie in M_n, where

$$n = m - p^j + 1 \leqslant \rho_{k-j} + p^j - 1 - p^j + 1 = \rho_{k-j};$$

so that $M_n \leqslant \Omega_{k-j}(H)$; that is, these p^j-fold commutators are of order p^{k-j} at most. Applying the formula (2.1), we see that $(X_0 Q)^{p^k} = X_0^{p^k} = B$, since all the other terms are equal to 1. Thus $X_0 Q$ satisfies (6) if X_0 does so. Obviously it is not of importance in this argument that X_0 should transform H precisely in accordance with θ, but merely that it should leave invariant each of the subgroups M_i. And this is true of *every* element of the coset $X_0 H$. Thus the solutions of (6) which lie in $X_0 H$ constitute a certain number of cosets of the subgroup M_m, of order p^m. This proves Theorem 1 completely in case (ii).

Finally, in the general case (iii), we may apply Theorem (2.5) to the group $G_0 = \{H, \theta\}$ obtained (as a subgroup of the holomorph of H, for example) by adjoining to H the automorphism θ. Thus G_0 is a p-group. If the L_i are as in (2.5), then, by the corollary (2.51), θ and $Z\theta$ have the same p^k-th power for all Z of L_k. But the condition that θ and $Z\theta$ have the same p^k-th power is simply that Z satisfies (7) and it follows from (7) that X_0 and $X_0 Z$ also have the same p^k-th power. Thus with X_0, every

element of $X_0 L_k$ satisfies (6). Now the same G_0 may be used whatever the choice of X_0 in the coset $X_0 H$, since we have not been required to know the exact order of θ in G_0. Hence, however X_0 is chosen in this coset, if it is a solution of (6), every element of $X_0 L_k$ will also be a solution of (6). Thus the solutions of (6) which lie in $X_0 H$ constitute a certain number of cosets of L_k. By Theorem (2.5), either L_k is of order $p^{k(p-1)}$ and then the number of solutions we are concerned with will also be divisible by this number; or else $L_k = \Omega_k(H)$ consists of *all* the elements of H which satisfy $X^{p^k} = 1$ and is of order p^{ρ_k} and then our solutions will be in number a multiple of p^{ρ_k}. Thus Theorem 1 is completely proved.

4. *The subgroup problem.*

4.1. As the simplest application of the theorem proved in the preceding section, we shall now consider the classical problem of the number of subgroups with certain specified properties which lie in a given group G of finite order.

It will be convenient if we summarise first the most important known facts; we ignore particular groups and mention only those results which have a wide application.

4.11. If p^l is the highest power of the prime p to divide the order of G, then the number of subgroups of order p^l in G is congruent to 1 (mod p) and these are all conjugate in G (Sylow). If G is soluble, the number in question is a product of prime-powers each of which is congruent to 1 (mod p)†.

4.12. If p^k divides the order of G, then the number of subgroups of order p^k in G is congruent to 1 (mod p) (Frobenius)‡.

4.13. If G is a p-group of order p^l, but not cyclical, and if p is odd, then, for $0 < k < l$, the number of subgroups of order p^k in G is congruent to $1+p$ (mod p^2) (Kulakoff)§.

4.14. If p is odd and $k > 1$ and if the p-Sylow subgroups of G are not cyclical, then the number of cyclical subgroups of order p^k in G is congruent to 0 (mod p) (Miller)‖.

---

† P. Hall, "A note on soluble groups", *Journal London Math. Soc.*, 3 (1928), 98–105.

‡ G. Frobenius, "Verallgemeinerung des Sylow'schen Satzes", *Berliner Sitzungsberichte* (1895), 981–993.

§ A. Kulakoff, *Math. Annalen*, 101 (1931), 779; see also, for a simpler proof, **C**, p. 40.

‖ G. A. Miller, "An extension of Sylow's theorem", *Proc. London Math. Soc.* (2), 2 (1904), 142–143.

4.15. If G is of order p^l with $l > \frac{1}{2}k(k-1)$, then the number of Abelian subgroups of order p^k in G is congruent to 1 (mod p) (Miller)†.

It will be observed that in all of these theorems except‡ (4.13), the modulus of the congruence is always the first power of p. Again, with the exception of (4.11), (4.14), and the case $k = 1$ of (4.13), the class of subgroups whose number is in question consists of several different types.

The main results which we shall prove below have neither of these drawbacks; a typical form of theorem is as follows : the number of subgroups of G which are isomorphic with H is congruent to a certain specified number modulo some specified power of p.

In (4.3) we deal with the case where H is an elementary Abelian p-group. In (4.4), the group H can be any group which is not the direct product of an elementary Abelian group with a group which coincides with its derived group. A particular case of this is given in (4.41), viz. H a cyclical group of order p^k with $k > 1$ [the case $k = 1$ being a particular case of (4.3), viz. (4.31)]; this Theorem (4.41) generalizes Miller's result (4.14).

These theorems depend on the main theorem; though in the interest of simplicity we have discarded those refinements which could have been obtained by making use of the "localization clause" of that theorem. We give, finally, in Theorem (4.6), a generalization of Kulakoff's theorem from the case of a p-group to the case of general G. This stands in precisely the same relation to Kulakoff's theorem as the result (4.12) of Frobenius to the elementary theorem that the number of subgroups of given order in a p-group is congruent to 1 (mod p). Theorem (4.6) depends only to a very slight extent on the theorems which precede.

4.2. To save circumlocutions, we shall introduce the following

Definition (4.2). Given any group G of finite order, any prime p, and any positive integer k, the number $m_k = m_k(G) = m_k(G, p)$ which occurs in the following theorems may be chosen according to any of the rules:

(i) if the p-Sylow subgroup J of G is regular, we may take $m_k = \rho_k$, where p^{ρ_k} is the order of $\Omega_k(J)$;

(ii) if J is irregular and the elements of order less than or equal to p^k in J form a subgroup $\Omega_k(J)$ of order p^{ρ_k}, we may take $m_k = \min\{\rho_k, k(p-1)\}$;

† G. A. Miller, "Number of Abelian subgroups in every prime power group", *Amer. Journal of Math.*, 51 (1929), 31.

‡ Another exception is the theorem of Tazawa referred to in (2.4) above; but this concerns a somewhat special category of groups. Compare also the continuation of this paper, in which a result is proved concerning the number of subgroups of order p^m, $m > 1$, in the groups in question.

(iii) if J is irregular and the elements of order less than or equal to p^k in J do not form a subgroup, we may take $m_k = k(p-1)$;

(iv) if L is any regular subgroup of J, we may take

$$m_k = \min_{j=0}^{k} \{\rho_j(L) + p^{k-j} - 1\},$$

where $p^{\rho_j(L)}$ is the order of $\Omega_j(L)$.

The case (iv) overlaps the other three, and if we have the necessary information, the best result will always be obtained by taking the largest permissible value of m_k.

For $k = 1$, we have the simple rule: $m_1 = \omega(J)$ *or* $p-1$ *according as* J *is regular or not.*

LEMMA 4.21. *Let*

$$f = g = \ldots = 1 \tag{1}$$

be any system of word-equations in the symbols $X, A_1, A_2, \ldots, A_\lambda$; *and let their degrees in* X *be all congruent to* 0 (mod p^k). *Let* (C) *be any assigned class of conjugate ordered sets of* λ *elements of* G.

Then the total number of ordered sets $(X, A_1, A_2, \ldots, A_\lambda)$ *of elements of* G *which* (i) *satisfy the equations* (1), *and* (ii) *are such that* $(A_1, A_2, \ldots, A_\lambda)$ *belongs to* (C), *is congruent to* 0 (mod p^{m_k}), *where* $m_k = m_k(G, p)$ *is defined as in* (4.2).

This formulation corresponds to the form of Frobenius's theorem given in Burnside's *Theory of groups*. It is, of course, much less precise than Theorem 1.

To prove the lemma, let $(A_1, \ldots, A_\lambda)$ be any fixed member of the class (C) and G_A its centraliser in G. Then, if the index of G_A in G is μ, (C) will consist of μ distinct members; and each of these will correspond to the same number of solutions of (1), say ν. Thus we have to show that $\mu\nu \equiv 0 \pmod{p^{m_k}}$.

Let H be any p-Sylow subgroup of G_A, and let J be a p-Sylow subgroup of G containing H. If $(J:H) = p^b$ is the index of H in J, we have $\mu \equiv 0 \pmod{p^b}$.

Apply Theorem 1 to the system of equations (1), neglecting the distribution of the solutions; and consider in turn each of the four cases of definition (4.2).

(i) Here we apply Theorem 1 (i), and obtain $\nu \equiv 0 \pmod{p^m}$, where $m = \rho_k(H)$. Obviously $m_k = \rho_k = \rho_k(J) \leqslant b + \rho_k(H)$ and the result follows.

(ii) Here the elements of order less than or equal to p^k in H form a subgroup $\Omega_k(H)$ of order $p^{\rho_k(H)}$, namely the meet of H with $\Omega_k(J)$. Theorem 1 (iii) gives $\nu \equiv 0 \pmod{p^m}$, where $m = \min\{\rho_k(H),\ k(p-1)\}$. Here again $\rho_k = \rho_k(J) \leqslant b + \rho_k(H)$ and so $m_k \leqslant m + b$.

(iii) Here we may obviously assume that the elements of order less than or equal to p^k in H still form a subgroup $\Omega_k(H)$ of order $p^{\rho_k(H)}$, since otherwise the result follows at once by Theorem 1 (iii). Now, by (2.5), J has in the present case a subgroup L_k of order $p^{k(p-1)}$ whose elements are all of order less than or equal to p^k; and $\Omega_k(H)$ certainly contains the meet of H with L_k. Hence $k(p-1) \leqslant b + \rho_k(H)$.

(iv) Let L' be the meet of H with the given regular subgroup L of J. Then, for each value of j, we have $\rho_j(L) \leqslant \rho_j(L') + b$, since $\Omega_j(L')$ is simply the meet of H with $\Omega_j(L)$. Theorem 1 (ii), applied with J replaced by L', gives $\nu \equiv 0 \pmod{p^m}$, where

$$m = \min_{j=0}^{k} \{\rho_j(L') + p^{k-j} - 1\}.$$

Hence here also we have $m + b \geqslant m_k$.

This completes the proof of the lemma, and we pass now to the "Anzahl" theorems themselves.

4.3. We deal first with the number of elementary Abelian subgroups of prime-power order. The result is:

THEOREM 4.3. *For given $n \geqslant 1$, the number of elementary Abelian subgroups $A_p{}^n$ of order p^n contained in the group of finite order G is congruent to*

$$\phi_{\sigma+n-1,\,n} \pmod{p^\sigma},$$

where $\sigma = m_1 - \frac{1}{2}n(n-1)$, and $m_1 = \omega(J)$ or $p-1$, according as the p-Sylow subgroup J of G is regular or not, and finally $\phi_{a,\,n}$ denotes the number of $A_p{}^n$ contained in an $A_p{}^a$.

It may be supposed that $\sigma > 0$, since otherwise the theorem is void; thus we can assert nothing when n exceeds a certain value depending only on J.

The value of $\phi_{a,\,n}$ is well known; it is given by

$$\frac{(p^a-1)(p^{a-1}-1)\dots(p^{a-n+1}-1)}{(p^n-1)(p^{n-1}-1)\dots(p-1)}.$$

In the theorem we have chosen to write $\phi_{\sigma+n-1,\,n}$; but it would have been equally correct to write $\phi_{\alpha,\,n}$ with any $\alpha \geqslant \sigma+n-1$. This is because†

$$\phi_{\alpha+\beta,\,n} \equiv \phi_{\alpha,\,n} \pmod{p^{\alpha-n+1}}$$

for all positive β. Thus what the theorem asserts is, roughly, that the number of $A_p{}^n$ in G is congruent modulo a certain power of p to the number of $A_p{}^n$ in a *sufficiently large* $A_p{}^\alpha$.

Proof of (4.3). Let λ_n denote the number of $A_p{}^n$ in G, and let μ_n denote the number of solutions in G of the system of equations

$$\left.\begin{aligned} X_i{}^p &= 1 \qquad (i = 1, 2, \ldots, n), \\ X_i{}^{-1} X_j{}^{-1} X_i X_j &= 1 \quad (i, j = 1, 2, \ldots, n); \end{aligned}\right\} \tag{1}$$

that is, the number of distinct ordered sets $(X_1, \ldots, X_n)$ of n mutually permutable elements of G whose orders divide the prime p.

Clearly every solution of (1) will generate an $A_p{}^k$ for some k in $0 \leqslant k \leqslant n$. Also, any given $A_p{}^k$ of G will be generated by exactly

$$\psi_{n,\,k} = (p^n-1)(p^n-p) \ldots (p^n-p^{k-1})$$

distinct solutions of (1); for this is the number of distinct homomorphisms of a given $A_p{}^n$ into a given $A_p{}^k$, viz. $\phi_{n,\,n-k} = \phi_{n,\,k}$, multiplied by the order $\psi_{k,\,k}$ of the group of automorphisms of an $A_p{}^k$.

Hence, if we take $\psi_{0,\,0} = \mu_0 = 1$, we have, for each $n = 0, 1, 2, \ldots$, the identity

$$\mu_n = \sum_{k=0}^{n} \psi_{n,\,k} \lambda_k, \tag{2}$$

irrespective of the nature of G.

Now, by (4.21), we have $\mu_n \equiv 0 \pmod{p^{m_1}}$, where $m_1 = \omega(J)$ or $p-1$, as in the theorem‡. Thus (2) gives

$$\sum_{k=0}^{n} \psi_{n,\,k} \lambda_k \equiv 0 \pmod{p^{m_1}}. \tag{3}$$

We may now assert that the system of congruences (3) determines uniquely for each k the class to which λ_k belongs (mod $p^{\sigma(k)}$), where $\sigma(k) = m_1 - \frac{1}{2}k(k-1)$. For $\psi_{n,\,k}$ is divisible by $p^{\frac{1}{2}k(k-1)}$ and by no higher

† (C, 1.42).

‡ The relevant case of (4.21) can be deduced very simply from the results of section (2.4), without any appeal to the main theorem.

power of p; so that, if our assertion holds for all $k < n$, it will hold for λ_n also, by the n-th congruence (3).

But these congruences are valid when G is replaced by any $A_p{}^\alpha$ having $\alpha \geqslant m_1$. Hence λ_n must be congruent mod p^σ, $\sigma = \sigma(n)$, to the number of $A_p{}^n$ in an $A_p{}^\alpha$, for any $\alpha \geqslant m_1$; *i.e.*,

$$\lambda_n \equiv \phi_{\alpha, n} \pmod{p^\sigma}.$$

This, by the remark made above, is the required result; for obviously $\alpha \geqslant m_1 \geqslant \sigma + n - 1$.

The simplest case of (4.3) is when $n = 1$, when we have

COROLLARY 4.31. *The number of subgroups of order p in G is congruent to*

$$1 + p + p^2 + \ldots + p^{m_1 - 1} \pmod{p^{m_1}},$$

where $m_1 = \omega(J)$ or $m_1 = p - 1$ according as the p-Sylow subgroup J of G is regular or not.

For comparison, we may mention here the following

COROLLARY 4.32. *The number of subgroups of order p^2 in G is congruent to*

$$\frac{1}{(p-1)(p^2-1)} \pmod{p^{m_1 - 1}}.$$

Here m_1 is as in (4.31). For a subgroup of order p^2 is either an $A_p{}^2$ or else is cyclical. But the number of $A_p{}^2$ is congruent to $\phi_{m_1, 2} \pmod{p^{m_1 - 1}}$ by Theorem (4.3). And, obviously,

$$\phi_{m_1, 2} \equiv \frac{1}{(p-1)(p^2-1)} \pmod{p^{m_1 - 1}};$$

while on the other hand the number of cyclical subgroups of order p^2 is obtained by dividing by $p(p-1)$ the number of solutions of $X^{p^2} = 1$ which do *not* satisfy $X^p = 1$. The number of solutions of these two equations is, however, in both cases a multiple of p^{m_1}. Thus the number of cyclical subgroups of order p^2 is congruent to $0 \pmod{p^{m_1 - 1}}$; and the result (4.32) follows.

4.4. We pass now to a much more general result. This is

THEOREM 4.4. *Let H be any group of finite order, let H' be its derived group, and let H'' be the group obtained from H' by adjoining every element of order a prime from the central of H; let p^k be the highest power of p which is an invariant of the Abelian group H/H'', and let p^a be the highest power of p which divides the order of the group of automorphisms of H.*

Then, if G is any group of finite order, the number of subgroups of G which are isomorphic with H is congruent to $0 \pmod{p^{m_k - a}}$, *where* $m_k = m_k(G, p)$ *is defined as in* (4.2).

We may suppose that $k > 0$ and that $m_k > a$, since otherwise the theorem is void.

Let H^* be a subgroup of H containing H'' and such that H/H^* is cyclic of order p^k; by hypothesis, such an H^* exists. Let $H^* = \{B_1, B_2, \ldots, B_\lambda\}$. We may suppose that G has a subgroup K^* isomorphic with H^*, since otherwise there will be no subgroup isomorphic with H. Let

$$K^* = \{A_1, A_2, \ldots, A_\lambda\},$$

where A_i corresponds to B_i.

If $H = \{H^*, Y\}$, we may express the generating relations of H by taking first those of H^*, and adjoining the relations which express Y^{p^k} and the $Y^{-1} B_i Y$ in terms of the B_j's: let these be $Y^{p^k} = f(B)$ and $Y^{-1} B_i Y = g_i(B)$, $i = 1, 2, \ldots, \lambda$. Consider the equations

$$\left.\begin{aligned} X^{p^k} &= f(A) \\ X^{-1} A_i X &= g_i(A) \end{aligned}\right\} \quad (i = 1, 2, \ldots, \lambda). \tag{1}$$

Every solution X of (1) in G gives rise to a subgroup $K = \{K^*, X\}$ which is isomorphic with H. For the A_i generate K^* and $K^* \sim H^*$. Also the generators X and the A_i of K satisfy the defining relations of H. Hence $K \simeq H/L$, where L is a certain self-conjugate subgroup of H, having only the identity in common with H^*. But H^* contains the derived group H' of H. Hence L belongs to the central of H. If $L \neq 1$, it would contain elements other than the identity of order a prime from the central of H. But all such elements belong to $H'' \leqslant H^*$. Hence $L = 1$ and $K \simeq H$ as stated.

Now let (C) be the class of conjugates to which the set $(A_1, \ldots, A_\lambda)$ belongs, and apply Lemma (4.21) to the equations (1). The total number of distinct ordered sets $(X, A_1, \ldots, A_\lambda)$ for which now $(A_1, \ldots, A_\lambda)$ is an arbitrary member of (C) and which satisfy (1) will, therefore, be a multiple of p^{m_k}. If in place of (C) we take any other class (C') of ordered sets $(A_1', \ldots, A_\lambda')$, for which $A_i' \longleftrightarrow B_i$ gives an isomorphism between $\{A_1', \ldots, A_\lambda'\}$ and H^*, the same result applies. And hence the *total* number of solutions of (1) in G, for which the A_i generate a group isomorphic with H^*, is also congruent to 0 mod p^{m_k}. And as we have seen, for each such solution, X and the A_i together generate a group isomorphic with H. But clearly every subgroup of G isomorphic with H can be obtained in this way, and each of them will arise from precisely the same number of solutions of (1), viz. from α, where α is the order of the group of automorphisms of H;

for if (X, A_i) and (X', A_i') are any two solutions of (1) of the required kind and for which $K = \{X, A_i\} = \{X', A_i'\}$, then the correspondences $X \longleftrightarrow X'$ and $A_i \longleftrightarrow A_i'$, $i = 1, 2, \ldots, \lambda$, will give rise to an automorphism of K.

It follows that if the total number of subgroups of G isomorphic with H is n, and the total number of relevant solutions of (1) is n', then

$$n' = n\alpha.$$

Since $n' \equiv 0 \pmod{p^{m_k}}$ and $\alpha \not\equiv 0 \pmod{p^{a+1}}$, we deduce that $n \equiv 0 \pmod{p^{m_k - a}}$, as required.

The most important particular case of (4.4) is when H is cyclical of order p^{k+1} with $k > 0$. Here $a = k$ and accordingly we have

COROLLARY 4.41. *The number of cyclical subgroups of order p^{k+1} in G is, for $k > 0$, a multiple of $p^{m_k - k}$. If the p-Sylow subgroups of G are irregular, this number is certainly congruent to* $0 \pmod{p^{p-2}}$.

The second clause follows from the fact that if G has any cyclical subgroups of order p^{k+1} at all, then the $\rho_k(J)$ of (4.2) (ii), if it exists, must certainly be at least equal to $p+k-2$, by (2.41). Since

$$k(p-1) \geqslant p+k-2$$

also, we have $m_k \geqslant p+k-2$ in every case. Hence the result.

It is easy to see that (4.41) implies the Theorem (4.14) due to Miller.

We may remark that Theorem (4.4) will, for a suitable choice of G, give a positive result *except when H is the direct product of a group which coincides with its derived group and of a number of elementary Abelian groups.* For these exceptional types of H, we have only been able to obtain a non-trivial assertion for the case of the A_{p^n}, dealt with in section (4.3).

4.5. The only groups of order p^n to which Theorem (4.4) does not apply are the A_{p^n}; and these are covered by (4.3). But the highest power of p which divides the order of the group of automorphisms of a group of order p^n is always less than or equal to $p^{\frac{1}{2}n(n-1)}$. Hence

THEOREM 4.5. *For given $n \geqslant 1$, the total number of subgroups of order p^n in G is congruent to $\phi_{\sigma+n-1,\,n} \pmod{p^\sigma}$, where $\sigma = m_1 - \frac{1}{2}n(n-1)$ and $m_1 = m_1(G, p)$.*

More generally, we may state the following: given any class of p-groups (P), let p^a be the highest power of p which divides the order of the group of automorphisms of any one of them; then the number of subgroups of G which are isomorphic with one of the group (P) is congruent to mod $p^{m_1 - a}$ to the number of such subgroups which are contained in a sufficiently large

A_{p^a}. Of course, an A_{p^a} only has elementary subgroups, so that if none of the groups (P) is elementary, the number in question is congruent to 0.

4.6. We pass finally to the generalization of the Theorems (4.12) and (4.13) of Frobenius and Kulakoff. We need first a result which is probably familiar.

LEMMA 4.61. *If the p-Sylow subgroups of G are cyclical and of order p^l, then the number of subgroups of order p^k in G is congruent to* 1 $(\mathrm{mod}\ p^{l-k+1})$ $(0 \leqslant k \leqslant l)$.

For if $\lambda_i p^i$ is the number of p-Sylow subgroups which meet a given one in a subgroup of order p^{l-i}, $0 < i \leqslant l$, the λ_i are integers and the number of distinct Sylow subgroups in G is

$$n_l = 1+\lambda_1 p+\lambda_2 p^2+\ldots+\lambda_l p^l.$$

Also, since they are cyclical, a given subgroup of order p^k is contained in exactly

$$\nu_k = 1+\lambda_1 p+\lambda_2 p^2+\ldots+\lambda_{l-k} p^{l-k}$$

of these n_l Sylow subgroups. Hence, if n_k is the total number of subgroups of order p^k, we have

$$n_k \nu_k = n_l,$$

$$(n_k-1)\nu_k \equiv 0 \pmod{p^{l-k+1}},$$

or, since $\nu_k \not\equiv \mathrm{mod}\ p$,

$$n_k \equiv 1 \pmod{p^{l-k+1}}.$$

THEOREM 4.6. *If p is an odd prime, and if the p-Sylow subgroups of G are of order p^l and not cyclical, then, for $0 < k < l$, the total number of subgroups of order p^k in G is congruent to* $1+p \pmod{p^2}$.

Let n_k be the number in question. By (4.31) we have $n_1 \equiv 1+p \pmod{p^2}$. Hence we need only prove that, for $1 < k < l$, $n_k \equiv n_{k-1} \pmod{p^2}$.

Let P_k denote a typical subgroup of order p^k of G; and let $n_k = n_k' + n_k''$, where n_k' is the number of cyclical P_k. Each of these cyclical P_k contains exactly one P_{k-1}; while each of the remaining n_k'' non-cyclical P_k contains a number of P_{k-1} which is congruent to $1+p \pmod{p^2}$. Hence the total number of times M_k that a P_k of G contains a P_{k-1} is congruent to $n_k'+n_k''(1+p) \pmod{p^2}$. But, by (4.14), $n_k' \equiv 0 \pmod{p}$. Hence

$$M_k \equiv n_k(1+p) \pmod{p^2}. \tag{1}$$

On the other hand, let $n_{k-1} = r+s+t$, where r is the number of P_{k-1} for which the factor group N_{k-1}/P_{k-1} of P_{k-1} in its normaliser N_{k-1} in G is

not cyclical, and s is the number of the remaining P_{k-1} for which the index of N_{k-1} in G is divisible by p. Then, by (4.31), the contribution of the r P_{k-1}'s first mentioned to the number M_k is congruent to $r(1+p)$ (mod p^2).

Suppose next that P_{k-1} is a particular one of the s next mentioned. This P_{k-1} is then one of (say) λp conjugates in G, each of which is contained in the same number, say $1+\mu p$, of subgroups P_k. The contribution of this class of conjugates to M_k is therefore $\lambda p(1+\mu p)$. Since this is congruent to $\lambda p(1+p)$ (mod p^2), it follows that the s P_{k-1}'s of the kind we are now considering contribute in all a number congruent to $s(1+p)$ (mod p^2) to M_k.

Finally, let P_l be a p-Sylow subgroup of G and consider the number Λ of times one of the remaining t P_{k-1}'s is contained in a P_l. Each P_{k-1} is contained in 1 (mod p) of the P_l. So the number Λ is congruent to t (mod p). But each P_l contains 0 (mod p) of these P_{k-1}; for P_l/P_{k-1} is cyclical of order p^2 at least and P_l is not cyclical, by hypothesis. Hence $\Lambda \equiv 0$ (mod p). Therefore, $t \equiv 0$ (mod p). But each of these P_{k-1} is contained in 1 (mod p^2) of the P_k; for the p-Sylow subgroups of N_{k-1}/P_{k-1} are cyclical of order p^2 at least and so have 1 (mod p^2) subgroups of order p by the Lemma (4.61). It follows that the contribution of these t P_{k-1}'s to the number M_k is congruent to t (mod p^2) or, since $t \equiv 0$ (mod p), it is congruent to $t(1+p)$ (mod p^2).

Altogether, therefore, we have

$$M_k \equiv (r+s+t)(1+p) \pmod{p^2}. \tag{2}$$

Comparison with (1) gives $n_{k-1} = r+s+t \equiv n_k \pmod{p^2}$, as required.

The reader will observe that we have not made any appeal in this proof to Kulakoff's theorem itself; and the use which we have made of (4.31) could easily have been avoided.

King's College,
Cambridge.

ON THE (C, k) SUMMABILITY OF A TRIGONOMETRICAL INTEGRAL

FRANTIŠEK WOLF.

[Received 30 March, 1935.—Read 25 April, 1935.]

1. The aim of this paper is to deduce theorems of equiconvergence between the Cesàro sums of a trigonometrical integral

$$\int_0^\infty e^{i\lambda x}\,dA(\lambda) \tag{1.1}$$

and a certain classical Fourier series

$$\sum_{n=-\infty}^{\infty} a_n e^{inax} \tag{1.2}$$

in a finite interval (a, b).

In each theorem we start from two principal conditions. One of them is necessary, and states, essentially, that the integral (1.1) formally integrated a sufficient number of times,

$$\int_0^\infty \frac{e^{i\lambda x}}{(i\lambda)^n}\,dA(\lambda), \tag{1.3}$$

represents (by convergence or summability) a function which in (a, b) is the n-th integral of an integrable function $f(x)$,

$$\left(\int^x dx\right)^n f(x). \tag{1.4}$$

This condition establishes a latent relation between (1.1) and $f(x)$, which becomes evident by equiconvergence between the (C, k) sums of (1.1) and the (C, k) sums of (1.2), the Fourier series of $f(x)$ in (a, b), provided that a second condition is fulfilled. In the first three theorems, this condition is

$$A(m_1)-A(m) = o(m^k) \tag{1.5}$$

uniformly for $m_1 \subset (m, m+1)$.

In the last theorem it is supposed that (1.1) is summable (C, k) for $x = x_0$, an arbitrary point.

The integrals having the form of Stieltjes integrals may, however, be more general. It is sufficient to suppose the existence of finite and integrable $A(\lambda)$, $A(\lambda-0)$, and $A(\lambda+0)$. Then $A(\lambda)$ must be bounded in (a, b) and the integrals may be defined by

$$(1.6) \qquad \int_a^b \phi(\lambda)\, dA(\lambda) = \phi(b)\, A(b) - \phi(a)\, A(a) - \int_a^b \phi'(\lambda)\, A(\lambda)\, d\lambda.$$

In this paper the functions $\phi(\lambda)$ have always a continuous derivative.

The theorems may be extended to series of orthogonal functions which have a certain asymptotic expansion; for instance, to polynomials of Legendre, Laguerre, and Hermite. I have established this for simple convergence in a previous memoir*. The method there given is applicable also to summability (C, k). I have given there also some theorems concerning the uniqueness of a generalized trigonometrical development, which may be immediately adapted to trigonometrical integrals. The general problem of the uniqueness of a summable trigonometrical integral will be treated in a separate paper.

From theorems proved in the present paper we can deduce analogous results for trigonometrical series

$$\Sigma a_{\lambda_n} e^{i\lambda_n x},$$

where $\lambda_n \to \infty$, which have not hitherto been stated in this general form.

The analogues of the following theorems for trigonometrical series

$$\tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx)$$

have been known for a long time in a somewhat less general form. Probably the first writer to deduce a theorem of this kind was M. Riesz†, but instead of the condition for the integrated series, he supposed the possibility of summing the series by Poisson's method.

Of fundamental importance also is the theory of "restricted Fourier series", developed by W. H. Young‡.

Later came a result from A. Zygmund§, similar to that of Riesz, and theorems of other authors.

---

* "Contribution à la théorie des séries trigonométriques généralisées, etc.", *Publications de la Faculté d. Sc. de l'Université Masaryk à Brno*, No. 130 (1931), 1–34.

† *Comptes rendus*, 149 (1909), 909.

‡ *Proc. London Math. Soc.* (2), 17 (1918), 195 and 353.

§ *Math. Zeitschrift*, 24 (1926), 47.

I am obliged to Dr. L. S. Bosanquet for numerous suggestions and criticisms, in particular for those which helped me to give certain proofs quite a different and simpler form.

2. LEMMA. *Let $\phi(\lambda)$ be an indefinite integral. Then*

$$(2.1)\quad \left|\int_m^{m_1}\phi(\lambda)\,dA(\lambda)\right| \leqslant \Big[|\phi(m_1)|+\operatorname*{var}_{\lambda\subset(m,\,m_1)}\phi(\lambda)\Big]\max_{m<\xi\leqslant m_1}|A(\xi)-A(m)|.$$

If $\phi'(\lambda)$ exists and is continuous, and $m_1-m\leqslant 1$, then

$$(2.2)\quad \left|\int_m^{m_1}\phi(\lambda)\,dA(\lambda)\right| \leqslant \Big[|\phi(m_1)|+\max_{\lambda\subset(m,\,m_1)}|\phi'(\lambda)|\Big]\max_{m<\xi\leqslant m_1}|A(\xi)-A(m)|.$$

This is an immediate consequence of (1.6).

3. LEMMA. *Let*

(1) *k be positive and let l be a non-negative integer.*

(2) *$A(\lambda)=0$ for $\lambda\subset(0,\,1)$.*

Then

$$(3.1)\quad I(\omega)=\int_0^{\omega}e^{ix\lambda}(\omega-\lambda)^k\,dA(\lambda)$$

$$=\frac{d^{l+2}}{dx^{l+2}}\,\frac{2\Gamma(k+1)}{\pi}\int_0^{\infty}\frac{dA(\lambda)}{(i\lambda)^{l+2}}\,e^{ix\lambda}\int_0^{\infty}t^{-(1+k)}\,C_{1+k}(\omega t)\cos\lambda t\,dt.*$$

We define $C_{1+k}(t)$ in the manner of Hobson†,

$$(3.2)\quad C_{1+k}(t)=\frac{t^k}{\Gamma(k)}\int_0^1(1-u)^{k-1}\sin tu\,du=\frac{t^{k+1}}{\Gamma(k+1)}\int_0^1(1-u)^k\cos tu\,du.$$

We require the following properties:

$$(3.3)\qquad t^{-(1+k)}\,C_{1+k}(t)=O(t^{-2})+O(t^{-(1+k)})$$

---

* The infinite quasi-Stieltjes integral is defined by

$$\int_0^{\infty}\phi(\lambda)\,dA(\lambda)=\lim_{a\to\infty}\int_0^a\phi(\lambda)\,dA(\lambda).$$

† *Theory of functions of a real variable*, 2 (1926), 565, § 375.

for large values of t,

$$\lim_{t\to 0} t^{-(1+k)} C_{1+k}(t) = \frac{1}{\Gamma(k+2)}, \tag{3.4}$$

$$\frac{d}{dt} C_{1+k}(t) = C_k(t), \tag{3.5}$$

and, finally,

$$\int_0^\infty t^{-(1+k)} C_{1+k}(t) \cos tx\, dt = \frac{\pi}{2\Gamma(1+k)} (1-|x|)^k \qquad (|x|\leqslant 1), \tag{3.6}$$

$$= 0 \qquad (|x| > 1).$$

In this relation change t into ωt, and x into λ/ω. Then, after multiplying by $e^{ix\lambda} 2\Gamma(1+k)\omega^k/\pi$, we have

$$\frac{2\Gamma(1+k)}{\pi} e^{ix\lambda} \int_0^\infty t^{-(1+k)} C_{1+k}(\omega t) \cos \lambda t\, dt = e^{ix\lambda} (\omega-\lambda)^k \qquad (|\lambda| \leqslant \omega), \tag{3.7}$$

$$= 0 \qquad (|\lambda| > \omega).$$

From this we deduce immediately

$$\int_0^\omega \frac{dA(\lambda)}{(i\lambda)^{l+2}} e^{ix\lambda} (\omega-\lambda)^k \tag{3.8}$$

$$= \frac{2\Gamma(k+1)}{\pi} \int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}} e^{i\lambda x} \int_0^\infty t^{-(1+k)} C_{1+k}(\omega t) \cos \lambda t\, dt.$$

From what has been said of the Stieltjes integrals used here, it is evident that we may perform the differentiation under the sign of the first integral. and so we obtain (3.1).

4. LEMMA. *If $k > 0$ and n is the greatest integer less than $\frac{1}{2}(k+2)$, then*

$$C_{1+k}(t) = \frac{t^{k-1}}{\Gamma(k)} - \frac{t^{k-3}}{\Gamma(k-2)} + \ldots + (-1)^n C_{1+k-2n}(t). \tag{4.1}$$

If k is an even integer, then

$$C_{1+k-2n}(t) = C_1(t) = \sin t.$$

If k is an odd integer, then

$$C_{1+k-2n}(t) = C_0(t) = \cos t. \tag{4.3}$$

For k non-integral and $k-2m-1<0$, we get

$$(4.4)\quad C_{1+k}(t)=\sum_{p=0}^{m-1}\frac{(-1)^p t^{k-2p-1}}{\Gamma(k-2p)}-\frac{(-1)^{m+[k]}}{\Gamma(k-2m)}\times\int_t^\infty u^{k-2m-1}\sin(t-u)\,du+\sin\left(t-\frac{k}{2}\pi\right).$$

Proof. From (3.2) we can show, by partial integration, that

$$(4.5)\qquad C_{k+1}(t)=\frac{t^{k-1}}{\Gamma(k)}-C_{k-1}(t).$$

By applying this relation n times, we obtain (4.1).

If (4.4) is valid for a certain k, then, by differentiation, we get

$$C_k(t)=C'_{1+k}(t)=\sum_{p=0}^{m-1}\frac{(-1)^p t^{k-1-2p-1}}{\Gamma(k-1-2p)}-\frac{(-1)^{m+[k]}}{\Gamma(k-2m)}\times\int_t^\infty u^{k-2m-1}\cos(t-u)\,du+\cos\left(t-\frac{k\pi}{2}\right),$$

and, by partial integration,

$$C_k(t)=\sum_{p=0}^{m-1}\frac{(-1)^p t^{k-1-2p-1}}{\Gamma(k-1-2p)}+\frac{(-1)^{m+[k]}}{\Gamma(k-1-2m)}\times\int_t^\infty u^{k-1-2m-1}\sin(t-u)\,du+\sin\left(t-\frac{(k-1)\pi}{2}\right).$$

But this is (4.4) with $k-1$ instead of k. Now, if (4.4) has been proved for $0<k<1$, it is valid by (4.1) also for $2l<k<2l+1$, l being any positive integer. Then, as we have just proved, it is valid also for $k-1$, which is in the interval $(2l-1,\ 2l)$. Hence (4.4) is true for all non-integral positive values of k.

To prove (4.4) for $0<k<1$ let us start from (3.2).

$$(4.6)\quad C_{1+k}(t)=\frac{t^k}{\Gamma(k)}\int_0^1(1-u)^{k-1}\sin tu\,du=\frac{1}{\Gamma(k)}\int_0^t(t-u)^{k-1}\sin u\,du$$

$$=\frac{1}{\Gamma(k)}\int_0^t u^{k-1}\sin(t-u)\,du$$

$$=\frac{1}{\Gamma(k)}\int_0^\infty u^{k-1}\sin(t-u)\,du-\frac{1}{\Gamma(k)}\int_t^\infty u^{k-1}\sin(t-u)\,du.$$

It is well known that*

$$\frac{1}{\Gamma(k)}\int_0^\infty u^{k-1}\sin(t-u)\,du = \sin\left(t-\frac{k\pi}{2}\right), \tag{4.7}$$

and, by repeated partial integrations of the last integral in (4.6), we get (4.4).

5. LEMMA. *If*

(5.1) (1) $A(m_1)-A(m)=o(m^h)$ *uniformly for* $m_1 \subset (m, m+1)$,

then the integral

$$\int_0^\infty \frac{e^{i\lambda x}}{(i\lambda)^{l+2}}\,dA(\lambda) = F(x) \tag{5.2}$$

converges for $l=[h]$, *uniformly for* x *in any finite interval. If, furthermore,*

(2) $F(x)$ *is such that, in* (a, b),

$$F(x) = \left(\int_a^x dx\right)^{l+2} f(x)+a_0x^{l+1}+a_1x^l+\ldots+a_{l+1}, \tag{5.3}$$

then

$$K(\omega) = \frac{2\Gamma(k+1)}{\pi}\frac{d^{l+2}}{dx^{l+2}}\int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}}e^{i\lambda x}\int_0^\epsilon [t^{-(1+k)}-\psi(t)]\,C_{1+k}(\omega t)\cos\lambda t\,dt \tag{5.4}$$

$$= \frac{\Gamma(k+1)}{\pi}\int_0^\epsilon [t^{-(1+k)}-\psi(t)]\,C_{1+k}(\omega t)[f(x+t)+f(x-t)]\,dt,$$

for $\psi(t)$ *and* $\psi'(t)$ *continuous and* $x \subset (a+\epsilon, b-\epsilon)$.

Proof. Let us show that

$$\int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}}\,[t^{-(1+k)}-\psi(t)]\,C_{1+k}(\omega t)\,e^{i\lambda x}\cos\lambda t \tag{5.5}$$

converges uniformly for all values of t in the interval $(0, \epsilon)$. Thus it will be proved that we may invert the order of integration in $K(\omega)$, because the integration $\int_0^\epsilon dt$ may then be carried out under the sign of integration in (5.5).

* The relation connects the imaginary parts of the two members of

$$e^{it}\int_0^\infty u^{k-1}e^{-iu}\,du = e_{it}\,e^{-\frac{1}{2}k\pi i}\,\Gamma(k).$$

In fact, for $n \subset (m, m+1)$, we have

$$(5.6)\quad \int_m^n e^{i\lambda x} \frac{\cos \lambda t}{(i\lambda)^{l+2}}\, dA(\lambda) = \frac{A(n)-A(m)}{i^{l+2} n^{l+2}} \cos nt\, e^{inx}$$

$$-\int_m^n [A(\lambda)-A(m)] \frac{d}{d\lambda}\left(\frac{\cos \lambda t\, e^{i\lambda x}}{(i\lambda)^{l+2}}\right) d\lambda = O(m^{h-[h]-2})(1+|x|)$$

uniformly for all x. The coefficient $(1+|x|)$ comes from the derivation of $e^{i\lambda x}$ in the integral.

The function $$[t^{-(1+k)}-\psi(t)]\, C_{1+k}(\omega t),$$

being continuous for $t \subset (0, \epsilon)$ by (3.4) and independent of λ, is bounded, and hence the uniform convergence of (5.5) is proved.

Hence, by (5.2),

$$(5.7)\quad K(\omega) = \frac{\Gamma(k+1)}{\pi} \frac{d^{l+2}}{dx^{l+2}}$$

$$\times \int_0^\epsilon (t^{-(1+k)}-\psi(t))\, C_{1+k}(\omega t)\, dt \int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}} [e^{i\lambda(x+t)}+e^{i\lambda(x-t)}]$$

$$= \frac{\Gamma(k+1)}{\pi} \frac{d^{l+2}}{dx^{l+2}} \int_0^\epsilon [t^{-(1+k)}-\psi(t)] C_{1+k}(\omega t)[F(x+t)+F(x-t)]\, dt.$$

For $x \subset (a+\epsilon, b-\epsilon)$ the arguments of $F(x)$ are in (a, b), where this function is the $(l+2)$-th integral of $f(x)$. Thus we can differentiate $l+1$ times under the sign of integration. But even the last differentiation may be executed in the same manner, as can be seen after integrating by parts, the function $[t^{-(1+k)}-\psi(t)]\, C_{1+k}(\omega t)$ being an indefinite integral. We have thus completed the proof.

6. LEMMA. *For a continuous $\psi(t)$, we have for $\omega \to \infty$, uniformly for all values of x,*

$$(6.1)\quad \frac{1}{\omega^k} \int_{\frac{1}{2}\epsilon}^\epsilon \psi(t)\, C_{1+k}(\omega t)\, [f(x+t)+f(x-t)]\, dt$$

$$= [O(\omega^{-1})+O(\omega^{-k})] \int_{x-\epsilon}^{x+\epsilon} |f(t)|\, dt.$$

From (3.3) we have, for $t \geqslant \frac{1}{2}\epsilon$,

$$\omega^{-k}\, C_{1+k}(\omega t) = O(\omega^{-1})+O(\omega^{-k}).$$

The rest of the proof is elementary.

7. LEMMA. *If*

$$(7.01)\quad \begin{cases} (1)\ A(m_1)-A(m)=o(m^h)\ \textit{uniformly for} \\ \qquad\qquad m_1 \subset (m,\ m+1),\ h \geqslant 0,\ l=[h], \\ (2)\ \psi(t)\ \textit{and its}\ l+3\ \textit{derivatives are continuous in}\ (0,\ \infty),\ \psi(t)=0 \\ \quad \textit{for}\ t \subset (0,\ \tfrac{1}{2}\epsilon),\ \psi(t)=t^{-(1+k)}\ \textit{for}\ t \subset (\epsilon,\ \infty),\ k>0, \end{cases}$$

then

$$(7.02)\quad L=\frac{d^{l+2}}{dx^{l+2}}\int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}}\, e^{i\lambda x}\int_0^\infty \psi(t)\, e^{\pm i(\omega\pm\lambda)t}\,dt = o(\omega^h)(1+|x|)$$

uniformly for all values of x.

Proof. First consider the expression with $e^{+i(\omega+\lambda)t}$. From condition (2) it follows that

$$(7.03)\quad \int_0^\infty \psi(t)\, e^{i(\omega+\lambda)t}\,dt = \frac{i^{l+3}}{(\omega+\lambda)^{l+3}}\int_0^\infty \psi^{(l+3)}(t)\, e^{i(\omega+\lambda)t}\,dt = O\left(\frac{1}{(\omega+\lambda)^{l+3}}\right),$$

and

$$(7.04)\quad \frac{d}{d\lambda}\left[e^{i\lambda x}\int_0^\infty \psi(t)\, e^{i(\omega+\lambda)t}\,dt\right]$$

$$= ixe^{i\lambda x}\int_0^\infty \psi(t)\, e^{i(\omega+\lambda)t}\,dt + ie^{i\lambda x}\int_0^\infty \psi(t)\, te^{i(\omega+\lambda)t}\,dt$$

$$= O\left(\frac{1+|x|}{(\omega+\lambda)^{l+3}}\right).$$

From this and (2.2) we can deduce that the integral

$$(7.05)\qquad \int_0^\infty dA(\lambda)\, e^{i\lambda x}\int_0^\infty \psi(t)\, e^{i(\omega+\lambda)t}\,dt$$

converges uniformly for all values of x in any finite interval. Hence we may carry out the differentiations in L under the sign of integration and we have in (7.05) a new expression for L.

From (2.2), (7.01), (7.03), (7.04) we have, again,

$$(7.06)\qquad \int_n^{n+1} dA(\lambda)\, e^{i\lambda x}\int_0^\infty \psi(t)\, e^{i(\omega+\lambda)t}\,dt = \frac{o(n^h)}{(\omega+n)^{l+3}}\,(1+|x|),$$

and hence

$$(7.07)\quad L(\omega)=O\left(\sum_{n=0}^{\infty}\frac{n^h}{(\omega+n)^{l+3}}\,(1+|x|)\right)=O\left(\sum_{n=0}^{\infty}(\omega+n)^{-3}(1+|x|)\right)$$

$$= O(\omega^{-2})(1+|x|)=o(\omega^l)\,(1+|x|).$$

We have thus proved (7.02) in one case.

Now we have to consider the case of $e^{i(\omega-\lambda)x}$ instead of $e^{i(\omega+\lambda)x}$. For $|\omega-\lambda|>1$ we have relations analogous to (7.03) and (7.04).

We write

$$(7.08)\quad L=\int_0^\infty dA(\lambda)\,e^{i\lambda x}\int_0^\infty \psi(t)\,e^{i(\omega-\lambda)t}\,dt$$

$$=\int_{\omega-2}^{\omega+2} dA(\lambda)\,e^{i\lambda x}\int_0^\infty \psi(t)\,e^{i(\omega-\lambda)t}\,dt+\int_0^{'\infty} dA(\phi)\,e^{i\lambda x}\int_0^\infty \psi(t)\,e^{i(\omega-\lambda)}\,dt$$

$$=L_1+L_2,$$

where $\int_0^{'\infty}$ is written for $\int_0^{\omega-2}+\int_{\omega+2}^\infty$.

For the application of (2.1) to L_1 we want

$$(7.09)\qquad \max_{\omega-2\leqslant\lambda\leqslant\omega+2}\left|\int_0^\infty \psi(t)\,e^{i(\omega-\lambda)}\,dt\right|\leqslant\int_0^\infty|\psi(t)|\,dt,$$

which is finite by the definition of $\psi(t)$.

Also

$$(7.10)\quad \operatorname*{var}_{\omega-2\leqslant\lambda\leqslant\omega+2}\int_0^\infty \psi(t)\,e^{i(\omega-\lambda)t}\,dt$$

$$=\operatorname*{var}_{-2\leqslant\lambda\leqslant2}\int_0^\infty\psi(t)\,e^{i\lambda t}\,dt\leqslant\int_{-2}^2 d\lambda\left|\int_0^\infty\psi(t)\,te^{i\lambda t}\,dt\right|$$

$$\leqslant\int_{-2}^2 d\lambda\int_0^\epsilon\psi(t)\,t\,dt+\int_{-2}^2 d\lambda\left|\int_\epsilon^\infty t^{-k}\,e^{i\lambda t}\,dt\right|\leqslant C_1+\int_{-2}^2|\lambda^{k-1}|\,d\lambda\left|\int_{\xi\lambda}^\infty t^{-k}\,e^{it}\,dt\right|.$$

But
$$\int_{\epsilon\lambda}^\infty t^{-k}\,e^{it}\,dt=O(\lambda^{1-k})+O(|\log\lambda|).$$

Hence the last integral in (7.10) is finite and the variation there is bounded. Now from (2.1), (7.01) we get

$$(7.11)\qquad L_1=o(\omega^h)(1+|x|),$$

and, in the same manner as in (7.07), we get

$$(7.12)\qquad L_2=\sum_{n=0}^{\infty}{}'\frac{o(n^h)}{|\omega-n|^{l+3}}(1+|x|),$$

where the members with $|\omega-n|\leqslant1$ are omitted. For $n>2\omega$ we use the inequality

$$n/(n-\omega)\leqslant2,$$

and for $n\leqslant2\omega$
$$n^h\leqslant2^h\omega^h,$$

and we get

$$(7.13)\qquad L_2 = \left[2^l \sum_{n>2\omega}^{\infty}{}' \frac{o(1)}{|\omega-n|^2} + o(\omega^h)\, 2^h \sum_{n=0}^{2\omega}{}' \frac{1}{|\omega-n|^{l+3}}\right](1+|x|)$$
$$= [o(\omega^{-2})+o(\omega^h)](1+|x|) = o(\omega^h)(1+|x|).$$

The proof for the case with $e^{-i(\omega\pm\lambda)t}$ instead of $e^{i(\omega\pm\lambda)t}$ is similar.

8. LEMMA*. *If conditions* (1) *and* (2) *of Lemma* 7 *are satisfied, and if*

$$(8.1)\qquad D(t) = \frac{1}{\Gamma(k-m)} \int_t^{\infty} u^{k-m} \sin(t-u)\, du,$$

$$(8.2)\qquad m > k+l+3,$$

then

$$(8.3)\qquad I = \frac{d^{l+2}}{dx^{l+2}} \int_0^{\infty} \frac{dA(\lambda)}{(i\lambda)^{l+2}}\, e^{i\lambda x} \int_0^{\infty} \psi(t)\, D(\omega t) \cos\lambda t\, dt = o(1)(1+|x|),$$

for $\omega \to \infty$, *uniformly for all values of* x.

From the definition of $\psi(t)$ in Lemma 7, we find that

$$(8.4)\qquad \int_0^{\infty} \psi(t)\, D(\omega t) \cos\lambda t\, dt$$
$$= \int_{\frac{1}{2}\epsilon}^{\infty} \psi(t)\, D(\omega t) \cos\lambda t\, dt$$
$$= \frac{1}{\lambda^{l+3}} \int_{\frac{1}{2}\epsilon}^{\infty} [\psi(t)\, D(\omega t)]^{(l+3)} \cos(\lambda t + \overline{l+3}\, \tfrac{1}{2}\pi)\, dt$$
$$= \frac{1}{\lambda^{l+3}} \sum_{p=0}^{l+3} c_p\, \omega^p \int_{\frac{1}{2}\epsilon}^{\infty} \psi^{(l+3-p)}(t)\, D^{(p)}(\omega t) \cos(\lambda t + \overline{l+3}\, \tfrac{1}{2}\pi)\, dt.$$

Now we have

$$D'(t) = \frac{1}{\Gamma(k-m)} \int_t^{\infty} u^{k-m} \cos(t-u)\, du = O(t^{k-m})$$

and

$$D''(t) = -\frac{t^{k-m}}{\Gamma(k-m)} - \frac{1}{\Gamma(k-m)} \int_t^{\infty} u^{k-m} \sin(t-u)\, du$$
$$= -\frac{t^{k-m}}{\Gamma(k-m)} - D(t) = O(t^{k-m}).$$

From the last relation we can deduce generally

$$(8.5)\qquad D^{(p)}(t) = O(t^{k-m}).$$

---

* I owe to Dr. L. S. Bosanquet the idea of using in the proof the asymptotic formula for $C^{1+k}(t)$. It was thus possible to simplify the originally very complicated proof.

From (8.4) we now get

$$(8.6)\quad \int_0^\infty \psi(t)\, D(\omega t)\cos\lambda t\, dt = \frac{1}{\lambda^{l+3}} \sum_{p=0}^{l+3} c_p\, \omega^{p+k-m} \int_{\frac{1}{2}\epsilon}^\infty \psi^{(l+3-p)}(t)\, O(t^{k-m})\, dt$$

$$= \frac{1}{\lambda^{l+3}} \sum_{p=0}^{l+3} c_p\, \omega^{p+k-m} \int_{\frac{1}{2}\epsilon}^\infty O(t^{-(1+k+l+3-p)+k-m})\, dt$$

$$= \frac{1}{\lambda^{l+3}} \sum_{p=0}^{l+3} c_p\, \omega^{p+k-m} \int_{\frac{1}{2}\epsilon}^\infty O(t^{-1-(l+3-p)-m})\, dt.$$

By reason of (8.2) all the integrals are convergent, and so, finally,

$$(8.7)\quad \int_0^\infty \psi(t)\, D(\omega t)\cos\lambda t\, dt = \frac{O(\omega^{l+3+k-m})}{\lambda^{l+3}}.$$

In the same manner, we get

$$(8.8)\quad \frac{\lambda}{d\lambda}\int_0^\infty \psi(t)\, D(\omega t)\cos\lambda t\, dt = -\int_0^\infty \psi(t)\, t\, D(\omega t)\sin\lambda t\, dt = \frac{O(\omega^{l+3+k-m})}{\lambda^{l+3}}.$$

By reasoning similar to that which follows (7.05), we deduce, for I in (8.3), that

$$I = \int_0^\infty dA(\lambda)\, e^{i\lambda x} \int_0^\infty \psi(t)\, D(\omega t)\cos\lambda t\, dt,$$

this being a uniformly convergent integral for x in any finite interval.

Also, in the same manner as in (7.06) and (7.07), we find that

$$I = O(\omega^{l+3+k-m}) \sum_{n=1}^{\infty} \frac{o(n^h)}{n^{l+3}}\,(1+|x|)$$

$$= O(\omega^{l+3+k-m})\, O\Big(\sum_{n=1}^{\infty} n^{-2}\Big)(1+|x|) = o(1)(1+|x|),$$

by (8.2). Thus (8.3) is proved.

9. LEMMA. *If the conditions of Lemma 7 are satisfied, then, uniformly for all values of x,*

$$(9.1)\quad M = \frac{d^{l+2}}{dx^{l+2}} \int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}}\, e^{i\lambda x} \int_0^\infty \psi(t)\, C_{1+k}(\omega t)\cos\lambda t\, dt$$

$$= \frac{\omega^{k-1}}{\Gamma(k)}\,\phi_{k-1}(x) + \frac{\omega^{k-3}}{\Gamma(k-2)}\,\phi_{k-3}(x) + \cdots$$

$$+ \frac{\omega^{k-2n+1}}{\Gamma(k-2n+2)}\,\phi_{k-2n+1}(x) + o(\omega)(1+|x|),$$

where $n = [\frac{1}{2}(k+1)]$, $\phi_m(x)$ *is continuous, and*

$$(9.2) \qquad \phi_m(x) = \int_0^\infty dA(\lambda)\, e^{i\lambda x} \int_0^\infty \psi(t)\, t^m \cos \lambda t\, dt = O(x).$$

The integrals for $\phi_m(x)$ converge absolutely and uniformly for x in any finite interval.

For k integral we apply to $C_{1+k}(\omega t)$ the relation (4.1):

$$(9.3) \quad M = \frac{\omega^{k-1}}{\Gamma(k)} \frac{d^{l+2}}{dx^{l+2}} \int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}}\, e^{i\lambda x} \int_0^\infty \psi(t)\, t^{k-1} \cos \lambda t\, dt$$

$$- \frac{\omega^{k-3}}{\Gamma(k-2)} \frac{d^{l+2}}{dx^{l+2}} \int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}}\, e^{i\lambda x} \int_0^\infty \psi(t)\, t^{k-3} \cos \lambda t\, dt + \ldots$$

$$+ (-1)^n \frac{d^{l+2}}{dx^{l+2}} \int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}}\, e^{i\lambda x} \int_0^\infty \psi(t)\, C_{k-2n+1}(\omega t) \cos \lambda t\, dt.$$

To the last member, where $C_{k-2n+1}(\omega t)$ is $\sin \omega t$ or $\cos \omega t$, we can apply Lemma 7, and we find that it is $o(\omega^k)(1+|x|)$. In the other members we may differentiate as usual under the sign of integration, and so we get (9.1) and (9.2) for k integral.

For k non-integral we use (4.4) with

$$(9.4) \qquad 2m+1 > k+l+3,$$

i.e.

$$m > \tfrac{1}{2}(k+l+2).$$

Then

$$(9.5) \quad M = \sum_{p=0}^{m-1} \frac{(-1)^p\, \omega^{k-2p-1}}{\Gamma(k-2p)} \frac{d^{l+2}}{dx^{l+2}} \int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}}\, e^{i\lambda x} \int_0^\infty \psi(t)\, t^{k-2p-1} \cos \lambda t\, dt$$

$$- (-1)^{m+[k]} \frac{d^{l+2}}{dx^{l+2}} \int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}}\, e^{i\lambda x} \int_0^\infty \psi(t)\, D(\omega t) \cos \lambda t\, dt$$

$$+ \frac{d^{l+2}}{dx^{l+2}} \int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}}\, e^{i\lambda x} \int_0^\infty \psi(t) \sin\left(\omega t - \frac{k\pi}{2}\right) \cos \lambda t\, dt,$$

where

$$(9.6) \qquad D(t) = \frac{1}{\Gamma(k-2m)} \int_0^\infty u^{k-2m-1} \sin(t-u)\, du.$$

As usual, we may perform the differentiation under the signs of integration in the sum in (9.5), and we get members of the form

$$\omega^{k-2p-1}\, \phi_{k-2p-1}(-x)/\Gamma(k-2p),$$

where $\phi_k(x)$ is given by (9.2). The first n members are explicitly given in (9.1); the others are included in $o(\omega^h)(1+|x|)$. By reason of (9.4), we

may apply Lemma 8 to the member with $D(\omega t)$ in (9.5), and we find it to be $o(\omega^k)(1+|x|)$. The same is evidently true of the last member in (9.5) by Lemma 7. Thus (9.1) is proved.

10. THEOREM. *If*

(1) $A(m_1)-A(m)=o(m^h)$ *uniformly for* $m_1 \subset (m, m+1)$, $h \geqslant 0$, $l=[h]$,

(2) $$\int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}}\, e^{i\lambda x} = F(x)$$

is a function which is the* $(l+2)$*-th integral of an integrable function* $f(x)$ *in* (a, b), *then, for* $k>0$,

(10.1) $$\int_0^\omega e^{i\lambda x}\left(1-\frac{\lambda}{\omega}\right)^k dA(\lambda) - \frac{\Gamma(k+1)}{\pi\omega^k}\int_0^\epsilon t^{-(1+k)} C_{1+k}(\omega t)[f(x+t)+f(x-t)]\,dt$$

$$= O(\omega^{-1})+o(\omega^{h-k}),$$

uniformly for values of x *in any finite interval contained in* $(a+\epsilon, b-\epsilon)$.

Now let

(10.2) $$\sum_{n=-\infty}^{\infty} a_n e^{ina x}$$

be the Fourier series of $f(x)$ in (a, b). Then

(10.3) $$a=(b-a)/2\pi.$$

It is well known that†

(10.5) $$\sum_{|n|<\omega/a} a_n\left(1-\frac{|n|a}{\omega}\right)^k e^{ina x}$$

$$-\frac{\Gamma(k+1)}{\pi\omega^k}\int_0^\epsilon t^{-(1+k)} C_{1+k}(\omega t)[f(x+t)+f(x-t)]\,dt$$

converges for $\omega\to\infty$ to zero, uniformly for all values of x, if $f(x)$ is defined outside (a, b) as a periodic function of period $b-a$. Therefore, (10.1) establishes the equi-convergence of the first member with the (C, k) sums of (10.2).

---

* The integral converges uniformly because of (1).

† Hobson, *Theory of functions of a real variable*, 2 (1926), § 372.

Proof of the theorem. We use Lemma 3 and the definition of $\psi(t)$ in Lemma 7.

$$(10.6)\quad \frac{I(\omega)}{\omega^k} = \frac{2\Gamma(k+1)}{\pi\omega^k}\frac{d^{l+2}}{dx^{l+2}}\int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}}\, e^{i\lambda x}\int_0^\infty t^{-(1+k)}\, C_{1+k}(\omega t)\cos\lambda t\, dt$$

$$= \frac{2\Gamma(k+1)}{\pi\omega^k}\frac{d^{l+2}}{dx^{l+2}}\int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}}\, e^{i\lambda x}\int_0^\epsilon [t^{-(1+k)} - \psi(t)]\, C_{1+k}(\omega t)\cos\lambda t\, dt$$

$$+ \frac{2\Gamma(k+1)}{\pi\omega^k}\frac{d^{l+2}}{dx^{l+2}}\int_0^\infty \frac{dA(\lambda)}{(i\lambda)^{l+2}}\, e^{i\lambda x}\int_0^\infty \psi(t)\, C_{1+k}(\omega t)\cos\lambda t\, dt.$$

To the first integral apply Lemmas 5 and 6, and to the second, Lemmas 8 and 9.

We get

$$(10.7)\quad \frac{I(\omega)}{\omega^k} = \frac{\Gamma(k+1)}{\pi\omega^k}\int_0^\epsilon t^{-(1+k)}\, C_{1+k}(\omega t)[f(x+t)+f(x-t)]\, dt$$

$$+ O(\omega^{-1}) + O(\omega^{-k}) + \frac{\omega^{-1}}{\Gamma(k)}\,\phi_{k-1}(x)$$

$$+ \frac{\omega^{-3}}{\Gamma(k-2)}\,\phi_{k-3}(x) + \ldots + o(\omega^{h-k}).$$

This is valid uniformly in any finite $(a+\epsilon, b-\epsilon)$. Hence the theorem can be deduced immediately.

11. THEOREM. *If the conditions of Theorem* 10 *are satisfied, then*

$$(11.1)\quad \int_0^\omega e^{i\lambda x}\left(1-\frac{\lambda}{\omega}\right) dA(\lambda)$$

$$- \frac{\Gamma(k+1)}{\pi\omega^k}\int_0^\epsilon [t^{-(1+k)} - \psi(t)]\, C_{1+k}(\omega t)[f(x+t)+f(x-t)]\, dt$$

$$= o(\omega^{h-k})(1+|x|) + \frac{\omega^{-1}}{\Gamma(k)}\,\phi_{k-1}(x) + \frac{\omega^{-3}}{\Gamma(k-2)}\,\phi_{k-3}(x) + \ldots$$

$$+ (-)^n \frac{(\omega)^{-2n+1}}{\Gamma(k-2n+2)}\,\phi_{k-2n+1},$$

uniformly in $(a+\epsilon, b-\epsilon)$. *Here a and b may be infinite.*

The function $\psi(t)$ is defined as in Lemma 7. The functions $\phi_m(x)$ are given by (9.2).

The proof is similar to that of Theorem 10, but we do not need Lemma 5.

12. THEOREM. *If the conditions of Theorem* 10 *are satisfied, then we have, for* $k=0$,

$$(12.1)\quad \int_0^\omega e^{i\lambda x}\,dA(\lambda)-\frac{2}{\pi}\int_0^\epsilon [t^{-2}-\psi(t)]\sin(\omega+\tfrac{1}{2})t\sin\tfrac{1}{2}t[f(x+t)+f(x-t)]\,dt$$
$$=o(\omega^h)(1+|x|),$$

uniformly for all values of x *in an interval* (a', b') *contained in* $(a+\epsilon, b-\epsilon)$; a *and* b *may be infinite.*

We have, uniformly for x *in a finite* (a', b'),

$$(12.2)\quad \int_0^\omega e^{i\lambda x}\,dA(\lambda)-\frac{1}{\pi}\int_0^\epsilon \frac{\sin\omega t}{t}[f(x+t)+f(x-t)]\,dt=o(\omega^h).$$

From Theorem 11 we get, for $k=1$,

$$\int_0^\omega dA(\lambda)e^{i\lambda x}(\omega-\lambda)-\frac{1}{\pi}\int_0^\epsilon [t^{-2}-\psi(t)](1-\cos\omega t)[f(x+t)+f(x-t)]\,dt$$
$$=o(\omega^h)(1+(|x|).$$

If we write the same relation with $\omega+1$ instead of ω and take the difference of the two, we have

$$\int_0^\omega e^{i\lambda x}\,dA(\lambda)+\int_\omega^{\omega+1} e^{i\lambda x}(\omega+1-\lambda)\,dA(\lambda)$$
$$-\frac{2}{\pi}\int_0^\epsilon [t^{-2}-\psi(t)]\sin(\omega+\tfrac{1}{2})t\sin\tfrac{1}{2}t[f(x+t)+f(x-t)]\,dt$$
$$=o(\omega^h)(1+|x|).$$

By Lemma 2, the second integral in the first member is $o(\omega^h)(1+|x|)$, and so we get (12.1).

To deduce (12.2) we notice that

$$(12.3)\quad \left\{2[t^{-2}-\psi(t)]\sin(\omega+\tfrac{1}{2})t\sin\tfrac{1}{2}t-\frac{\sin\omega t}{t}\right\}f(x+t)$$

can be written in the form

$$(12.4)\quad \alpha(t)\sin\omega t+\beta(t)\cos\omega t,$$

where $\alpha(t)$, $\beta(t)$ are two integrable functions. Hence, if we integrate (12.3) with regard to t in $(-\epsilon, \epsilon)$, the integral will be $o(1)$ for large values of ω.

Thus we may write

$$\frac{1}{\pi}\int_0^{\epsilon}\frac{\sin\omega t}{t}[f(x+t)+f(x-t)]\,dt$$

instead of

$$\frac{2}{\pi}\int_0^{\epsilon}[t^{-2}-\psi(t)]\sin(\omega+\tfrac{1}{2})t\sin\tfrac{1}{2}t\,[f(x+t)+f(x-t)]\,dt,$$

and we get the second part of the theorem. The interval of uniformity must be finite, because of the factor $f(x+t)$ in $\alpha(t)$ and $\beta(t)$.

13. LEMMA. *If*

$$(13.1)\qquad \begin{cases} (1)\quad A(\lambda)=0 \quad \textit{for} \quad \lambda\subset(0,\ 1) \\ \textit{and} \\ (2)\quad \displaystyle\int_0^{\omega}\left(1-\frac{\lambda}{\omega}\right)^k dA(\lambda)=O(\omega^h), \quad k \textit{ integral}, \end{cases}$$

then, for $l=k+[h]$,

$$(13.2)\qquad \lim_{\omega\to\infty}\int_0^{\omega}\frac{e^{i\lambda x}}{(i\lambda)^{l+2}}\left(1-\frac{\lambda}{\omega}\right)^k dA(\lambda)=F(x)$$

for all x, *the convergence being uniform in any finite interval of* x.

Proof. Let $$A_n(\lambda)=\frac{1}{n!}\int_0^{\lambda}(\lambda-\mu)^n\,dA(\mu).$$

Then $$A_n(\lambda)=\int_0^{\lambda}A_{n-1}(\mu)\,d\mu,$$

and our condition is equivalent to

$$(13.3)\qquad A_k(\omega)=O(\omega^{h+k}).$$

By partial integrations we have

$$(13.4)\qquad \phi(x,\omega)=\frac{1}{\omega^k}\int_0^{\omega}\frac{dA(\lambda)}{(i\lambda)^{l+2}}e^{i\lambda x}(\omega-\lambda)^k$$

$$=\frac{(-1)^k}{i^l\omega^k}\int_0^{\omega}A_k(\lambda)\frac{d^{k+1}}{d\lambda^{k+1}}\left[\frac{e^{i\lambda x}}{\lambda^{l+2}}(\omega-\lambda)^k\right]d\lambda$$

$$+\frac{A_k(\omega)}{\omega^k}\frac{e^{i\omega x}}{i^{l+2}\omega^{l+2}}\,k!.$$

By reason of (13.3) the last member is $O(\omega^{-k-1})$, uniformly for all values of x. On carrying out the differentiations, we get

$$(13.5)\quad \phi(x,\omega)=\frac{1}{\omega^k}\Sigma c_{m,n}\int_1^\omega A_k(\lambda)\frac{(ix)^m e^{i\lambda x}}{\lambda^{l+2+n}}(\omega-\lambda)^{m+n-1}d\lambda+O(\omega^{-k-1}),$$

where $n\geqq 0,\ m\geqq 0,\ 1\leqq m+n\leqq k+1$.

But the integrals of the form

$$(13.6)\qquad \int_0^\infty \frac{A_k(\lambda)}{\lambda^{l+2+n}}e^{i\lambda x}d\lambda$$

converge absolutely. From this we see that, for $\omega\to\infty$, $\phi(x,\omega)$ converges uniformly in any finite interval of x.

14. LEMMA. *If the function*

$$(14.1)\qquad F(x)=\lim_{\omega\to\infty}\frac{1}{\omega^k}\int_0^\infty\frac{e^{i\lambda x}}{(i\lambda)^{l+2}}(\omega-\lambda)^k dA(\lambda)$$

is a $(l+2)$-th integral in (a,b), the same is true of

$$(14.2)\qquad \phi(x)=\int_0^\infty\frac{A_k(\lambda)}{\lambda^{l+2}}e^{i\lambda x}d\lambda.$$

Proof. We may take $a, b>0$. The integral

$$(14.3)\qquad \frac{1}{\omega^{m+n-1}}\int_0^\infty A_k(\lambda)\frac{(ix)^m e^{i\lambda x}}{\lambda^{l+2+n}}(\omega-\lambda)^{m+n-1}d\lambda$$

has as its limit for $\omega\to\infty$ the convergent integral (13.6). Hence we see that in (13.5) the limit of any member with $m+n-1<k$ is zero for $\omega\to\infty$. Thus, differentiating $(\omega-\lambda)^k$ in (13.4) gives rise to members whose limits are zero. Therefore, from (13.4),

$$(14.4)\quad F(x)=\lim_{\omega\to\infty}\phi(x,\omega)=\lim_{\omega\to\infty}\frac{(-1)^k}{i^l\omega^k}\int_0^\omega A_k(\lambda)(\omega-\lambda)^k\frac{d^{k+1}}{d\lambda^{k+1}}\left[\frac{e^{i\lambda x}}{\lambda^{l+2}}\right]d\lambda.$$

But, using the relation

$$\frac{d^{k+1}}{d\lambda^{k+1}}\left[\frac{e^{i\lambda x}}{(\lambda x)^{l+2}}\right]=\left(\frac{x}{\lambda}\right)^{k+1}\frac{d^{k+1}}{dx^{k+1}}\left[\frac{e^{i\lambda x}}{(\lambda x)^{l+2}}\right],$$

we get

$$(14.5)\quad F(x)=\lim_{\omega\to\infty}\frac{(-1)^k x^{k+l+3}}{i^l\omega^k}\frac{d^{k+1}}{dx^{k+1}}\left[\frac{1}{x^{l+2}}\int_0^\omega A_k(\lambda)\frac{e^{i\lambda x}}{\lambda^{k+l+3}}(\omega-\lambda)^k d\lambda\right].$$

The differentiation may also be performed under the sign of integration and,

after letting $\omega \to \infty$, we get

$$(14.6) \qquad F(x) = \frac{(-1)^k}{i^l} x^{k+l+3} \frac{d^{k+1}}{dx^{k+1}} \left[\frac{1}{x^{l+2}} \int_0^\infty A_k(\lambda) \frac{e^{i\lambda x}}{\lambda^{k+l+3}} d\lambda \right].$$

The condition imposed on $F(x)$ ensures that $x^{-k-l-3} F(x)$ is also an $(l+2)$-th integral in (a, b). From the last relation we see that

$$\frac{1}{x^{l+2}} \int_0^\infty A_k(\lambda) \frac{e^{i\lambda x}}{\lambda^{k+l+3}} d\lambda$$

is a $(k+l+3)$-th integral in (a, b), and the same is true after the omission of the factor x^{-l-2}.

But, since

$$(14.7) \qquad \phi(x) = \int_0^\infty A_k(\lambda) \frac{e^{i\lambda x}}{(i\lambda)^{l+2}} d\lambda = \frac{d^{k+1}}{d\lambda^{k+1}} \int_0^\infty A_k(\lambda) \frac{e^{i\lambda x}}{(i\lambda)^{k+l+3}} d\lambda,$$

we see that $\phi(x)$ is an $(l+2)$-th integral in (a, b). The lemma is therefore proved.

15. THEOREM. *If*

$$(15.01) \qquad (1) \quad \int_0^\omega \left(1 - \frac{\lambda}{\omega}\right)^k e^{i\lambda x_0} dA(\lambda) = A + o(\omega^h),$$

k is a non-negative integer, $l = k + [h]$, $h \geqslant 0$;

$$(15.02) \qquad (2) \quad A(\lambda) = 0 \quad \textit{for} \quad \lambda \subset (0, 1);$$

and

$$(15.03) \qquad F(x) = \lim_{\omega \to \infty} \int_0^\omega \left(1 - \frac{\lambda}{\omega}\right)^k \frac{e^{i\lambda x}}{(i\lambda)^{l+2}} dA(\lambda)$$

is an $(l+2)$-th integral of an integrable function $f(x)$ in (a, b), then, for $m \geqslant k$,

$$(15.04) \qquad \int_0^\omega e^{i\lambda x} \left(1 - \frac{\lambda}{\omega}\right)^m dA(\lambda) - \frac{\Gamma(m+1)}{\pi \omega^m} \int_0^\epsilon t^{-(1+m)} C_{1+m}(\omega t) [f(x+t) + f(x-t)] dt = o(\omega^{k+h-m}) + o(1)$$

for $\omega \to \infty$. The convergence is uniform for values of x in any finite interval contained in $(a+\epsilon, b-\epsilon)$.

Note. The condition for k to be integral is not essential. It merely simplifies the proof.

Proof. Without loss of generality we may suppose that $x_0 = 0$ and $A = 0$. The latter assumption can be justified by adding to $A(\lambda)$ a step-function with one jump $-A$ at $\lambda = 1$.

With the notation of § 13, we have

$$A_k(\lambda) = o(\lambda^{k+h}). \tag{15.05}$$

Using integration by parts, we get

$$\frac{1}{\omega^m}\int_0^\infty dA(\lambda)e^{i\lambda x}(\omega-\lambda)^m = \frac{(-1)^m}{\omega^m}\int_0^\omega A_k(\lambda)\frac{d^{k+1}}{d\lambda^{k+1}}[e^{i\lambda x}(\omega-\lambda)^m]\,d\lambda + \left[k!\,\frac{A_k(\omega)}{\omega^m}\,e^{i\omega x}\right]. \tag{15.06}$$

The last member does not appear when $m > k$, and when $k = m$ it is $o(\omega^{k+h-m})$. Hence

$$\int_0^\omega dA(\lambda)e^{i\lambda x}\left(1-\frac{\lambda}{\omega}\right)^m = (-1)^{m+1}\sum_{n=1}^{k+1}\binom{k+1}{n}(ix)^n(-1)^{k+1-n}\frac{\Gamma(m+1)}{\Gamma(m+n-k)}\frac{1}{\omega^m} \times\int_0^\omega A_k(\lambda)e^{i\lambda x}(\omega-\lambda)^{m+n-k-1}\,d\lambda + o(\omega^{h+k-m}). \tag{15.07}$$

To the formal integral

$$\int_0^\infty A_k(\lambda)\,e^{i\lambda x}\,dA, \tag{15.08}$$

we apply Theorem 10. Let

$$B(\lambda) = \int_0^\lambda A_k(\mu)\,d\mu; \tag{15.09}$$

then $B(\lambda)$ satisfies condition (1) of the theorem with $k+h$ instead of h in virtue of (15.05). In consequence of the relation

$$\int_0^\infty dB(\lambda)\,\frac{e^{i\lambda x}}{(i\lambda)^{l+2}} = \int_0^\infty A_k(\lambda)\,\frac{e^{i\lambda x}}{(i\lambda)^{l+2}}\,d\lambda,$$

condition (2), and Lemma 14, condition (2) of Theorem 10 is also fulfilled. From the result of this theorem (where we write $m+n-k+1$

instead of k, and $k+h$ instead of h) we get

$$(15.10)\quad \frac{1}{\omega^m}\int_0^\omega A_k(\lambda)\,e^{i\lambda x}(\omega-\lambda)^{m+n-k-1}\,d\lambda - \frac{\Gamma(m+n-k)}{\pi\omega^m}\int_0^\omega t^{-(m+n-k)}\,C_{m+n-k}(\omega t)[\phi(x+t)+\phi(x-t)]\,dt = o(\omega^{h+k-m})+O(\omega^{-1}),$$

where

$$(15.11)\qquad \phi(x)=\frac{d^{l+2}}{dx^{l+2}}\int_0^\infty A_k(\lambda)\,\frac{e^{i\lambda x}}{(i\lambda)^{l+2}}\,d\lambda.$$

Hence, by (15.07),

$$(15.12)\quad \int_0^\omega dA(\lambda)e^{i\lambda x}\left(1-\frac{\lambda}{\omega}\right)^m = \frac{(-1)^{m+1}}{\pi\omega^m}\,\Gamma(m+1)\sum_{n-1}^{k+1}\binom{k+1}{n}(ix)^n(-1)^{k+1-n} \times\int_0^\epsilon t^{-(m+n-k)}\,C_{m+n-k}(\omega t)[\phi(x+t)+\phi(x-t)]dt+o(\omega^{h+k-m})+O(\omega^{-1}).$$

Now we can show that

$$(15.13)\quad D=\frac{1}{\omega^{p+1}}\int_0^\epsilon t^{-(1+p)}\,C_{1+p}(\omega t)\,[\phi(x+t)+\phi(x-t)]\,dt=o(1),$$

uniformly for x in $(a+\epsilon,\ b-\epsilon)$.

In fact, by Lemma 14, $\phi(x)$ is an integrable function in (a, b). Now use (3.3) for $\omega t>\omega\delta$, where δ is any positive number less than ϵ, and use (3.4) for $\omega t\leqslant\omega\delta$. Then there is a constant C_0 such that

$$(15.14)\qquad |D|\leqslant C_0\int_0^\delta|\phi(x+t)+\phi(x-t)|\,dt+O(\omega^{-2}).$$

Since δ is arbitrarily small, it is easy to deduce (15.13) from (15.14). This result simplifies (15.12) considerably and we get

$$(15.15)\quad \int_0^\omega dA(\lambda)\,e^{i\lambda x}\left(1-\frac{\lambda}{\omega}\right)^m = (-1)^{m+1}\,\frac{\Gamma(m+1)}{\pi\omega^m}\,(ix)^{k+1} \times\int_0^\epsilon t^{-(1+m)}\,C_{1+m}(\omega t)[\phi(x+t)+\phi(x-t)]\,dt+o(\omega^{h+k-m})+o(1).$$

We now show that, for

$$\Phi(x) = (-1)^m (ix)^{k+1} \phi(x), \tag{15.16}$$

we have

$$E = (-1)^m \frac{\Gamma(m+1)}{\pi\omega^m} (ix)^{k+1} \int_0^\epsilon t^{-(1+m)} C_{1+m}(\omega t)[\phi(x+t)+\phi(x-t)]\,dt \tag{15.17}$$
$$- \frac{\Gamma(m+1)}{\pi\omega^m} \int_0^\epsilon t^{-(1+m)} C_{1+m}(\omega t)[\Phi(x+t)+\Phi(x-t)]\,dt = o(1)$$

uniformly for $x \subset (a+\epsilon, b-\epsilon)$. Indeed, we have

$$|(-1)^m (ix)^m \phi(x+t) - \Phi(x+t)| \leqslant C_1 t \phi(x+t) \tag{15.18}$$

for $x \subset (a+\epsilon, b-\epsilon)$ and $|t| \leqslant \epsilon$. Then

$$|E| \leqslant \frac{C_1 \Gamma(m+1)}{\pi\omega^m} \int_0^\epsilon t^{-m} C_{1+m}(\omega t)[\phi(x+t)+\phi(x-t)]\,dt. \tag{15.19}$$

Now we apply (3.3) and (3.4) to $(\omega t)^{-m-1} C_{1+m}(\omega t)$, the first for $\omega t \geqslant 1$, and the second for $\omega t \leqslant 1$.

$$|E| \leqslant \frac{C_0 C_1 \Gamma(m+1)}{\pi} \Big[\omega^{-1} \int_\delta^\epsilon t^{-1} |\phi(x+t)+\phi(x-t)|\,dt \tag{15.20}$$
$$+ \omega^{-1} \int_{1/\omega}^\delta t^{-1} |\phi(x+t)+\phi(x-t)|\,dt$$
$$+ \omega \int_0^{1/\omega} t\,|\phi(x+t)+\phi(x-t)|\,dt\Big]$$
$$\leqslant \frac{C_0 C_1 \Gamma(m+1)}{\pi} \Big[\omega^{-1} \int_\delta^\epsilon t^{-1} |\phi(x+t)+\phi(x-t)|\,dt$$
$$+ \int_0^\delta |\phi(x+t)+\phi(x-t)|\,dt + \int_0^{1/\omega} |\phi(x+t)+\phi(x-t)|\,dt\Big].$$

Since δ can be arbitrarily small we obtain (15.17). This enables us to write (15.15) in the form

$$\int_0^\omega dA(\lambda)\, e^{i\lambda x} \Big(1 - \frac{\lambda}{\omega}\Big)^m - \frac{\Gamma(m+1)}{\pi\omega^m} \tag{15.21}$$
$$\times \int_0^\epsilon t^{-(1+m)} C_{1+m}(\omega t)[\Phi(x+t)+\Phi(x-t)]\,dt = o(\omega^{h+k-m}) + o(1).$$

If we write

$$\Phi_n(x) = \Big(\int_0^x dx\Big)^n \Phi(x) \quad [c \subset (a+\epsilon, b-\epsilon)], \tag{15.22}$$

we get, by integration of (15.21),

$$(15.23)\quad \int_0^{\omega} \frac{e^{i\lambda x}}{(i\lambda)^{l+2}} \left(1-\frac{\lambda}{\omega}\right)^m dA(\lambda) - \frac{\Gamma(m+1)}{\pi\omega^m} \int_0^{\epsilon} t^{-(1+m)} C_{1+m}(\omega t) \times [\Phi_{l+2}(x+t)+\Phi_{l+2}(x-t)]\,dt + P_{l+1}(x, \omega) = o(1)+o(\omega^{k+h-m}).$$

Since $\phi(x)$, by (15.11), and hence also $\Phi(x)$, by (15.16), is independent of m, we may take $h+k=m$, and then the second member of (15.23) is $o(1)$. Moreover, $P_{l+1}(x, \omega)$ denotes a polynomial in x, of degree $l+1$, whose coefficients are functions of ω. Since $m \geqslant k$, it follows from Lemma 13 that the first integral in (15.23) converges to $F(x)$ uniformly in any finite interval. But, since $\Phi_{l+2}(x)$ is an integral, the second expression in (15.23) converges to $\Phi_{l+2}(x)$, uniformly for $x \subset (a+\epsilon, b-\epsilon)$. But then $P_{l+1}(x, \omega)$ must also have a limit, which must be also a polynomial of degree $l+1$: $P_{l+1}(x)$. We have already remarked that the limit of the second member is zero. Therefore

$$F(x)-\Phi_{l+2}(x)+P_{l+1}(x)=0,$$

so that

$$f(x)-\Phi(x)=0,$$

almost everywhere in (a, b).

But then (15.21) is identical with (15.04), and so the theorem is proved.

Prague, XII, Lucemburská, 11/V.
Czechoslovakia.

SOME RELATIONS BETWEEN DIFFERENT KINDS OF RIEMANN SUMMABILITY

By B. KUTTNER.

[Received 17 June, 1935.—Read 20 June, 1935.]

1. *Introduction.*

We shall say that the series

$$\sum_{n=1}^{\infty} a_n \tag{1}$$

is summable (R, p) to s, where p is a positive integer, if

$$\sum_{n=1}^{\infty} a_n \left(\frac{\sin nh}{nh}\right)^p \tag{2}$$

converges for all values of h in some neighbourhood of the origin, and if its sum tends to s as $h \to 0$. We shall say that the series (1) is summable (R', p) to s if

$$h \sum_{n=1}^{\infty} s_n \left(\frac{\sin nh}{nh}\right)^p \tag{3}$$

converges for all values of h in some neighbourhood of the origin, and if its sum tends to $A_p s$ as $h \to 0$, where*

$$s_n = \sum_{m=1}^{n} a_m$$

and where

$$A_p = \lim_{h \to 0} h \sum_{n=1}^{\infty} \left(\frac{\sin nh}{nh}\right)^p.$$

A result due to Riemann† can be stated in the form that a series which converges to 0 is summable $(R', 2)$ to 0. Summability $(R', 2)$ has also been considered by Wiener‡.

---

* s_n will be given this meaning throughout this paper.

† See, *e.g.*, Hobson, **2**, 648 [Theorem I (2)].

‡ Wiener, **12**.

We shall say that the series (1) is approximately summable (R, p) to s if there is a set of points E having unit density at the origin such that the series (2) converges at all points of E, and such that its sum tends to s as $h \to 0$ through points of E. We define approximate summability (R', p) in the obvious way. The definition of approximate summability (R, p) and (R', p) is due to Rajchman and Zygmund*.

Several papers have been written dealing with the relations between Riemann and Cesàro summability†. In most of these, merely summability (R, p) has been considered, but only a slight modification of the arguments is sufficient to obtain corresponding results for summability (R', p). Some results on the relations between different kinds of Riemann summability have been obtained by Verblunsky‡. He has shown that, if $a_n = o(n^3)$, and if

$$\sum_{n=1}^{\infty} a_n \left(\frac{\sin nh}{nh}\right)^2 \tag{4}$$

is Abel summable to $\phi(h)$ for h in some neighbourhood of the origin, where $\phi(h) \to s$ as $h \to 0$, then $\sum_{n=1}^{\infty} a_n$ is summable (R, p) to s for any $p > 2$. This evidently includes the result that a series summable $(R, 2)$ is summable (R, p) for $p > 2$. He has also obtained a similar theorem which includes the result that a series summable $(R, 1)$ is summable (R, p) for $p > 1$. The corresponding results for summability (R', p) can be obtained by methods similar to Verblunsky's; thus a series summable $(R', 2)$ is summable (R', p) for $p > 2$, and a series summable $(R', 1)$ is summable (R', p) for $p > 1$.

In this paper we prove the following theorems.

A series summable $(R, 2)$ *is approximately summable* $(R', 2)$ *to the same sum.*

A series summable $(R, 2)$ *is summable* (R', p) *to the same sum if* $p > 2$.

A series summable $(R', 2)$ *is approximately summable* $(R, 2)$ *to the same sum.*

A series summable $(R', 2)$ *is summable* (R, p) *to the same sum if* $p > 2$.

It can also be proved that a series summable $(R, 1)$ is approximately summable $(R', 1)$, and is summable (R', p) for $p > 1$; also that a series summable $(R', 1)$ is approximately summable $(R, 1)$, and is summable

---

* Rajchman and Zygmund, **10**.

† Kuttner, **4**, and references there given.

‡ Verblunsky, **11**.

(R, p) for $p > 1$. The proofs of these results are, apart from one difference which will be indicated in § 4, precisely similar to those of the four theorems of this paper.

It should be remarked that it is not true that a series summable $(R, 2)$ is summable $(R', 2)$; nor is it true that a series summable $(R', 2)$ is summable $(R, 2)$. Examples to demonstrate this have been obtained by Marcinkiewicz*; somewhat simpler examples are given in § 5 of this paper. Examples are also given in § 5 to show that results corresponding to the theorems of this paper do not hold for summability $(R, 3)$ or $(R', 3)$.

Throughout this paper, c will be used to indicate a constant; it may be a different constant at each occurrence. We shall write

$$\int_0^A [F(h+x)-F(h-x)] \cot \tfrac{1}{2}x\, dx$$

for $$\lim_{\epsilon \to 0} \int_\epsilon^A [F(h+x)-F(h-x)] \cot \tfrac{1}{2}x\, dx;$$

this limit exists almost everywhere if $F(x)$ is integrable†.

2. *Preliminary lemmas.*

LEMMA 1. *Suppose that $[F(x)]^2$ is integrable in $(-2A, 2A)$, where $A \leqslant \pi$. Then*

$$\int_{-A}^A dh \left\{ \int_0^A [F(h+x)-F(h-x)] \cot \tfrac{1}{2}x\, dx \right\}^2 \leqslant K \int_{-2A}^{2A} [F(x)]^2 dx,$$

where K is an absolute constant.

This follows at once from a known theorem‡.

COROLLARY. *If $[F(x)]^2$ is integrable in $(C-A, D+A)$, then*

$$\int_C^D dh \left\{ \int_0^A [F(h+x)-F(h-x)] \cot \tfrac{1}{2}x\, dx \right\}^2 \leqslant 2K \int_{C-A}^{D+A} [F(x)]^2 dx.$$

LEMMA 2. *Suppose that $F(h)$ is an integrable function which is $o(h^k)$, where $k > -\frac{1}{2}$, for small h. Let*

$$\phi(h) = \int_0^h [F(h+x)-F(h-x)] \cot \tfrac{1}{2}x\, dx.$$

---

* Marcinkiewicz, 7.

† Plessner, 8.

‡ Besikovitch, 1; Lusin, 6.

Then

(i) *At a set of points having unit density at the origin,* $\phi(h) = o(h^k)$.

(ii) *If* $K(h)$ *is any function such that* $K(h+x)-K(h) = O(x^\lambda)$ *uniformly in* h, *where* $\lambda > 0$,

$$\int_0^h \phi(t)\, K(t)\, dt = o(h^{k+1}).$$

(i) Let

$$\phi_H(h) = \int_0^H [F(h+x)-F(h-x)] \cot \tfrac{1}{2}x\, dx.$$

Then, by Lemma 1,

$$\int_{-H}^{H} [\phi_H(h)]^2\, dh \leqslant K \int_{-2H}^{2H} [F(h)]^2\, dh$$
$$\leqslant [\eta(H)]^3 H^{2k+1},$$

where $\eta(H) \to 0$ as $H \to 0$. It follows that the set of points in $(\frac{1}{2}H, H)$ at which $|\phi_H(h)| \geqslant H^k \eta(H)$ has measure less than or equal to $H\eta(H)$. But it is evident that, for h in $(\frac{1}{2}H, H)$, $\phi(h)-\phi_H(h) = o(H^k)$, and the result follows.

(ii) It follows from Lemma 1, Corollary, that, for fixed h,

$$\int_\epsilon^h dt \left\{ \int_0^\epsilon [F(t+x)-F(t-x)] \cot \tfrac{1}{2}x\, dx \right\}^2$$

is bounded. Hence*

$$(5) \qquad \int_0^h \phi(t)\, K(t)\, dt = \lim_{\epsilon \to 0} \int_\epsilon^h K(t)\, dt \int_\epsilon^t [F(t+x)-F(t-x)] \cot \tfrac{1}{2}x\, dx.$$

It is therefore sufficient to show that the integral on the right of (5) is $o(h^{k+1})$ uniformly in ϵ. This integral is equal to

$$\int_\epsilon^h \cot \tfrac{1}{2}x\, dx \int_x^h K(t)\, [F(t+x)-F(t-x)]\, dt$$
$$= \int_\epsilon^h \cot \tfrac{1}{2}x\, dx \left\{ -\int_0^{2x} K(t+x)\, F(t)\, dt + \int_{2x}^{h-x} F(t)[K(t-x)-K(t+x)]\, dt \right.$$
$$\left. + \int_{h-x}^{h+x} K(t-x)\, F(t)\, dt \right\},$$

whence the result follows†.

---

* Hobson, 2, 301.

† We use the results that $K(t)$ is bounded, and that $K(t-x)-K(t+x) = O(x^\lambda)$.

LEMMA 3. *Suppose that the series*

$$\sum_{n=1}^{\infty} \alpha_n \cos nh + \beta_n \sin nh$$

converges to an integrable function $f(h)$ for h in (α, β). Then the series

$$\sum_{n=1}^{\infty} \frac{1}{n} (\alpha_n \sin nh - \beta_n \cos nh)$$

is the Fourier series of a function which is, in (α, β), an indefinite integral of $f(h)$.

This result is included in a more general theorem given by Hobson*. It can be deduced at once from a result due to Zygmund†.

LEMMA 4. *Suppose that $G(h)$ is an integrable function which is, in $(0, \eta)$, an indefinite integral of a function $F(h)$. Suppose that, for small h, $F(h) = o(h^k)$, where $k > -1$, and that $G(0) = 0$. Let the Fourier series of $G(h)$ be*

$$\tfrac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos nh + \beta_n \sin nh).$$

Then the series

$$\sum_{n=1}^{\infty} n(\alpha_n \cos nh + \beta_n \sin nh), \tag{6}$$

i.e. the series obtained by differentiating term by term the series allied with the Fourier series of $G(h)$, is summable $(C, 1)$ to

$$-\frac{1}{4\pi} \int_{2h}^{\pi} [G(x) \operatorname{cosec}^2 \tfrac{1}{2}(x-h) + G(-x) \operatorname{cosec}^2 \tfrac{1}{2}(x+h)]\, dx$$

$$-\frac{1}{2\pi} \int_{0}^{h} [F(h+x) - F(h-x)] \cot \tfrac{1}{2}x\, dx + o(h^k)$$

almost everywhere in $(0, \frac{1}{2}\eta)$.

The n-th Cesàro mean of order 1 of the series (6) is

$$\frac{1}{\pi} \int_{-\pi}^{\pi} G(x) \frac{d}{dh} \left[-\tfrac{1}{2} \cot \tfrac{1}{2}(x-h) + \frac{1}{4n} \frac{\sin n(x-h)}{\sin^2 \frac{1}{2}(x-h)} \right] dx.$$

---

* Hobson, 3.

† Zygmund, 13, 93.

It follows from the Riemann-Lebesgue theorem that the portion of the integral outside $(0, 2h)$ tends to

$$-\frac{1}{4\pi}\left\{\int_{2h}^{\pi}+\int_{-\pi}^{0}\right\} G(x)\operatorname{cosec}^2\tfrac{1}{2}(x-h)\,dx$$

$$=-\frac{1}{4\pi}\int_{2h}^{\pi}[G(x)\operatorname{cosec}^2\tfrac{1}{2}(x-h)+G(-x)\operatorname{cosec}^2\tfrac{1}{2}(x+h)]\,dx+o(h^k)$$

as $n\to\infty$. In order to deal with the integral from 0 to $2h$, we may, if $0\leqslant h\leqslant\frac{1}{2}\eta$, integrate by parts. We then obtain

$$(7)\quad \frac{1}{\pi}\left[G(x)\left\{\tfrac{1}{2}\cot\tfrac{1}{2}(x-h)-\frac{1}{4n}\frac{\sin n(x-h)}{\sin^2\frac{1}{2}(x-h)}\right\}\right]_0^{2h}$$

$$-\frac{1}{\pi}\int_0^{2h}F(x)\left\{\tfrac{1}{2}\cot\tfrac{1}{2}(x-h)-\frac{1}{4n}\frac{\sin n(x-h)}{\sin^2\frac{1}{2}(x-h)}\right\}dx.$$

For almost all h, the integral occurring in (7) tends to*

$$-\frac{1}{2\pi}\int_0^h[F(h+x)-F(h-x)]\cot\tfrac{1}{2}x\,dx$$

as $n\to\infty$, and the lemma follows.

LEMMA 5. *Suppose that the conditions of Lemma 4 are satisfied, and that the series* (6) *is summable* $(C, 1)$ *to* $K(h)$ *almost everywhere in* $(0, \frac{1}{2}\eta)$. *Then, for* h *in* $(0, \frac{1}{2}\eta)$, *the series*

$$(8)\qquad \sum_{n=1}^{\infty}(\alpha_n\sin nh-\beta_n\cos nh)$$

converges to

$$c+\int_0^h K(t)+o(h^{k+1}).$$

For h in $(0, \eta)$, the series (8) converges† to

$$-\frac{1}{2\pi}\int_0^{\pi}[G(h+x)-G(h-x)]\cot\tfrac{1}{2}x\,dx$$

$$=-\frac{1}{2\pi}\int_{2h}^{\pi}[G(x)\cot\tfrac{1}{2}(x-h)-G(-x)\cot\tfrac{1}{2}(x+h)]\,dx+o(h^{k+1}).$$

* Plessner, 8.

† See, *e.g.*, Hobson, 2, 693.

But, by Lemma 4,

$$K(h) = -\frac{1}{4\pi}\int_{2h}^{\pi}[G(x)\operatorname{cosec}^2\tfrac{1}{2}(x-h)+G(-x)\operatorname{cosec}^2\tfrac{1}{2}(x+h)]\,dx$$
$$-\frac{1}{2\pi}\int_0^h[F(h+x)-F(h-x)]\cot\tfrac{1}{2}x\,dx+o(h^k)$$

almost everywhere. Using Lemma 2 (ii), we see that

$$\int_0^h K(t)\,dt = c-\frac{1}{2\pi}\int_{2h}^{\pi}[G(x)\cot\tfrac{1}{2}(x-h)-G(-x)\cot\tfrac{1}{2}(x+h)]\,dx+o(k^{k+1}),$$

and the lemma follows.

It may be noted that in the special case in which $G(h)$ is everywhere the indefinite integral of $F(h)$, the series (6) is the series allied with the Fourier series of $F(h)$. Lemma 4 is then an immediate consequence of the known result* that the allied series is almost everywhere summable $(C, 1)$ to

$$-\frac{1}{2\pi}\int_0^{\pi}[F(h+x)-F(h-x)]\cot\tfrac{1}{2}x\,dx,$$

while Lemma 5 states, in effect, that this result may be integrated term by term.

LEMMA 6. *Suppose that the conditions of Lemma* 4 *are satisfied, and that* $k<3$. *Then*

(i) *If* $G(h)$ *is odd, the series* (6) *is summable* $(C, 1)$ *to*

$$-\frac{h}{2\pi}\int_{2h}^{\pi}G(x)\operatorname{cosec}^2\tfrac{1}{2}x\cot\tfrac{1}{2}x\,dx$$
$$-\frac{1}{2\pi}\int_0^h[F(h+x)-F(h-x)]\cot\tfrac{1}{2}x\,dx+o(h^k)$$

almost everywhere in $(0, \frac{1}{2}\eta)$.

(ii) *If* $G(h)$ *is even, and if* $0<k<2$, *the series* (6) *is summable* $(C, 1)$ *to*

$$-\frac{1}{2\pi}\int_0^h[F(h+x)-F(h-x)]\cot\tfrac{1}{2}x\,dx+c+o(h^k)$$

almost everywhere in $(0, \frac{1}{2}\eta)$.

(iii) *If* $G(h)$ *is odd, the series* (8), *i.e. the series allied with the Fourier series of* $G(x)$, *converges to*

$$c-\frac{h^2}{4\pi}\int_{2h}^{\pi}G(x)\operatorname{cosec}^2\tfrac{1}{2}x\cot\tfrac{1}{2}x\,dx+o(h^{k+1})$$

for h *in* $(0, \frac{1}{2}\eta)$.

* Plessner, 8.

(iv) *If* $G(h)$ *is even, and if* $0<k<2$, *the series* (8) *converges to* $ch+o(h^{k+1})$ *for* h *in* $(0, \frac{1}{2}\eta)$.

Since, for $x \geqslant 2h$,

$$\operatorname{cosec}^2 \tfrac{1}{2}(x-h)-\operatorname{cosec}^2 \tfrac{1}{2}(x+h) = 2h \operatorname{cosec}^2 \tfrac{1}{2}x \cot \tfrac{1}{2}x+O\left(\frac{h^3}{x^5}\right),$$

$$\operatorname{cosec}^2 \tfrac{1}{2}(x-h)+\operatorname{cosec}^2 \tfrac{1}{2}(x+h) = 2 \operatorname{cosec}^2 \tfrac{1}{2}x+O\left(\frac{h^2}{x^4}\right),$$

the results (i) and (ii) follow at once from Lemma 4. The results (iii) and (iv) follow* from (i) and (ii), with the aid of Lemmas 5 and 2 (ii).

COROLLARY. *The results* (iii) *and* (iv) *may be formally integrated any number of times.*

This follows at once with the aid of Lemma 3.

LEMMA 7. *Suppose that* $G(h)$ *is an even function, which is, in the open interval* $(0, 2\eta)$, *the indefinite integral of a function* $F(h)$. *Suppose that, for small* h, $F(h) = o(1/h)$. *Let the Fourier series of* $G(h)$ *be*

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty} a_n \cos nx.$$

Then the series

$$-\sum_{n=1}^{\infty} \frac{a_n}{n} \cos nh$$

converges to $c+o(h)$ *for small* h.

The series

$$\sum_{n=1}^{\infty} a_n \sin nh$$

converges to

$$-\frac{1}{2\pi}\int_0^{\pi} \{G(h+x)-G(h-x)\} \cot \tfrac{1}{2}x\,dx$$

$$= -\frac{1}{2\pi}\int_0^{h} \cot \tfrac{1}{2}x\,dx \int_{h-x}^{h+x} F(z)\,dz-\frac{1}{2\pi}\int_h^{\eta} \cot \tfrac{1}{2}x\,dx \int_{x-h}^{x+h} F(z)\,dz$$

$$+\frac{1}{2\pi}\int_{\eta-h}^{\eta+h} G(x) \cot \tfrac{1}{2}(x+h)\,dx+\frac{1}{2\pi}\int_{\eta+h}^{\pi} G(x)[\cot \tfrac{1}{2}(x+h)-\cot \tfrac{1}{2}(x-h)]\,dx,$$

and it may easily be shown that this expression tends to 0 as h tends to 0. This result may be formally integrated, and the lemma follows.

COROLLARY. *The result of the lemma may be formally integrated.*

---

* Of course, these results may also be obtained directly.

3. *Proof of the theorems.*

THEOREM I. *If the series* $\sum_{n=1}^{\infty} a_n$ *is summable* $(R, 2)$ *to* s, *then it is approximately summable* $(R', 2)$ *to* s.

The series $$\sum_{n=1}^{\infty} \frac{a_n}{n^2} \sin^2 nh$$

converges in some neighbourhood of the origin, say $(-\frac{1}{2}\eta, \frac{1}{2}\eta)$. It follows* that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} \tag{9}$$

converges. We may evidently suppose, without loss of generality, that the sum of the series (9) is zero, and that $s = 0$. We deduce that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nh \tag{10}$$

converges in $(-\eta, \eta)$ to $f(h)$, say, where $f(h) = o(h^2)$. It follows, by Lemma 3, that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^3} \sin nh$$

is the Fourier series of a function $g(h)$, say, which is, in $(-\eta, \eta)$, an indefinite integral of $f(h)$. Thus

$$a_n = \frac{n^3}{\pi} \int_{-\pi}^{\pi} g(x) \sin nx \, dx. \tag{11}$$

Now $$\sum_{n=1}^{N} \frac{s_n}{n^2} \cos nh = \sum_{n=1}^{N} a_n \psi(n) - s_N \psi(N+1),$$

where

$$\psi(n) = \sum_{m=n}^{\infty} \frac{\cos mh}{m^2} = \tfrac{1}{2} \frac{\cos nh}{n^2} - \tfrac{1}{2} \cot \tfrac{1}{2}h \frac{\sin nh}{n^2} + \tfrac{1}{2} \operatorname{cosec}^2 \tfrac{1}{2}h \frac{\cos nh}{n^3} + O\left(\frac{1}{n^4}\right)$$

for fixed h. Since the series (9) converges, it follows that $s_n = o(n^2)$, and hence that the series

$$\sum_{n=1}^{\infty} \frac{s_n}{n^2} \cos nh \tag{12}$$

* Kuttner, 4, 276 (Lemma 2).

converges whenever

$$\sum_{n=1}^{\infty} a_n \left\{ \tfrac{1}{2} \frac{\cos nh}{n^2} - \tfrac{1}{2} \cot \tfrac{1}{2}h \frac{\sin nh}{n^2} + \tfrac{1}{2} \operatorname{cosec}^2 \tfrac{1}{2}h \frac{\cos nh}{n^3} \right\}$$

does so. Since the series (10) converges in $(-\eta, \eta)$, it follows that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^3} \cos nh$$

does so; it also follows* that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} \sin nh,$$

and hence the series (12), converges at almost all those points in $(-\eta, \eta)$ at which it is summable $(C, 1)$. It is therefore sufficient to consider the sum $(C, 1)$ of the series (12).

In the rest of the proof, equations involving the sums of infinite series refer to the sums $(C, 1)$, and hold at a set of points having unit density at the origin.

It follows from (11) that

$$s_n = \tfrac{1}{2}a_n + \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \left\{ -\tfrac{1}{2}n^3 \cot \tfrac{1}{2}x \cos nx + \tfrac{3}{4}n^2 \frac{\sin nx}{\sin^2 \tfrac{1}{2}x} \right.$$

$$\left. + \tfrac{3}{4}n \frac{\cos \tfrac{1}{2}x}{\sin^3 \tfrac{1}{2}x} \cos nx - \frac{(2+\cos x)}{8 \sin^4 \tfrac{1}{2}x} \sin nx \right\} dx$$

$$= \tfrac{1}{2}a_n + s_n^{\,1} + s_n^{\,2} + s_n^{\,3} + s_n^{\,4} \quad \text{(say)}.$$

We have already seen that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nh = o(h^2).$$

By Lemmas 6 (ii) and 2 (i),

$$\sum_{n=1}^{\infty} \frac{s_n^{\,1}}{n^2} \cos nh = c + o(h).$$

By Lemma 6 (iii),

$$\sum_{n=1}^{\infty} \frac{s_n^{\,2}}{n^2} \cos nh = c + o(h).$$

By Lemma 7,

$$\sum_{n=1}^{\infty} \frac{s_n^{\,3}}{n^2} \cos nh = c + o(h).$$

* Kuttner, 5.

Finally, integrating by parts, we have

$$\sum_{n=1}^{\infty} \frac{s_n{}^4}{n^2} \cos nh = -\frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos nh \int_0^{\pi} \left\{ \int_x^{\pi} \frac{g(y)(2+\cos y)}{\sin^4 \frac{1}{2}y} \, dy \right\} \cos nx \, dx$$

$$= c+o(h) \quad \text{by Lemma 7.}$$

We deduce that $$\sum_{n=1}^{\infty} \frac{s_n}{n^2} \cos nh = c+o(h),$$

and the theorem follows.

THEOREM II. *If the series* $\sum_{n=1}^{\infty} a_n$ *is summable* $(R, 2)$ *to* s, *then it is summable* (R', p) *to* s *for any* $p \geqslant 3$.

We make the same simplifications, and use the same notation, as in Theorem I, and thus have

$$s_n = \tfrac{1}{2}a_n + s_n{}^1 + s_n{}^2 + s_n{}^3 + s_n{}^4.$$

Using Lemma 6 (iv), Lemma 6, Corollary, and Lemma 7, Corollary, we can show, as in Theorem I, that

$$\sum_{n=1}^{\infty} \frac{s_n}{n^3} \sin nh = ch+o(h^2).$$

This result may evidently be formally integrated any number of times, whence we deduce that, if p is odd and greater than or equal to 3,

$$\sum_{n=1}^{\infty} \frac{s_n}{n^p} \sin nh = ch+ch^3+\ldots+ch^{p-2}+o(h^{p-1}),$$

and that, if p is even and greater than or equal to 4,

$$\sum_{n=1}^{\infty} \frac{s_n}{n^p} \cos nh = c+ch^2+\ldots+ch^{p-2}+o(h^{p-1}).$$

The theorem follows.

THEOREM III. *If the series* $\sum_{n=1}^{\infty} a_n$ *is summable* $(R', 2)$ *to* s, *then it is approximately summable* $(R, 2)$ *to* s.

As in the proof of Theorem I, we can show that

(13) $$\sum_{n=1}^{\infty} \frac{s_n}{n^2} \cos nh$$

converges in an interval $(-\eta, \eta)$, say, to a function $f(h)$. We may suppose,

without loss of generality, that $s = 0$, and that the sum of the series (13) for $h = 0$ is zero, so that $f(h) = o(h)$. By Lemma 3,

$$\sum_{n=1}^{\infty} \frac{s_n}{n^3} \sin nh$$

is the Fourier series of a function $g(h)$, say, which is, in $(-\eta, \eta)$, the indefinite integral of $f(h)$.

Now,

$$\sum_{n=1}^{N} \frac{a_n}{n^2} \cos nh = 2 \sin^2 \tfrac{1}{2}h \sum_{n=1}^{N} \frac{s_n}{n^2} \cos nh + \sin h \sum_{n=1}^{N} \frac{s_n}{n^2} \sin nh$$

$$+ \sum_{n=1}^{N} \frac{(2n+1)s_n}{n^2(n+1)^2} \cos(n+1)h + \frac{s_N}{(N+1)^2} \cos(N+1)h.$$

We can thus show, as in Theorem I, that it is sufficient to consider the sum $(C, 1)$ of the series

$$(14) \quad 2 \sin^2 \tfrac{1}{2}h \sum_{n=1}^{\infty} \frac{s_n}{n^2} \cos nh + \sin h \sum_{n=1}^{\infty} \frac{s_n}{n^2} \sin nh + \sum_{n=1}^{\infty} \frac{(2n+1)s_n}{n^2(n+1)^2} \cos(n+1)h.$$

In the rest of the proof, equations involving the sums of infinite series refer to the sums $(C, 1)$. Certain equations hold only at a set of points having unit density at the origin; such equations will be indicated by appending the letter (D) to them. The remaining equations hold almost everywhere in some neighbourhood of the origin.

We have already seen that

$$(15) \qquad \sum_{n=1}^{\infty} \frac{s_n}{n^2} \cos nh = f(h) = o(h).$$

It follows that

$$\sum_{n=1}^{\infty} \frac{s_n}{n^2} [\sin(n+1)h - \sin(n-1)h] = 2f(h) \sin h = o(h^2).$$

This equation may be integrated term by term, whence

$$(16) \qquad \sum_{n=1}^{\infty} \frac{s_n}{n^2} \left[\frac{\cos(n+1)h}{n+1} - \frac{\cos(n-1)h}{n-1} \right] = c + o(h^3).$$

By Lemmas 6 (i) and 2 (i),

$$(17) \quad \sum_{n=1}^{\infty} \frac{s_n}{n^2} \sin nh = h\psi(h) - \frac{1}{2\pi} \int_0^h [f(h+x) - f(h-x)] \cot \tfrac{1}{2}x \, dx + o(h)$$

$$= h\psi(h) + o(h) \quad (D),$$

where $$\psi(h) = -\frac{1}{2\pi}\int_{2h}^{\pi} g(x)\operatorname{cosec}^2 \tfrac{1}{2}x \cot \tfrac{1}{2}x\, dx.$$

Hence

$$\sum_{n=1}^{\infty} \frac{s_n}{n^2}[\sin(n+1)h + \sin(n-1)h] = 2h\psi(h) - \frac{1}{\pi}\cos h \int_0^h [f(h+x)-f(h-x)] \cot \tfrac{1}{2}x\, dx + o(h).$$

Hence, by Lemmas 5 and 2 (ii),

$$\sum_{n=1}^{\infty} \frac{s_n}{n^2}\left[\frac{\cos(n+1)h}{n+1} + \frac{\cos(n-1)h}{n-1}\right] = c - h^2\psi(h) + o(h^2). \tag{18}$$

In a similar manner, by carrying out two term by term integrations, we can show that

$$\sum_{n=1}^{\infty} \frac{s_n}{n^2}\left[\frac{\cos(n+1)h}{(n+1)^2} - \frac{\cos(n-1)h}{(n-1)^2}\right] = c + o(h^3); \tag{19}$$

$$\sum_{n=1}^{\infty} \frac{s_n}{n^2}\left[\frac{\cos(n+1)h}{(n+1)^2} + \frac{\cos(n-1)h}{(n-1)^2}\right] = c + o(h^3). \tag{20}$$

From equations (15), (16), (17), (18), (19), and (20) it follows that the sum of the series (14) is $c+o(h^2)$ (D), and the theorem follows.

THEOREM IV. *If the series* $\sum_{n=1}^{\infty} a_n$ *is summable* $(R', 2)$ *to* s, *then it is summable* (R, p) *to* s *for any* $p \geqslant 3$.

We make the same simplifications, and use the same notation, as in Theorem III. Thus, for small h,

$$\sum_{n=1}^{\infty} \frac{s_n}{n^3}\sin nh = g(h) = o(h^2), \tag{21}$$

and, by Lemma 6 (iii),

$$-\sum_{n=1}^{\infty} \frac{s_n}{n^3}\cos nh = A + \tfrac{1}{2}h^2\psi(h) + o(h^2), \tag{22}$$

where A is a constant. We deduce that

$$\sum_{n=1}^{\infty} \frac{s_n}{n^3}\cos(n+1)h = -A - \tfrac{1}{2}\{\psi(h)-A\}h^2 + o(h^2). \tag{23}$$

The series (23) is a Fourier series, since its coefficients are $o(1/n)$. We may therefore integrate term by term, whence

$$\sum_{n=1}^{\infty} \frac{s_n}{n^3} \frac{\sin(n+1)h}{n+1} = -Ah - \tfrac{1}{6}\{\psi(h) - A\}h^3 + o(h^3) \tag{24}$$

and

$$\sum_{n=1}^{\infty} \frac{s_n}{n^3} \frac{\sin(n+1)h}{(n+1)^3} = ch + \tfrac{1}{6}Ah^3 + o(h^3). \tag{25}$$

Again, from (21) and (22),

$$\sum_{n=1}^{\infty} \frac{s_n}{n^3} \sin(n+1)h = -Ah + o(h^2),$$

and, integrating term by term twice, we have

$$\sum_{n=1}^{\infty} \frac{s_n}{n^3} \frac{\sin(n+1)h}{(n+1)^2} = ch + \tfrac{1}{6}Ah^3 + o(h^3). \tag{26}$$

From (21), (22), (24), (25), and (26) we deduce that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{a_n}{n^3} \sin nh = \sum_{n=1}^{\infty} \frac{s_n}{n^3} \Big\{ & 2\sin^2 \tfrac{1}{2}h \sin nh - \sin h \cos nh \\ & + \sin(n+1)h \left[\frac{3}{(n+1)} - \frac{3}{(n+1)^2} + \frac{1}{(n+1)^3} \right] \Big\} \\ = ch + o(h^3).\end{aligned}$$

It follows at once that $\sum_{n=1}^{\infty} a_n$ is summable $(R, 3)$, and summability (R, p) for $p > 3$ can be deduced in the same way as in Theorem II.

4. *Note on summability* $(R, 1)$.

In the course of the proof of Theorem I, we required the result that $s_n = o(n^2)$. Similarly, in order to prove that a series summable $(R, 1)$ is approximately summable $(R', 1)$, we require to show that the hypothesis implies $s_n = o(n)$. Since it is not necessarily true that

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

is convergent, we must find some other method of proof.

Using the theorems on the formal multiplication of trigonometric series due to Rajchman and Zygmund*, we can show that† it is sufficient to

* See Rajchman, **9**, or Zygmund, **13**.

† As in Kuttner, **4**.

prove the result in two special cases; the first is the case in which the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \sin nh \tag{27}$$

is a Fourier series, and the second that in which the series (27) converges to 0 for all values of h in some neighbourhood of the origin. The result in the first case is almost trivial; in the second case it follows from a result due to Zygmund*.

5. *Negative results.*

We now obtain an example of a series summable $(R', 2)$ but not summable $(R, 2)$.

Let $\{h_m\}$ be a sequence of numbers decreasing to zero, and let $\sum_{m=1}^{\infty} \eta_m$ be an absolutely convergent series. Let

$$s_n = n \sum_{m=1}^{\infty} \eta_m h_m \sin nh_m,$$

so that

$$a_n = 2n \sum_{m=1}^{\infty} \eta_m h_m \sin \tfrac{1}{2}h_m \cos(n-\tfrac{1}{2})h_m + \sum_{m=1}^{\infty} \eta_m h_m \sin(n-1)h_m.$$

Then

$$\sum_{n=1}^{\infty} \frac{s_n}{n^2} \cos nh = \sum_{m=1}^{\infty} \eta_m h_m \sum_{n=1}^{\infty} \frac{1}{n} \sin nh_m \cos nh$$

$$= \tfrac{1}{2} \sum_{m=1}^{\infty} (\pi - h_m)\eta_m h_m - \tfrac{1}{2}\pi \sum_{h_m \leqslant h}' \eta_m h_m,$$

the dash indicating that the term $h_m = h$, if it occurs, is multiplied by $\frac{1}{2}$. Since

$$\sum_{h_m \leqslant h}' \eta_m h_m = o(h),$$

it follows that the series $\sum_{n=1}^{\infty} a_n$ is summable $(R', 2)$ to 0. But it can easily be verified that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} \sin^2 nh$$

diverges for $h = \frac{1}{2}h_m$; thus $\sum_{n=1}^{\infty} a_n$ is not summable $(R, 2)$.

---

* Zygmund, 13, 87 [Théorème I (2)].

Similarly, by taking

$$s_n = \sum_{m=1}^{\infty} \eta_m [n \sin \tfrac{1}{2} h_m \cos(n+\tfrac{1}{2}) h_m - \tfrac{1}{2} \sin n h_m],$$

so that

$$a_n = -2n \sum_{m=1}^{\infty} \eta_m \sin^2 \tfrac{1}{2} h_m \sin n h_m,$$

we obtain an example of a series summable $(R, 2)$ but not summable $(R', 2)$.

We next give examples to show that the analogues of Theorems I and III for summability $(R, 3)$ and $(R', 3)$ are false. Let $(\delta)_n$ denote

$$\delta(\delta+1) \ldots (\delta+n-1),$$

and let A denote $2\pi/\log 3$. We note that, if $\Re \delta < 1$,

$$\sum_{n=1}^{\infty} \frac{(\delta)_n}{n!} \sin n h = \sin \tfrac{1}{2}\pi\delta \; h^{-\delta} + ch^{1-\delta} + ch^{2-\delta} + O(h^{3-\delta}),$$

and that

$$\sum_{n=1}^{\infty} \frac{\sin n h}{n^3} \left\{ \frac{n^3 (\delta)_n}{n!} - \frac{(n-1)^3 (\delta)_{n-1}}{(n-1)!} \right\}$$

$$= ch - \frac{(\delta+2)}{(\delta-1)} \cos \tfrac{1}{2}\pi\delta \; h^{1-\delta} + ch^{2-\delta} + ch^{3-\delta} + O(h^{4-\delta}).$$

Now let

$$s_n = \frac{n^3}{n!} \{(Ai-1)_n + \alpha (Ai-2)_n\},$$

where α is a constant. Then

$$\sum_{n=1}^{\infty} \frac{s_n}{n^3} \sin n h = ch^{-Ai+1} + c_1 h^{-Ai+2} + O(h^3),$$

and

$$\sum_{n=1}^{\infty} \frac{a_n}{n^3} \sin n h = ch + c_2 h^{-Ai+2} + ch^{-Ai+3} + O(h^4),$$

where c_1 and c_2 are constants, and where $c_2 \neq 0$. By suitable choice of α, we can make $c_1 = 0$. Since

$$(3h)^{-Ai+1} = 3h^{-Ai+1},$$

we see that $\sum_{n=1}^{\infty} a_n$ is then summable $(R', 3)$. It is not, however, approximately summable $(R, 3)$.

Again, if we put

$$s_n = \frac{n^3}{n!} \{(Ai)_n + \alpha (Ai-1)_n + \beta (Ai-2)_n\},$$

where α and β are constants, we have

$$\sum_{n=1}^{\infty} \frac{s_n}{n^3} \sin nh = c_1 h^{-Ai} + ch^{-Ai+1} + ch^{-Ai+2} + O(h^3),$$

and

$$\sum_{n=1}^{\infty} \frac{a_n}{n^3} \sin nh = ch + ch^{-Ai+1} + c_2 h^{-Ai+2} + c_3 h^{-Ai+3} + O(h^4),$$

where c_1, c_2, and c_3 are constants, and where $c_1 \neq 0$. By suitable choice of α and β we can make $c_2 = c_3 = 0$. We then have an example of a series summable $(R, 3)$, but not approximately summable $(R', 3)$.

The same two examples show that the analogues of Theorems II and IV are false.

References.

1. A. S. Besikovitch, " Sur la nature des fonctions à carré sommable et des ensembles mesurables ", *Fundamenta Math.*, 4 (1923), 172–195.
2. E. W. Hobson, *Functions of a real variable*, 2 (1926), 2 ed.
3. ————, " On the integration of trigonometric series ", *Journal London Math. Soc.*, 2 (1927), 164–166.
4. B. Kuttner, " The relation between Riemann and Cesàro summability ", *Proc. London Math. Soc.* (2), 38 (1935), 273–283.
5. ————, " A theorem on trigonometric series ", *Journal London Math. Soc.*, 10 (1935), 131–135.
6. N. Lusin, " Sur la convergence des séries trigonométriques de Fourier ", *Comptes rendus* 156 (1913), 1655–1658.
7. J. Marcinkiewicz, " On Riemann's two methods of summation ", *Journal London Math. Soc.*, 10 (1935), 268–272.
8. A. Plessner, " Zur Theorie der konjugierten trigonometrischen Reihen ", *Mitteilungen des Math. Seminars der Univ. Giessen*, 10 (1923), 1–36.
9. A. Rajchman, " Sur la multiplication des séries trigonométriques et sur une classe remarquable d'ensembles fermés ", *Math. Annalen*, 95 (1926), 389–408.
10. A. Rajchman and A. Zygmund, " Sur la relation du procédé de sommation de Cesàro et celui de Riemann ", *Bull. de l'Acad. Polonaise* (1925), 69-80.
11. S. Verblunsky, " On the theory of trigonometric series (VI) ", *Proc. London Math. Soc.* (2), 38 (1935), 284–326.
12. N. Wiener, " Note on a new type of summability ", *American Journal of Math.*, 45 (1923), 83–86.
13. A. Zygmund, " Sur la théorie riemannienne des séries trigonométriques ", *Math. Zeitschrift*, 24 (1926), 47–104.

The University,
Birmingham.

THE LOWER BOUNDS OF HERMITIAN QUADRATIC FORMS IN ANY QUADRATIC FIELD

By A. OPPENHEIM.

[Received 23 January, 1933.—Read 16 February, 1933.]

In a previous note† I obtained upper limits for the lower bounds of indefinite binary Hermitian quadratic forms for variables which are integers in $K(i)$. Here I consider the general quadratic field $K(\sqrt{m})$, where m is any quadratfrei integer (positive or negative), unity excepted, and I obtain precise results for certain sets of values of m, including also fields in which the unique factorization theorem does not hold, that is to say, fields whose class number is greater than 1.

To find these theorems I use a theorem‡ about quaternary quadratic forms of signature zero. The relevant part may be stated as follows:

THEOREM A. *Let f be a real quaternary quadratic form of zero signature, and so of positive determinant Δ, in rational integral variables. Let $L(f)$ be the lower bound of $|f|$ for all sets of values of the variables other than the zero set. Then, if $\Delta \leqslant 10L^4$, Δ/L^4 necessarily has one of the following values*§:

$$(1) \qquad \tfrac{9}{4},\ \tfrac{17}{4},\ \tfrac{25}{4},\ \tfrac{117}{16},\ \tfrac{33}{4},\ 9,\ \tfrac{625}{64}.$$

Also in each case except the last, f/L must have integral coefficients. In the last case, $2f/L$ must have integral coefficients.

For fields in which the class-number is unity, we may avoid using Theorem A and use instead the corresponding results due to Markoff‖ and Dickson‖ for indefinite ternary quadratic forms.

† *Quart. J. of Math.* (Oxford), 3 (1932), 10–14.

‡ Oppenheim, *Proc. London Math. Soc.* (2), 37 (1934), 63–81.

§ The values $\frac{9}{4}$, $\frac{17}{4}$ were given by me in *Annals of Math.*, 32 (1931), 271–298 (272). I have since found that the value $\frac{9}{4}$ is due to Markoff, *Bull. de l'Acad. des Sci. de St. Pétersbourg* (5), 16 (1902), 97–108.

‖ See Dickson, *Studies in the theory of numbers* (Chicago, 1930), 8, 108–133.

2. The integers in $K(\sqrt{m})$ are of the form

$$p = p_1 + p_2(\lambda + \mu\sqrt{m})\dagger, \tag{1}$$

where p_1, p_2 are rational integers and

$$\lambda = \mu = \tfrac{1}{2} \text{ if } m \equiv 1 \pmod 4, \quad \lambda = 0, \quad \mu = 1 \text{ if } m \not\equiv 1 \pmod 4. \tag{2}$$

Consequently, to the binary Hermitian quadratic form of determinant Δ, with variables which are integers in $K(\sqrt{m})$,

$$\phi = a(x + \tilde{\alpha}y)(\tilde{x} + \alpha\tilde{y}) + \frac{\Delta}{a} y\tilde{y}, \tag{3}$$

where a, Δ, α_1, α_2 are real and $\alpha = \alpha_1 + \alpha_2(\lambda + \mu\sqrt{m})$, corresponds the quaternary quadratic form in rational integral variables x_1, x_2, y_1, y_2,

$$f = a(X_1^2 - \mu^2 m X_2^2) + \frac{\Delta}{a}(Y_1^2 - \mu^2 m\, Y_2^2), \quad \Delta_f = \mu^4 m^2 \Delta^2, \tag{4}$$

where

$$\begin{cases} X_1 = x_1 + \lambda x_2 + (\alpha_1 + \lambda\alpha_2)(y_1 + \lambda y_2) - \mu^2 m\, \alpha_2 y_2, \\ X_2 = x_2 - \alpha_2 y_1 + \alpha_1 y_2, \\ Y_1 = y_1 + \lambda y_2, \quad Y_2 = y_2. \end{cases} \tag{5}$$

Plainly ϕ and f have the same absolute lower bound L. If $m > 0$, ϕ is always indefinite whether Δ is positive or negative, for $x\tilde{x}$, the norm of the integer x, can be positive or negative in a real quadratic field. If, however, $m < 0$, then for ϕ to be indefinite we require $\Delta < 0$. The case $\Delta = 0$ is excluded.

In any case, if ϕ is indefinite, f is a quaternary quadratic form of signature zero, determinant $\mu^4 m^2 \Delta^2$, and absolute lower bound L. By Theorem A, therefore, the assumption

$$\mu^4 m^2 \Delta^2 / L^4 \leqslant 10 \tag{6}$$

entails that $\mu^4 m^2 \Delta^2 / L^4$ has one of the seven values in (1; 1) and that the coefficients in $2f/L$ are integral.

The latter condition implies that $a/L, \ldots, \Delta/aL$ are rational, and therefore that Δ/L^2 is rational. The values $\frac{17}{4}$, $\frac{117}{16}$, and $\frac{33}{4}$ in (1; 1) cannot,

† When $m < 0$, $\sqrt{m} = i\sqrt{|m|}$. Also $\tilde{p} = p_1 + p_2(\lambda - \mu\sqrt{m})$.

therefore, be possible values of $\mu^4 m^2 \Delta^2/L^4$ and it remains to consider the other four values

$$\frac{9}{4},\ \frac{25}{4},\ 9,\ \frac{625}{64}. \tag{7}$$

It should now be noted that even when we have secured that the coefficients of f/L are integral, it does not necessarily follow that the lower bound of $|f|$ is L. To make sure of this point in the case of the first three numbers of (2; 7) it is necessary and sufficient also that f/L is not null.

The necessity is obvious. If it is not sufficient, then $|f/L|$ has minimum at least 2 and so, by (1; A),

$$2^4 \leqslant \tfrac{4}{9}\mu^4 m^2 \Delta^2/L^4,$$

which contradicts (2; 6).

In the case of $\frac{625}{64}$ [the last number of (2; 7)], we must secure (i) that the coefficients of $2f/L$ are integers, (ii) that $2f/L$ is not null, and (iii) that $2f/L$ does not represent $+1$ or -1. For if the minimum of $|2f/L|$ is now not 2, it is at least 3, and so

$$3^4 \leqslant \tfrac{4}{9} \cdot 2^4 \mu^4 m^2 \Delta^2/L^4 = \tfrac{625}{9},$$

a contradiction.

3. We investigate the possibilities

$$\frac{m\Delta}{L^2} = \frac{k\mu^{-2}}{4} \quad (k = \pm 6,\ \pm 10,\ \pm 12), \tag{1}$$

corresponding to the first three numbers in (2; 7), where $k > 0$ if $m < 0$ and $k\Delta > 0$ if $m > 0$.

Write, for brevity,

$$\beta_1 = \alpha_1 + \lambda\alpha_2, \quad \beta_2 = \mu\alpha_2, \tag{2}$$

and consider the coefficients of x_1^2, $x_1 y_1$, $x_1 y_2$, and y_1^2 in f,

$$a = Lu, \quad 2a\beta_1 = Lv, \quad 2a(\lambda\beta_1 - \mu m\beta_2) = Lr, \tag{3}$$

$$a(\beta_1^2 - m\beta_2^2) + \frac{\Delta}{a} = Lt. \tag{4}$$

By (1; A), u, v, r, t must be rational integers. Write

$$w = (\lambda v - r)/\mu \tag{5}$$

so that w is a rational integer. By (4), the rational integers u, v, w, t must satisfy the Diophantine equation

$$v^2 - \frac{w^2 - k\mu^{-2}}{m} = 4ut. \tag{6}$$

Necessary and sufficient conditions for (6) to be solvable are

$$(k\mu^{-2}|p) = 1 \text{ for every prime } p|m,\ p+k\mu^{-2}, \tag{7}$$

i.e.

$$(k|p) = 1 \text{ for } p|m,\ p+k\dagger,$$

the symbol being Legendre's.

For, if $m \equiv 1 \pmod 4$, so that $\mu = \frac{1}{2}$, if (7) holds, we can choose w so that $W = (w^2 - 4k)/m$ is an integer, $W \equiv 1$ or 0 modulo 4 according as w is odd or even. Choose any integer $v \equiv w \pmod 2$. Then integers u and t are determined by (6). Since v and w are of like parity, r, determined by (5), is also integral. It is easily verified that all the coefficients in f/L are integral.

Similarly, if $m \equiv 2$ or $3 \pmod 4$, so that $\lambda = 0$, $\mu = 1$, and if (7) holds, we can choose an integer w so that $W = (w^2 - k)/m$ is an integer with $W \equiv 1 \pmod 4$ when k is one of ± 6, ± 10, and $W \equiv 0 \pmod 4$ if k is ± 12.

With v odd or even accordingly, we can choose integers u and t to satisfy (6). Here $r = -w$, from (5), and the remaining coefficients of f/L are integral.

The coefficients of f/L being integral for any set of solutions of (6), it remains merely to secure that f/L is not null.

For this purpose write

$$k = Kk_1^2,\ K \text{ quadratfrei},\quad (m, K) = \delta,\ \delta > 0, \tag{8}$$

$$m = \delta M,\quad K = \delta K_1,$$

so that δ, M, K_1 are quadratfrei and co-prime in pairs.

Now $$4\mu^2 mu\delta f/L = MX^2 - \delta Y^2 + K_1 Z^2 - \delta M K_1 T^2,$$

where

$$X = 2\delta\mu u X_1,\quad Y = 2\mu^2 mu X_2,\quad Z = \delta k_1 Y_1,\quad T = \delta k_1 \mu Y_2.$$

The coefficients in f/L being assumed rational, to rational values of x_1, x_2, y_1, y_2 correspond rational values of X, Y, Z, T; and conversely. Also, if a form vanishes for rational values of the variables, it vanishes for

† For $\mu^{-2} = 1$ or 4 and k is even.

integral values of the variables; and conversely. Hence the form f/L (the coefficients of which are assumed rational) is or is not null according as the form

$$F = MX^2 - \delta Y^2 + K_1 Z^2 - \delta M K_1 T^2$$

is or is not null.

Suppose that F is null for integers X, Y, Z, T, not all zero. We may assume that $(X, Y, Z, T) = 1$. Let p be any prime which divides M. Then

$$K_1 Z^2 - \delta Y^2 \equiv 0 \pmod{p},$$

so that either $K_1\delta$ is a quadratic residue modulo p or else $Z = pZ_1$, $Y = pY_1$, Y_1, Z_1 integral, and therefore

$$X^2 - \delta K_1 T^2 \equiv 0 \pmod{p};$$

hence either p divides both X and T or else $K_1\delta$ is a quadratic residue modulo p. Hence necessary conditions for F to be null are

$$(K_1\delta \mid p) = 1 \quad \text{for every prime } p \mid M, \tag{9}$$

and similarly

$$(-K_1 M \mid q) = 1 \quad \text{for every prime } q \mid \delta, \tag{10}$$

$$(M\delta \mid r) = 1 \quad \text{for every prime } r \mid K_1. \tag{11}$$

Conversely, if (9), (10), (11) are satisfied, F is null. For, by a classical theorem of Legendre†, the form

$$MX^2 - \delta Y^2 + K_1 Z^2$$

is then null since M, $-\delta$, K_1 are not all of like sign; *a fortiori*, F is null.

Now conditions (9) are (with our values for k) equivalent to (7), and (7) must be assumed to ensure that the coefficients of f/L are integers.

Hence, if (7) holds, a necessary and sufficient condition that f/L is not null is that one at least of the characters in (10) and (11) be -1.

We may sum up the results of this section in

THEOREM B. *Let m be quadratfrei and not unity. Assume that the indefinite form in $K(\sqrt{m})$,*

$$\phi = a(x + \tilde{\alpha}y)(\tilde{x} + \alpha\tilde{y}) + \frac{\Delta}{a} y\tilde{y},$$

† See, *e.g.*, Dickson, *Introduction to the theory of numbers* (Chicago, 1929), Theorem 91, p. 117.

has absolute lower bound L such that

$$\frac{m\Delta}{L^2} = \frac{k}{4\mu^2},$$

where k has one of the values $\pm 6, \pm 10, \pm 12$ with

$$k > 0 \text{ if } m < 0, \quad k\Delta > 0 \text{ if } m > 0.$$

Then, necessarily, $(k|p) = 1$

for every prime $p|M$, and one at least of the characters

$$(-MK_1|q), \quad (M\delta|r)$$

has the value -1, where q and r are primes which divide δ and K_1 respectively, δ, K_1, M being defined by (8).

Further, we must have

$$\frac{\phi}{L} = ux\tilde{x} + \tfrac{1}{2}v(x\tilde{y}+\tilde{x}y) + \frac{w}{2\sqrt{m}}(x\tilde{y}-\tilde{x}y) + ty\tilde{y},$$

where u, v, w, t are (any) set of integers which satisfy the (solvable) equation (6).

If the class-number is unity, we may take $a = L$ and so $u = 1$, and $v = 0$ or 1.

4. We now investigate the last case of (2; 7),

(1) $$\frac{m\Delta}{L^2} = \frac{50\epsilon\mu^{-2}}{16} \quad (\epsilon = \pm 1),$$

where $\epsilon = 1$ if $m < 0$, $\epsilon\Delta > 0$ if $m > 0$.

Here the coefficients in $2f/L$ must be integers. As in (3; 3–6) we have now

$$2a = Lu, \quad 4a\beta_1 = Lv, \quad 4a(\lambda\beta_1 - \mu m\beta_2) = Lr, \quad \mu w = \lambda v - r,$$

$$2a(\beta_1^2 - \mu m\beta_2^2) + \frac{2\Delta}{a} = Lt,$$

(6) $$v^2 - \frac{w^2 - k\mu^{-2}}{m} = 4ut \quad (k = 50\epsilon),$$

with u, v, w, t rational integers.

For (6) to be solvable we require

$$(2\epsilon\,|\,p) = 1, \quad p\,|\,m, \quad (p,\ 10) = 1. \tag{7}$$

If (7) is satisfied, integral solutions of (6) exist and any set of solutions gives rise to integral coefficients for $2f/L$.

Again, as in § 3, $2f/L$ is null if and only if conditions (3; 9, 10, 11) are satisfied with 2ϵ playing the part of K [(3; 8)]. But here the conditions (3; 10, 11) are satisfied automatically. Hence a necessary and sufficient condition that $2f/L$ is not null is that, for some prime $p\,|\,M$, $(2\epsilon\,|\,p) = -1$. Compare with (4; 7). It follows that for (1) to be possible we must have

$$5\,|\,m, \quad m = 5M_1, \quad (2\epsilon\,|\,p) = 1, \quad p\,|\,M_1, \quad p \neq 2. \tag{8}$$

These conditions (8) ensure that $2f/L$ has integral coefficients and also that it is not null. By (1; A) its absolute minimum is either 1 or 2. It remains to secure, therefore, that $2f/L$ does not represent $+1$ or -1.

Now (8) shows that, in (6), $5\,|\,w$, $w = 5w_1$,

$$v^2 - 5A = 4ut, \quad AM_1 = w_1^2 - 2\epsilon\mu^{-2}. \tag{9}$$

Any integral solution of (9) in which one of u, t is a non-quadratic residue of 5 gives rise to a form $2f/L$ of minimum 2. For

$$2f/L \equiv ux_1^2 + vx_1y_1 + ty_1^2 \pmod 5, \tag{10}$$

and, by hypothesis, the binary quadratic form

$$\psi = ux_1^2 + vx_1y_1 + ty_1^2$$

of discriminant $5A$ has the character $(\psi\,|\,5) = -1$. Hence $2f/L$ cannot represent ± 1; its absolute minimum must be 2.

It is necessary to show that such solutions of (9) do exist. Choose an integer w_1 such that the integer A is congruent to 0 or 1 (mod 4) and is not a square. Clearly $(5, A) = 1$. We can find an odd prime P such that

$$(P\,|\,5) = -1, \quad (5A\,|\,P) = 1.$$

The congruence

$$V^2 \equiv 5A \pmod P$$

is then solvable with $V \equiv A \pmod 2$, $V^2 = 5A + 4PQ$, V, Q integers; the integers $u = P$, $t = Q$, $v = V$ satisfy our requirements.

It is clear also that no solution of (9) in which one of u, t is a quadratic residue of 5 will serve, for then the character $(\psi\,|\,5) = +1$, and so, by (10), $2f/L$ cannot represent ± 2; its absolute minimum must be 1.

To sum up we have

THEOREM C. *Let m be quadratfrei and not unity. Assume that the indefinite form in $K(\sqrt{m})$,*

$$\phi = a(x+\tilde{\alpha}y)(\tilde{x}+\alpha\tilde{y})+\frac{\Delta}{a}y\tilde{y},$$

has absolute lower bound L such that

$$\frac{m\Delta}{L^2}=\frac{25\epsilon}{8\mu^2} \quad (\epsilon = 1 \text{ or } -1),$$

with $\epsilon = 1$ *if* $m<0$, $\epsilon\Delta>0$ *if* $m>0$.

Then, necessarily,

$$m = 5M_1, \quad M_1 \text{ integral}, \quad (2\epsilon|p)=1 \text{ if } p|M_1, \quad p\neq 2.$$

Also we must have

$$\frac{2\phi}{L}=ux\tilde{x}+\tfrac{1}{2}v(x\tilde{y}+\tilde{x}y)+\frac{5w_1}{2\sqrt{m}}(x\tilde{y}-\tilde{x}y)+ty\tilde{y},$$

where u, v, w_1, t are solutions of the Diophantine equation

$$M_1(v^2-4ut)=5({w_1}^2-2\epsilon\mu^{-2})$$

such that one at least of u, t is congruent to $\pm 2 \pmod 5$.

5. It is convenient to restate the conditions on m which occur in B and C.

I. *When* $k=6$,

$$m = 2^\rho 3^\sigma M, \quad (M, 6)=1;$$

$$p|M, \quad p>0 \quad \text{implies} \quad p\equiv \pm 1, \pm 5 \pmod{24};$$

$$M\equiv(-1)^{1+\rho} \pmod 6.$$

I′. *When* $k=-6$, $M>0$;

$$p|M \quad \text{implies} \quad p\equiv 1, 5, 7, 11 \pmod{24};$$

$$M\equiv(-1)^{1+\rho+\sigma} \pmod 6.$$

II. *When* $k = 10$,

$$m = 2^{\rho} 5^{\sigma} M, \quad (M, 10) = 1;$$

$$p \mid M, \quad p > 0 \quad \text{implies} \quad p \equiv \pm 1, \pm 3, \pm 9, \pm 13 \pmod{40};$$

$$M \equiv \pm 3^{1+\rho+\sigma} \pmod{10}.$$

II′. *When* $k = -10$, $\qquad M > 0$;

$$p \mid M \quad \text{implies} \quad p \equiv 1, 7, 9, 11, 13, 19, 23, 37 \pmod{40};$$

$$M \equiv \pm 3^{1+\rho+\sigma} \pmod{10}.$$

III. *When* $k = 12$,

$$m = 3^{\sigma} M, \quad (M, 3) = 1;$$

$$p \mid M, \; p > 2 \quad \text{implies} \quad p \equiv \pm 1 \pmod{12};$$

$$M \equiv (-1)^{1+\sigma} \pmod{3}.$$

III′. *When* $k = -12$,

$$m = 2 \cdot 3^{\sigma} \cdot M, \quad M > 0, \quad (M, 6) = 1;$$

$$p \mid M, \quad p > 0 \quad \text{implies} \quad p \equiv 1 \pmod{6}.$$

IV. *When* $\epsilon = +1$,

$$m = 5 \cdot 2^{\rho} M, \quad (M, 10) = 1;$$

$$p \mid M, \quad p > 0 \quad \text{implies} \quad p \equiv \pm 1 \pmod{8}.$$

IV′. *When* $\epsilon = -1$, $\qquad M > 0$;

$$p \mid M \quad \text{implies} \quad p \equiv 1, 3 \pmod{8}.$$

Throughout m is quadratfrei and not unity.

We have obtained finally

THEOREM 1. *Let ϕ be an indefinite binary Hermitian quadratic form in variables which are integers in $K(\sqrt{m})$ of determinant $\Delta \neq 0$ and of absolute lower bound L. We can have*

$$(1) \qquad \frac{\mu^2 m \Delta}{L^2} = \tfrac{1}{4}k \quad (k = \pm 6, \pm 10, \pm 12), \qquad \frac{25\epsilon}{8} \quad (\epsilon = \pm 1),$$

where $k>0$, $\epsilon>0$ *if* $m<0$, $k\Delta>0$, $\epsilon\Delta>0$ *if* $m>0$, *only if* m *satisfies the conditions prescribed in the appropriate member of the set* I, ..., IV′, *and if the form* ϕ *is such that*

$$(2)\qquad \begin{cases} \dfrac{\phi}{L}=ux\tilde{x}+\frac{1}{2}v(x\tilde{y}+\tilde{x}y)+\dfrac{w}{2\sqrt{m}}(x\tilde{y}-\tilde{x}y)+ty\tilde{y} & (k), \\ \dfrac{2\phi}{L}=ux\tilde{x}+\ldots \qquad\qquad +ty\tilde{y} & (\epsilon), \end{cases}$$

where u, v, w, t *are integers which satisfy the Diophantine equation*

$$(3)\qquad v^2-\frac{w^2-k\mu^{-2}}{m}=4ut$$

with one of u, t *congruent to* $\pm 2 \pmod 5$ *when* $k=50\epsilon$.

For any other value of m *and for any other form* ϕ *we must have*

$$(4)\qquad \frac{\mu^2|m\Delta|}{L^2}>10.$$

I tabulate the values of m between -100 and 100 for which precise information is afforded by this theorem.

Case $k=6$. $\qquad L^2=\frac{2}{3}\mu^2 m\Delta$.

-1, 5, -19, 23, 29, -43, 47, 53, -67, 71, -73, -97; 95;

-3, 15, -57, 69, 87;

2, -10, 38, -46, -58, 86, -94;

6, -30.

Case $k=-6$. $\qquad L^2=-\frac{2}{3}\mu^2 m\Delta$.

5, 11, 29, 53, 59, 83; 35, 77;

30, 66;

2, 14, 62;

3, 21, 93.

Case $k=10$. $\qquad L^2=\frac{2}{5}\mu^2 m\Delta$.

± 3, ± 13, ± 37, ± 43, ± 53, ± 67, ± 83; ± 93;

± 30;

± 2, ± 62, ± 82; ± 78; ± 5.

Case $k=-10$. $L^2=-\frac{2}{5}\mu^2 m\Delta$.

7, 13, 23, 37, 47, 53; 77;

70;

2, 22, 38, 82;

5, 55, 95.

Case $k=12$. $L^2=\frac{1}{3}\mu^2 m\Delta$.

-1, 11, -13, 23, -37, 47, 59. -61, 71, -73, 83, -97;

2, -22, 26, -46, 74, -94;

3, -33, 39, -69;

-6, 66, -78.

Case $k=-12$. $L^2=-\frac{1}{3}\mu^2 m\Delta$.

2, 14, 26, 38, 62, 74, 86;

6, 42, 78.

Case $\epsilon=1$. $L^2=\frac{8}{25}\mu^2 m\Delta$.

± 5, ± 35, ± 85;

± 10, ± 70.

Case $\epsilon=-1$. $L^2=-\frac{8}{25}\mu^2 m\Delta$.

5, 15, 55, 85, 95;

10, 30.

Of particular interest is the case $m=10$. Here our theorem yields

$$L^2=\pm\tfrac{16}{5}\Delta \quad \text{or} \quad L^2<|\Delta|. \tag{5}$$

The forms ϕ for which the equation holds are given by

$$\frac{2\phi}{L}=ux\tilde{x}+\tfrac{1}{2}v(x\tilde{y}+\tilde{x}y)+\frac{\sqrt{10}}{2}w_2(x\tilde{y}-\tilde{x}y)+ty\tilde{y}, \tag{6}$$

where the rational integers u, v, w_2, t satisfy the equation

$$v^2 - 10w_2^2 \pm 5 = 4ut \tag{7}$$

with one of u, t congruent to $\pm 2 \pmod 5$. Clearly v must be odd and w_2 is odd or even according as the upper or lower sign is taken. In any case u and t are always odd and so at least 3 numerically.

Thus, although the absolute minimum of the form $2\phi/L$ in (6) is 2, no integral transformation in $K[\sqrt{(10)}]$ of value ± 1 will carry it into an equivalent form with leading coefficient 2. Naturally the class number in $K[\sqrt{(10)}]$ is greater than unity. In fields of class number unity the minimum is always properly represented.

A simple example of (6) is the form

$$3x\tilde{x} + \tfrac{3}{2}(x\tilde{y} + \tilde{x}y) + \sqrt{(10)}(x\tilde{y} - \tilde{x}y) - 3y\tilde{y}, \quad \Delta = -\tfrac{5}{4}, \quad L = 2, \tag{8}$$

which represents 2 for $x = 2$, $y = \sqrt{(10)} = w$ say. No integers r and s in $K(w)$ exist such that $2s - wr = 1$. In fact,

$$(2) = (2, w)^2, \ (w) = (2, w)(5, w),$$

and $(2, w)$ is a non-principal ideal of norm 2.

6. It will be observed that of the values of m, numerically less than 20,

$$-1, \pm 2, \pm 3, 5, 6, 7, -7, 11, -11, 13, 14, 19, -19,$$

for each of which fields the class number is unity, the only values for which our theorem fails to give precise information are -7, -11, 19. For these fields we may use a slight variation of the method referred to in § 1, already employed for $m = -1$.

We may now suppose in (2; 3) that $a = L$. Put $y_2 = 0$ in (2; 4). The resulting ternary quadratic form is indefinite of determinant $-\mu^2 mL\Delta$ and has absolute minimum L. With an obvious change of notation we may propose then the following problem:

Given that the indefinite ternary quadratic form

$$F = (x + \lambda y + \alpha z)^2 - \mu^2 m(y + \beta z)^2 + Dz^2, \tag{1}$$

where $D < 0$ if $m < 0$, has absolute minimum 1, determine the possible values of D, α, and β.

This problem I cannot solve. Certain results it is not difficult to obtain. These I give for $m = -7$ and $m = -11$.

Case $m = -7$. For any set of integers x, y, z not all zero we have, by hypothesis, $|F| \geqslant 1$. In particular, therefore, for any integer x either

$$(2) \qquad (x-\tfrac{1}{2}+\alpha)^2+\tfrac{7}{4}(1-\beta)^2-A \geqslant 1 \quad \text{or} \quad \leqslant -1 \quad (A=-D>0),$$

$$(3) \qquad (x+\alpha)^2+\tfrac{7}{4}\beta^2-A \geqslant 1 \quad \text{or} \quad \leqslant -1.$$

Denote the inequalities in (2) by P_x, N_x, the inequalities in (3) by $P_x{}^*$, $N_x{}^*$ respectively. We may assume that

$$(4) \qquad 0 \leqslant \alpha \leqslant \tfrac{1}{2}, \quad 0 \leqslant \beta \leqslant \tfrac{1}{2}$$

if we allow improper transformations.

$P_0{}^*$ implies $A \leqslant 0$. Hence $N_0{}^*$ holds. P_0, $N_0{}^*$ imply $\alpha = 0$, $\beta = 0$, $A = 1$, and F null. Hence N_0 holds. P_1 contradicts N_0. Therefore N_1 holds.

Now P_{-1}, N_1 imply

$$\alpha = 0, \quad A = \tfrac{5}{4}+\tfrac{7}{4}(1-\beta)^2,$$

and $F(1, 0, 1)$, $F(0, 1, 1)$ yield

$$(5) \qquad \beta = 0, \quad A = 3; \qquad \beta = \tfrac{2}{7}, \quad A = \tfrac{15}{7}.$$

Exclude the forms (5). Then N_{-1} holds, which is stronger than N_1. Since $P_1{}^*$ contradicts N_{-1}, $N_1{}^*$ must hold.

Now P^*_{-2}, $N_1{}^*$ imply $6\alpha \leqslant 1$; P_2, N_{-1} imply $3\alpha \geqslant 1$; and P^*_{-2}, N_2 imply $\beta \geqslant 2\alpha+\frac{4}{7}$, which contradicts (4). Hence P^*_{-2} is false, so N^*_{-2} must hold. We have the following possibilities:

$$(6) \qquad (N^*_{-2}, N_2) \quad \text{or} \quad (N_{-1}, N_1{}^*, N^*_{-2}, P_2).$$

It is easily seen that the first case gives $A \geqslant \frac{9}{2}$, with equality only when $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$, and that the second case gives $A \geqslant \frac{23}{7}$, with equality only when $\alpha = \frac{1}{2}$, $\beta = \frac{1}{7}$.

Each of the forms in (5) has absolute minimum unity. We obtain, therefore,

THEOREM 2. *In the field* $K(i\sqrt{7})$, *we have*

$$\frac{-\Delta}{L^2} = \tfrac{15}{7} \quad \textit{or} \quad 3, \quad \textit{or} \quad \geqslant \tfrac{23}{7}.$$

In the cases of equality, $\pm\phi/L$ is equivalent (properly or improperly) to the forms

$$x\tilde{x}+\frac{i}{\sqrt{7}}(x\tilde{y}-\tilde{x}y)-2y\tilde{y}, \quad -7\Delta=15L^2,$$

$$x\tilde{x}-3y\tilde{y}, \quad -\Delta=3L^2,$$

respectively.

Case $m=-11$. We use (2) and (3) with 11 in place of 7. P_0* contradicts $A>0$. From N_0*, P_1*, $2\alpha\geqslant 1$, $\alpha=\frac{1}{2}$,

$$A=\tfrac{5}{4}+\tfrac{11}{4}\beta^2.$$

From $F(0, 1, -1)$, $F(1, 1, -1)$, $F(1, -1, 2)$,

$$\beta=\tfrac{1}{11}, \quad A=\tfrac{14}{11}. \tag{7}$$

Apart from this case, we have N_1*, which is stronger than N_0*. If P_0 then

$$0\geqslant 3\alpha+\tfrac{11}{2}\beta, \quad \alpha=\beta=0, \quad A=2. \tag{8}$$

Exclude this form. Then N_0 holds. Since P_1 contradicts N_0, N_1 must hold.

If N_{-1}, then

$$A\geqslant 1+(\tfrac{3}{2}-\alpha)^2+\tfrac{11}{4}(1-\beta)^2,$$

$$A\geqslant 1+(1+\alpha)^2+\tfrac{11}{4}\beta^2 \qquad (N_1^*),$$

whence $A\geqslant\frac{13}{4}$.

If P_{-1}, then, with N_1,

$$4\alpha\leqslant 0, \quad \alpha=0, \quad A=\tfrac{5}{4}+\tfrac{11}{4}(1-\beta)^2, \quad \beta\leqslant\tfrac{4}{11} \quad (\text{from } N_1^*).$$

From $F(2, 0, 1)$, $F(3, 0, 2)$ we find

$$\beta\geqslant\tfrac{4}{11} \quad \text{or} \quad \tfrac{2}{11}\leqslant\beta\leqslant\tfrac{3}{11}.$$

Hence

$$A=2+\tfrac{4}{11} \quad \text{or} \quad A\geqslant\tfrac{119}{44}. \tag{9}$$

Each of the forms obtained in (7), (8), (9) has absolute minimum unity. We thus obtain

THEOREM 3. *In $K[i\sqrt{(11)}]$ we have*

$$-\frac{\Delta}{L^2}=\tfrac{14}{11}, \quad 2, \quad \tfrac{26}{11}, \quad \textit{or} \quad \geqslant\tfrac{119}{44}.$$

In the cases of equality, $\pm\phi/L$ is equivalent (properly or improperly) to the forms

$$x\tilde{x}+\tfrac{1}{2}(x\tilde{y}+\tilde{x}y)+\frac{i}{2\sqrt{11}}(x\tilde{y}-\tilde{x}y)-y\tilde{y},$$

$$x\tilde{x}-2y\tilde{y},$$

$$x\tilde{x}+\frac{2i}{\sqrt{11}}(x\tilde{y}-\tilde{x}y)-2y\tilde{y},$$

respectively.

For $m=-2$, I find that

$$(A,\alpha,\beta)=(\tfrac{5}{4},\tfrac{1}{2},0),\ (\tfrac{7}{4},\tfrac{1}{2},\tfrac{1}{2}),\ (\tfrac{5}{2},0,\tfrac{1}{2}) \quad \text{or} \quad A\geqslant 2{\cdot}71. \tag{10}$$

The corresponding theorem for $K(i\sqrt{2})$ is easy to write down, and is an improvement on our original theorem.

The results for some of the other values of m may be extended in the same way. It may be noted also that the method of § 6 can be adapted to deal with the case when the class-number is not unity and the minimum is not necessarily properly represented.

Raffles College,
Singapore, S.S.

INDEX.

INDEX OF PAPERS PUBLISHED IN THIS VOLUME ARRANGED ACCORDING TO NAMES OF AUTHORS

[The numbers refer to pages.]

Bailey, W. N. Some infinite integrals involving Bessel functions, 37.
Bosanquet, L. S., and Offord, A. C. A local property of Fourier series, 273.
Burnett, D. The distribution of molecular velocities and the mean motion in a non-uniform gas, 382.
Cox, H. R. On the synthesis and analysis of simply-stiff frameworks, 203.
Gwilliam, A. E. Cesàro means of power series (2), 345.
———— On Lipschitz conditions, 353.
Hall, P. On a theorem of Frobenius, 468.
Heawood, P. J. Failures in congruences connected with the four-colour map theorem, 189.
Hyslop, J. M. Some relations between the de la Vallée Poussin and Abel methods of summability, 449.
Kuttner, B. Some relations between different kinds of Riemann summability, 524.
Lawrence, B. E. The summability of double power series, 321.
Littlewood, D. E. Some properties of S-functions, 49.
———— On compound and induced matrices, 370.
Meijer, C. S. Integraldarstellungen aus der Theorie der Besselschen Funktionen, 1.
Offord, A. C. On Fourier transforms (II), 281.
Oppenheim, A. The lower bounds of Hermitian quadratic forms in any quadratic field, 541.
Phillips, Eric G. On the region of summability (A, λ, α) $(\alpha > 1)$ and $(A, \lambda \log \lambda, 1)$ of Dirichlet's series of type λ_n, 436.
Roth, L. Adjoint primals in four dimensions, 217.
———— Some formulae for primals in four dimensions (III), 365.
Sargent, W. L. C. On the Cesàro derivates of a function, 235.
Synge, J. L. Conditions satisfied by the vorticity and the stream-function in a viscous liquid moving in two dimensions between fixed parallel planes, 23.
Szymański, Piotr. On the integral representations of the Lommel functions, 71.
Verblunsky, S. On positive harmonic functions (*Second paper*), 290.
Watson, G. N. Singular moduli (3), 83.
Welchman, W. G. Special scrolls and involutions of canonical curves, 143.
Whipple, F. J. W. Relations between well-poised hypergeometric series of the type ${}_7F_6$, 336.
Whittaker, J. M. A theorem on meromorphic functions, 255.
Wolf, F. On the (C, k) summability of a trigonometrical integral, 502.

INDEX ACCORDING TO SUBJECTS.

[*The numbers refer to pages.*]

Algebra : Hall, P., 468; Littlewood, D. E., 49, 370.
Analytic Functions : Lawrence, B. E., 321; Whittaker, J. M., 255.
Arithmetic : Watson, G. N., 83.
Bessel Functions : Bailey, W. N., 37; Meijer, C. S., 1; Szymański, P., 71.
Dirichlet Series : Phillips, Eric G., 436.
Elasticity : Cox, H. R., 203.
Elliptic Functions : Watson, G. N., 83.
Forms, Quadratic : Oppenheim, A., 541.
Fourier Integrals : Offord, A. C., 281; Wolf, F., 502
Fourier Series : Bosanquet, L. S., and Offord, A. C., 273; Wolf, F., 502.
Functions of a Complex Variable : Lawrence, B. E., 321; Whittaker, J. M., 255.
Functions of a Real Variable : Bosanquet, L. S., and Offord, A. C., 273; Sargent, W. L. C., 235.
Geometry : Roth, L., 217, 365; Welchman, W. G., 143.
Groups, Theory of : Hall, P., 468.
Harmonic Functions : Verblunsky, S., 290.
Hydrodynamics : Synge, J. L., 23.
Hypergeometric Series : Whipple, F. J. W., 336.
Integral Calculus : Wolf, F., 502.
Integral Equations : Offord, A. C., 281.
Kinetic Theory of Gases : Burnett, D., 382.
Mathematical Physics : Burnett, D., 382; Cox, H. R., 203; Synge, J. L., 23.
Matrices : Littlewood, D. E., 370.
Numbers, Theory of : Oppenheim, A., 541.
Series, Theory of : Bosanquet, L. S., and Offord, A. C., 273; Gwilliam, A. E., 345, 353; Hyslop, J. M., 449; Kuttner, B., 524; Lawrence, B. E., 321; Phillips, Eric G., 436.
Statics : Cox, H. R., 203.
Topology : Heawood, P. J., 189.
Transforms : Offord, A. C., 281.
Trigonometrical Series : Bosanquet, L. S., and Offord, A. C., 273; Wolf, F., 502.
Viscous Liquid, Motion of : Synge, J. L., 23.

CORRIGENDA

THE CHARACTERS OF THE SYMMETRIC GROUP OF ORDER 11!

M. ZIA-UD-DIN.

[*Proc. London Math. Soc.* (2), 39 (1935), 200–204.]

The characteristics of $\{5\,3\,1^3\}$, for negative permutations, are to be read with signs changed.

ON THE LATTICE-POINTS ON CURVES OF GENUS

K. MAHLER.

[*Proc. London Math. Soc.* (2), 39 (1935), 431–466.]

P. 432, "Two cubic curves which have the same absolute invariant J can be transformed into one another by a birational transformation, and *this transformation has rational coefficients when there are points with rational coordinates on both curves.*" The second assertion is *not* true, but no use of it is made in the paper.

P. 441, line 5 from below: Change "z/a" into "$1/a$".

P. 445, lines 1, 3, 5, 7 from below: Change the signs "$=$" into "$\leqslant$".

P. 446, line 4: Replace "the last two inequalities" by "the last eight inequalities".

BIBL. LONDIN. UNIV.